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On certain geometric properties of generalized polylogarithm function

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Abstract

In this manuscript, we investigate the Hadamard product $H_f(a,b;z)$ of normalized analytic functions in the unit disc Δ and generalized second order polylogarithm function $G(a,b;z)$, where

$$G(a,b;z) = \sum_{n=1}^{\infty} \frac{(a+1)(b+1)}{(n+a)(n+b)} z^n, a,b \in \mathbb{C}\backslash\{-1,-2,\dots\}.$$

Further, we derive certain characteristics of the function $H_f(a,b;z)$ and obtain various sufficient conditions for the function $H_f(a,b;z)$ to be Janowski starlike. Also certain inequalities containing the function $H_f(a,b;z)$ are obtained.

Keywords

Analytic functions, Convolution, Subordination, Generalized polylogarithm function.

AMS Subject Classification

30C45, 30C80.

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1. Introduction

Let $\mathscr{H}$ signifies the class of analytic functions in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathscr{A}_n$ signifies the class of analytic functions in Δ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \tag{1.1}$$

and denote $\mathscr{A} := \mathscr{A}_1$.

Let f and g be analytic in Δ, then we say that f is subordinate to g in Δ (written $f \prec g$) if there exists a Schwarz function $w(z)$, analytic in Δ with

$$w(0) = 0, \qquad |w(z)| < 1 \qquad (z \in \Delta)$$

in a way that

$$f(z) = g(w(z)), \qquad (z \in \Delta).$$

Particularly, if the function g is univalent in Δ, then the subordination is similar to

$$f(0) = g(0) \qquad \text{or} \qquad f(\Delta) \subset g(\Delta).$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

be the Maclaurin series, the Hadamard product of f and g is defined by the power series

$$(f*g)z = f(z)*g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

The polylogarithm function [4] is defined as the analytic continuation of the Dirichlet series,

$$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \qquad (z \in \Delta, s \in \mathbb{C}).$$

Ramanujan [1] derived several properties of dilogarithms $Li_2(z)$ and trilogarithms $Li_3(z)$. Ponnusamy and Sabapathy [8] obtained certain geometric properties of generalized polylogarithms and determined the conditions on the parameters for the function to be univalent and starlike.

Let

$$G(a,b;z) = \sum_{n=1}^{\infty} \frac{(a+1)(b+1)}{(n+a)(n+b)} z^n, \; a,b \in \{-1,-2,\dots\}, \qquad (1.2)$$

be the generalized second order polylogarithm function, which significantly reduces to Lerch function of order 2, for $a = b$. For the values $a = b = 0$ (1.2) reduces to dilogarithm function [2] and to the identity function for $a = -1$ and $b = -1$.

In the convolution structure $H_f(a,b;z) = G(a,b;z) * f(z)$, for $a \neq b$, we get the following integral representation

$$H_f(a,b;z) = \frac{(a+1)(b+1)}{b-a} \int_0^1 t^{a-1}(1-t)^{b-a} f(tz)dt. \qquad (1.3)$$

It should be remarked that the convolution structure (1.3) is the generalization of many well known operators.

1. For $a = -\alpha$ and $b = 2-\alpha$, the function (1.3) reduces to the operator
$$I_\alpha(z) = \frac{(1-\alpha)(3-\alpha)}{2} \int_0^1 t^{-(\alpha+1)}(1-t^\alpha) f(tz)dt$$
with $0 \leq \alpha < 1$.
2. For the limiting case $b \to \infty$ $Re\ a > -1$, the function $H_f(a,b;z)$ represents the Bernardi transform [8, 11]
$$B_f(a,z) = \frac{a+1}{z^a} \int_0^z t^{\alpha-1} f(t)dt.$$

Now, it can easily be verified that the function $H_f(a,b;z)$ satisfies the differential equation

$$z^2 H_f''(a,b;z) + (a+b+1)zH_f'(a,b;z) + abH_f(a,b;z) = (a+1)(b+1)f(z). \qquad (1.4)$$

S. Ponnusamy [9] considered the differential equation (1.4) and discussed certain geometric properties of $H_f(a,b;z)$ that depend on the parameters a and b.

