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Application of Simpson's method for solving singular Volterra integral equation

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Abstract. In this paper, the numerical scheme for solving singular Volterra integral equation is obtained by considering non-variable subinterval and the function under the integrals were approximated by the Simpson's rule. The error bound for the numerical scheme is established where the scheme derived has convergence of order 3. The scheme obtained is compared with exact solution of the tested problems which shows that the scheme is effective.

Keywords: Singular Volterra integral equation, convergence order, Simpson's rule, exact solution, error bound.

1. Introduction and Background

Singular Volterra integral equation of this form

$$u(t) = \int_0^t \frac{s^{\mu-1}}{t^{\mu}} u(s)ds + g(t), \quad t \in (0, T], \tag{1.1}$$

with $\mu > 0$, $g(t) \in C[0, T]$ is a given function and the kernel is weakly type has been considered by many authors. Diogo $et\ al.$[3], investigated the application of product integration method for the numerical solutions base on graded meshes by Trapezoidal method. Diogo $et\ al.$ [1], utilized the analytic results for the existence and uniqueness solution of (1.1). Further more, Euler's and Trapezoidal methods were used to develop new schemes, comparison between them was made and error bound analysis were developed. Diogo $et\ al.$ [2], used a class of singular Volterra integral equation of the form (1.1) and obtained the numerical schemes which uses Euler method and Trapezoidal rules. The numerical approximation base on the product Euler scheme converges to the smooth solution but with poor order of convergence. However, Diogo and Lima [4], analyzed discrete supperconvergence properties of spline collocation results and for a certain choice of parameter the attainable convergence order of (1.1) was considered. Diogo and Lima [5], proved that a higher order attained at the meshes points by special choice of the collocation methods. Also Diogo [6], utilized the iterated methods on the collocation results.

In this article we consider the work in Diogo $et\ al.$ (2006) which we used the Simpson's method in the case of when $0 < \mu \leq 1$, Eqn (1.1) has a family of solutions in the space $C[0, T]$. The work has been organized as follows; In section 2, we derived the scheme by applying the Simpson's method. In section 3, we estimates the error bound analysis for the convergence results of the propose scheme. Also, in section 4 we tested the scheme by means of some examples and finally in section 5 the conclusion was given.

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2. Derivation of the Scheme by Simpson's rule Approach

2.1. Definitions of the basic concepts

We start by presenting some definitions, theorems and lemmas;

Definition 2.1. *A kernel is called separable if it can be expressed as the outer product of two variables (vectors). For examples*

$$u(t) = \int_0^t \frac{s^{\mu-1}}{t^\mu} u(s)ds + g(t), \quad t \in [0,T],$$

where $k(t,s) = \frac{s^{\mu-1}}{t^\mu}$ *that can be expressed as* $k(t,s) = \frac{1}{t^\mu} s^{\mu-1}$
otherwise, it is nonseparable.

Theorem 2.2. *Mean Value Theorem: Let* $u(x)$ *be a function which is continuous on the closed interval* $[a,b]$ *and which is differentiable at every point of (a,b). Then there is a point* $c \in (a,b)$ *such that*

$$u'(c) = \frac{u(b)-u(a)}{b-a},$$

Lemma 2.3. *Special Gronwall lemma: Let* (e_n) *and* (e_j) *be nonnegative sequences and* C *a nonnegative constant if*

$$u_n \leq C + \sum_{k=0}^{n-1} g_k u_k \quad for \quad n \geq 0,$$

then

$$u_n \leq C \prod_{j=0}^{n-1} (1+g_j) \leq \quad C \exp^{\left(\sum_{j=0}^{n-1} g_j\right)} \quad for \quad n \geq 0.$$

Lemma 2.4. *(i) If* $0 < \mu \leq 1$ *and* $g \in C^1[0,T]$ *(with* $g(0) = 0$ *if* $\mu = 1$*) then equation(1) has a family of solutions* $u \in C[0,T]$ *given by the formula*

$$u(t) = c_0 t^{1-\mu} + g(t) + \gamma + t^{1-\mu} \int_0^t s^{\mu-2}(g(s) - g(0))ds, \tag{2.1}$$

where

$$\gamma := \begin{cases} \frac{1}{\mu-1} g(0) & if \quad \mu < 1, \\ 0 & if \quad \mu = 1, \end{cases}$$

and c_0 *is an arbitrary constant. Out of this family of solutions there is one particular solution* $u \in C^1[0,T]$.

(ii) If $\mu \leq 1$ *and* $g \in C^m[0,T], m \geq 0$ *then the unique solution* $u \in C^m[0,T]$ *of (1) is given by*

$$u(t) = g(t) + t^{1-\mu} \int_0^t s^{\mu-2} g(s)ds.,$$

We note that Eqn (2.4) can be obtained from Eqn (2.1) with $c_0 = 0$. *Indeed; it follows from Eqn (2.1) that*

$$c_0 = \lim_{t \to 0^+} t^{\mu-1} u(t),$$

and this limit is zero when $\mu > 1$.

2.2. Derivation of the scheme

Let us reformulate (1.1) into a new form by choosing some fixed real number $\alpha > 0$. Substituting t by $t+\alpha$ in (1.1) we have

$$u(t+\alpha) = \int_0^{t+\alpha} \frac{s^{\mu-