Let S denote the subclass of $\mathscr{A}$ consisting of univalent functions. For $-1 \leq B < A \leq 1$, let $P[A,B]$ denote the class consisting of normalized analytic functions $p(z)$ satisfying $p(0) = 1$ and $p'(0) = 0$ in ways that

$$p(z) \prec \frac{1+Az}{1+Bz},$$

and its subclass $S^*[A,B]$ the class of Janowski starlike functions is defined by

$$S^*[A,B] = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right\}.$$

For suitable choices of the parameters A and B, certain well known subclasses are obtained as special cases of the class $S^*[A,B]$.

For $0 \leq \alpha < 1$, $S^*[1-2\alpha,-1] = S^*(\alpha)$ is the familiar class of starlike functions of order α, the class $S^*[1-\alpha,0] = \left\{ f \in \mathscr{A}; \left| \frac{zf'(z)}{f(z)} - 1 \right| < (1-\alpha) \right\}$ is denoted by $S_1^*(\alpha)$ and $S^* = S^*(0)$ denotes the class of starlike functions.

Recently many authors have investigated the sufficient conditions for functions to belong to $S^*[A,B]$ and to various subclasses of $S^*[A,B]$. The Janowski starlikeness of Bessel function and Kummer hypergeometric function are studied by Ravichandran et.al. [13]. Tuneski [14] obtained sufficient condition for a function to be Janowski starlike with respect to N-symmetric points.

Inspired by the aforementioned works, we obtain various sufficient conditions for the univalence and Janowski starlikeness of the function $H_f(a,b;z)$ and derive certain inequalities involving $H_f(a,b;z)$.

Lemma 1.1. *[6] If an analytic function f has the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots, (z \in \Delta)$ and satisfies the condition*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1$$

then f is univalent in Δ.

Lemma 1.2. *[5] If $f \in \mathscr{A}$ satisfies $\left| \frac{f(z)}{z} - 1 \right| < 1$, $(z \in \Delta)$, then $f(z)$ is univalent and starlike for $|z| < 1/2$.*

2. Main results

Theorem 2.1. *Let $f \in \mathscr{A}$, $a,b \in \mathbb{C} \backslash \{-1,-2,\dots\}$ and $-1 \leq B < A \leq 1$. If*

$$(1+A)^2(1+B)^2 + (1+A)^2(1+B)^3 Re(a+b) + (1+B)^4 Re\{F(z)\} > 0 \qquad (2.1)$$

and

$$\Big((1+A)^2(A-B) + (1+A)(1+B)Re(a+b) + (1+B)^2 ReF(z)\Big) \Big(2(A-B) + (1-A^2) + (1-A)(1-B)Re(a+b) + (1-B)^2 ReF(z)\Big) - \Big((1-A^2) + (1-AB)Im(a+b) + (1-B^2)ImF(z)\Big)^2 > 0, \qquad (2.2)$$

where $F(z)=\left\{ab+\frac{(a+1)(b+1)f(z)}{H_f(a,b;z)}\right\}$, *then* $H_f(a,b;z)\in S^*[A,B]$.

Proof. Let

$$q(z)=\frac{zH_f'(a,b;z)}{H_f(a,b;z)}.$$

Define the analytic functions

$$p(z)=-\frac{(1-A)-(1-B)q(z)}{(1+A)-(1+B)q(z)}, \tag{2.3}$$

then we have

$$q(z)=\frac{zH_f'(a,b;z)}{H_f(a,b;z)}=\frac{(1-A)+(1+A)p(z)}{(1-B)+(1+B)p(z)},$$
$$q'(z)=\frac{2(A-B)p'(z)}{((1-B)+(1+B)p(z))^2},$$

and

$$q''(z)=\frac{2(A-B)(1-B)+((1+B)p(z))p''(z)-4(1+B)(A-B)(p'(z))^2}{((1-B)+(1+B)(p(z))^3}.$$

From (1.4), we obtain

$$zq'(z)+q^2(z)+(a+b)q(z)+ab$$
$$=(a+1)(b+1)\frac{f(z)}{H_f(a,b;z)}. \tag{2.4}$$

Using (2.3) and simplifying (2.4) further, we obtain,

$$C(z)p^2(z)+D(z)p(z)+2(A-B)zp^{'}(z)+E(z)=0, \tag{2.5}$$

where

$$C(z)=(1+A)^2+(a+b)(1+A)(1+B)$$
$$+ab(1+B)^2+(a+1)(b+1)\frac{f(z)}{H_f(a,b;z)}(1+B)^2,$$
$$D(z)=2(1-A)^2+2(a+b)(1-AB)$$
$$+2ab(1-B)^2+2(a+1)(1-B^2)\frac{f(z)}{H_f(a,b;z)},$$

and

$$E(z)=(1-A)^2+(a+b)(1-A)(1-B)$$
$$+ab(1-B)^2+(a+1)(b+1)\frac{f(z)}{H_f(a,b;z)}(1-B)^2.$$

Define $\psi(r,s;z)$ such that,

$$\psi(r,s;z)=C(z)r^2+D(z)r+2(A-B)(z)s+E(z). \tag{2.6}$$

By letting $\Omega=\{0\}$, from (2.5), we get

$$\psi(p(z),zp'(z);z)\in\Omega.$$

Now,

$$Re\ \psi(i\rho,\sigma;z)$$
$$=Re(C(z)(i\rho)^2+D(z)i\rho+2(A-B)(z)\sigma+E(z))$$
$$\le -Re\ C(z)\rho^2-ImD(z)\rho+2(A-B)\left(-\left(\frac{1+\rho^2}{2}\right)\right)$$
$$+Re\ J(z)$$
$$=-\left[Re\ C(z)+\frac{2(A-B)}{2}\right]\rho^2-Im\ D(z)\rho-2(A-B)$$
$$+Re\ E(z)$$
$$=R\rho^2+S\rho+T=G(\rho),$$

where

$$R=-(Re\ C(z)+(A-B)),\ S=-Im\ D(z),$$
$$T=-2(A-B)+Re\ E(z).$$

Now, we observe that

$$\max_{\rho\in\mathbb{R}}G(\rho)=\frac{4RT-S^2}{4R},\qquad (R<0).$$

From (2.1) and (2.2), we have $Re\ \psi(i\rho,\sigma;z)<0$.

Therefore by the result [11](pg.35), $Re\ p(z)>0$,

that is,

$$-\frac{(1-A)-(1-B)q(z)}{(1+A)-(1+B)q(z)}\prec\frac{1+z}{1-z}.$$

Hence for an analytic function g in Δ with $g(0)=0$ such that

$$-\frac{(1-A)-(1-B)q(z)}{(1+A)-(1+B)q(z)}=\frac{1+g(z)}{1-g(z)}.$$

Hence

$$q(z)\prec\frac{1+Az}{1+Bz}.$$

In particular

$$\frac{zH_f'(a,b;z)}{H_f(a,b;z)}\in P[A,B],$$

that is

$$H_f(a,b;z)\in S^*[A,B].$$

□

The following theorem gives the sufficient condition that rely on the parameters a and b, for the function $H_f(a,b;z)$ to be in the subclass $S_1^*(\alpha)$.

Theorem 2.2. *If* $a,b\in\mathbb{C}\backslash\{-1,-2,\dots\}$, *with* $2|a+1||b+1|<(1-\alpha)(\alpha+ab+2)$ *then* $H_f(a,b;z)\in S_1^*(\alpha)$.

Proof. Let

$$p(z) = \frac{zH_f'(a,b;z)}{H_f(a,b;z)} - 1. \quad (2.7)$$

From (1.4), we have

$$zp'(z) + p^2(z) + p(z)(a+b+2) + (a+b+ab+1) - \frac{(a+1)(b+1)f(z)}{H_f(a,b;z)} = 0.$$

That is

$$\psi(p(z), zp'(z); z) = 0,$$

where

$$\psi(r,s,t) = r(r+(a+b+2)) + s + a + b + ab + 1 - \frac{(a+1)(b+1)f(z)}{H_f(a,b;z)}.$$

Now, we claim that

$$|p(z)| < (1-\alpha), \quad 0 \le \alpha < 1.$$

From the result [11](Pg.34), it is sufficient to show that,

$$\psi((1-\alpha)e^{i\theta}, Ke^{i\theta}; z) \notin \Omega,$$

where θ is real, $K \ge (1-\alpha)$, and $z \in \Delta$, for $\Omega = \{0\}$, $n = 1$ and $q(z) = (1-\alpha)z$.

Now,

$$\psi(1-\alpha)e^{i\theta}, Ke^{i\theta}; z) = Ce^{i\theta} - D, \quad (2.8)$$

where

$$C = (1-\alpha)[(1-\alpha)e^{i\theta} + (a+b+2)] + (1-\alpha)$$

and

$$D = -(a+b+ab+1) + \frac{(a+1)(b+1)f(z)}{H_f(a,b;z)}.$$

Also

$$\begin{aligned} |C| \ge Re\, C &> (1-\alpha)[(1-\alpha)e^{i\theta} + (a+b+2)] + (1-\alpha) \\ &\ge (1-\alpha)[(1-\alpha)\cos\theta + (a+b+2)] + (1-\alpha) \\ &\ge (1-\alpha)(\alpha + a + b + 2) = \beta, \text{ (say)}, \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} |D| &\le \left| (a+1)(b+1) - \frac{(a+1)(b+1)f(z)}{H_f(a,b;z)} \right| \\ &\le |(a+1)(b+1)| - \left| \frac{(a+1)(b+1)f(z)}{H_f(a,b;z)} \right| \\ &\le 2|(a+1)(b+1)| \le \beta, \end{aligned}$$

where β is as given in (2.9).
Then we have,

$$\begin{aligned} |\psi((1-\alpha)e^{i\theta}, Ke^{i\theta}; z)| &\ge |e^{i\theta}C - D| \\ &\ge |C| - |D| \\ &> \beta - \beta = 0. \end{aligned}$$

Therefore,

$$\psi((1-\alpha)e^{i\theta}, Ke^{i\theta}; z) \notin \Omega$$

Thus, $|p(z)| < (1-\alpha)$ in Δ and hence $H_f(a,b;z) \in S_1^*(\alpha)$. □

The proof of next theorem follows from the definitions of the function $H_f(a,b;z)$ and the class $S_1^*(\alpha)$, hence we give the statement alone.

Theorem 2.3. *Let $g : \Delta \to \mathbb{C}$ be defined by*

$$g(z) = \frac{H_f(a,b;z)}{z},$$

such that $\left|\frac{zg'(z)}{g(z)}\right| < (1-\alpha)$, for $\alpha \in [0,1/2]$ and $z \in \Delta$, then

$$H_f(a,b;z) \in S_1^*(\alpha)$$

3. Univalence and Starlikeness of $H_f(a,b;z)$

Theorem 3.1. *Let $a,b \in \mathbb{C}\backslash\{-1,-2,\dots\}$ such that*

$$|(a+1)(b+1)| \left| \frac{f(z)}{z} - 1 \right| < Re[(a+2)(b+2)] \quad (3.1)$$

then $H_f(a,b;z)$ is univalent and starlike for $|z| < 1/2$.

Proof. Let

$$p(z) = \frac{H_f(a,b;z)}{z} - 1,$$

then $p(z)$ is an analytic function with $p(0) = 0$.

Using the differential equation (1.4), we obtain the equation,

$$z^2p''(z) + zp'(z)(a+b+3) + p(z)(a+1)(b+1) + (a+1)(b+1) - (a+1)(b+1)\frac{f(z)}{z} = 0. \quad (3.2)$$

If we consider

$$\psi(r,s,t;z) = t + (a+b+3)s + (a+1)(b+1)r - (a+1)(b+1)\left(\frac{f(z)-1}{z}\right) \text{ and } \Omega = \{0\},$$

then (3.2) implies

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \text{ for all } z \in \Delta.$$

Further, for any $\theta \in \mathbb{R}$, $K \geq M$, $Re(Le^{-i\theta}) \geq 0$ and

$$M = \frac{|(a+1)(b+1)(\frac{f(z)}{z}-1)|}{Re(a+2)(b+2)}, \quad \text{we have}$$

$$\begin{aligned}
&|\psi(Me^{i\theta}, Ke^{i\theta}, L; z)| \\
&= |L + (a+b+3)Ke^{i\theta} + (a+1)(b+1)Me^{i\theta} \\
&\qquad - (a+1)(b+1)K(z)| \\
&\geq Re(a+b+3)M + Re(a+1)(b+1)M \\
&\quad - |(a+1)(b+1)K(z)| \\
&= Re[a+b+3+(a+1)(b+1)]M \\
&\quad - |(a+1)(b+1)K(z)| \\
&= Re[(a+2)(b+2)]M - |(a+1)(b+1)K(z)| \\
&= 0.
\end{aligned}$$

That is, $\psi(Me^{i\theta}, Ke^{i\theta}, L; z) \notin \Omega$.
Now by applying the result[11](Pg.34), we get

$$|p(z)| < M, \qquad |z| < 1/2.$$

The assertion of the theorem follows by using (3.1) and applying Lemma 1.2. □

Theorem 3.2. *For $a, b \in \mathbb{C}\backslash\{-1,-2,-3,\dots\}$. If $H_f(a,b;z)$ satisfies any one of the following conditions for all $z \in \Delta$:*

$$\left|\frac{z^2H_f'(a,b;z)}{[H_f(a,b;z)]^2}\left(\frac{(zH_f(a,b;z))''}{H_f'(a,b;z)} - \frac{2zH_f'(a,b;z)}{H_f(a,b;z)}\right)\right| < 1, \tag{3.3}$$

$$\left|\frac{(H_f(a,b;z)}{z^2H_f'(a,b;z)}\left[\frac{[zH_f(a,b;z)]''}{H_f'(a,b;z)} - \frac{2zH_f'(a,b;z)}{H_f(a,b;z)}\right]\right| < 1/4, \tag{3.4}$$

$$\left|\frac{\frac{(zH_f(a,b;z))''}{H_f'(a,b;z)} - \frac{2zH_f'(a,b;z)}{H_f(a,b;z)}}{\frac{z^2H_f'(a,b;z)}{(H_f(a,b;z))^2} - 1}\right| < 1/2, \tag{3.5}$$

$$Re\left\{\frac{z^2H_f'(a,b;z)}{(H_f(a,b;z))^2}\left(\frac{\frac{(zH_f(a,b;z))''}{H_f'(a,b;z)} - \frac{2zH_f'(a,b;z)}{H_f(a,b;z)}}{\frac{z^2H_f'(a,b;z)}{(H_f(a,b;z))^2} - 1}\right)\right\} < 1, \tag{3.6}$$

then $H_f(a,b;z)$ is univalent.

Proof. Let

$$\frac{z^2H_f'(a,b;z)}{(H_f(a,b;z))^2} = 1 + h(z), \tag{3.7}$$

then $h(z)$ is analytic in Δ and $h(0) = 0$.
Further differentiating (3.7), we obtain

$$\frac{[zH_f(a,b;z)]''}{H_f'(a,b;z)} - \frac{2zH_f'(a,b;z)}{H_f(a,b;z)} = \frac{zh'(z)}{1+h(z)}. \tag{3.8}$$

Hence, from (3.7) and (3.8), we have

$$\begin{aligned}
K_1(z) &= \frac{z^2H_f'(a,b;z)}{(H_f(a,b;z))^2}\left[\frac{[zH_f(a,b;z)]''}{H_f'(a,b;z)} - \frac{2zH_f'(a,b;z)}{H_f(a,b;z)}\right] \\
&= zh'(z). \qquad (3.9)\\
K_2(z) &= \frac{[H_f(a,b;z)]^2}{z^2H_f'(a,b;z)}\left[\frac{[zH_f(a,b;z)]''}{H_f'(a,b;z)} - \frac{2zH_f'(a,b;z)}{H_f(a,b;z)}\right] \\
&= \frac{zh'(z)}{(1+h(z))^2}. \qquad (3.10)
\end{aligned}$$

$$K_3(z) = \left[\frac{\frac{zH_f''(a,b;z)}{H_f'(a,b;z)} - \frac{2zH_f''(a,b;z)}{H_f(a,b;z)}}{\left[\frac{z^2H_f'(a,b;z)}{(H_f(a,b;z))^2} - 1\right]}\right] = \frac{zh'(z)}{h(z)}\frac{1}{1+h(z)}.$$

$$K_4(z) = \frac{z^2H_f'(a,b;z)}{[H_f(a,b;z)]^2}\left[\frac{\frac{[zH_f(a,b;z)]''}{H_f'(a,b;z)} - \frac{2zH_f'(a,b;z)}{H_f(a,b;z)}}{\frac{z^2H_f'(a,b;z)}{H_f(a,b;z)} - 1}\right] = \frac{zh'(z)}{h(z)}.$$

Now, suppose that there exist $z_0 \in \Delta$ such that

$$\max_{|z|<|z_0|} |h(z)| = |h(z_0)| = 1,$$

then by Jack's Lemma [3], we have

$$z_0h'(z_0) = \delta h(z_0),$$

where $\delta \in \mathbb{R}$ and $\delta \geq 1$. Therefore by letting $h(z_0) = e^{i\theta}$ in each equation of (3.9), we obtain that

$$|K_1(z_0)| = |z_0h'(z_0)| = |\delta h(z_0)| = |\delta e^{i\theta}| \geq 1 \tag{3.11}$$

$$\begin{aligned}
|K_2(z_0)| &= \left|\frac{z_0h'(z_0)}{(1+h(z_0))^2}\right| = \frac{\delta h(z_0)}{(1+h(z_0))^2} \\
&= \frac{\delta}{|1+e^{i\theta}|^2} \geq 1/4 \qquad (3.12)\\
|K_3(z_0)| &= \frac{z_0h'(z_0)}{h(z_0)}.\frac{1}{1+h(z_0)} = \left|\frac{\delta h(z_0)}{h(z_0)}.\frac{1}{1+h(z_0)}\right| \\
&= \left|\frac{\delta}{|1+e^{i\theta}|^2}\right| \geq 1/2 \qquad (3.13)
\end{aligned}$$

$$Re[K_4(z_0)] = Re\left\{\frac{z_0h'(z_0)}{h(z_0)}\right\} = Re\left\{\frac{\delta h(z_0)}{h(z_0)}\right\} = \delta \geq 1$$

which contradicts our assumption (3.3) to (3.6) respectively and hence $|h(z)| < 1$ for all $z \in \Delta$. Therefore, we obtain

$$\left|\frac{z^2H_f'(a,b;z)}{(H_f(a,b;z))^2} - 1\right| = |h(z)| < 1$$

which implies $H_f(a,b;z)$ is univalent, by Lemma 1.1. □

Theorem 3.3. *Let $c > 0, d \geq 0$ such that $c + 2d \leq 1$. If $H_f(a,b;z)$ satisfies*

$$Re\left\{\frac{(zH_f(a,b;z))''}{(H'_f(a,b;z)} - \frac{2zH'_f(a,b;z)}{H_f(a,b;z)}\right\} < \frac{c+d}{(1+c)(1-d)},\ z \in \Delta$$

then $H_f(a,b;z)$ is univalent in Δ.

Proof. Let

$$\frac{z^2H'_f(a,b;z)}{(H_f(a,b;z))^2} = \frac{1+ah(z)}{1-bh(z)}\ (z \in \Delta). \tag{3.14}$$

then $h(z)$ is analytic in Δ and $h(0) = 0$.
Differentiating (3.14), we get

$$\frac{(zH_f(a,b;z))''}{(H'_f(a,b;z)} - \frac{2zH'_f(a,b;z)}{H_f(a,b;z)} = \frac{(c+d)zh'(z)}{(1+ch(z))(1-dh(z))} = \mathscr{T}(z), (say). \tag{3.15}$$

If there exist $z_0 \in \Delta$ such that

$$\max_{|z|<|z_0|} |h(z)| = |h(z_0)| = 1.$$

Then from Lemma due to Jack [3] we have $z_0h'(z_0) = \delta h(z_0)$ and

$$Re\left(1 + \frac{z_0h''(z_0)}{h'(z_0)}\right) \geq \delta$$

Now letting $h(z_0) = e^{i\theta}$, $(\theta \in [0, 2\pi])$ in (3.15), we have

$$\begin{aligned}
&Re\{\mathscr{T}(z_0)\}\\
&= \delta(c+d)Re\left\{\frac{h(z_0)}{(1+ch(z_0))(1-dh(z_0))}\right\}\\
&= \delta Re\left\{\frac{1}{1-dh(z_0)} - \frac{1}{1+ch(z_0)}\right\}\\
&= \delta Re\left\{\frac{1-de^{-i\theta}}{1+d^2-2d\cos\theta} - \frac{1+ce^{-i\theta}}{1+c^2+2c\cos\theta}\right\}\\
&= \delta\left\{\frac{1}{2+\frac{d^2-1}{1-d\cos\theta}} - \frac{1}{2+\frac{c^2-1}{1+c\cos\theta}}\right\}
\end{aligned}$$

where $\theta \neq \cos^{-1}(-1/c)$ and $\theta \neq \cos^{-1}(1/d)$ we have

$$Re\{\mathscr{T}(z_0)\} > \frac{c+d}{(1+c)(1-d)}.$$

This contradicts the inequality given in the hypothesis and therefore $|h(z)| < 1$ for all $z \in \Delta$.
Thus, we have

$$\left|\frac{z^2H'_f(a,b;z)}{(H_f(a,b;z))^2} - 1\right| = \left|\frac{(c+d)h(z)}{1-dh(z)}\right| < \frac{c+d}{1-d} \leq 1,\ \ z \in \Delta.$$

In view of Lemma 1.1, it implies that $H_f(a,b;z)$ is univalent □

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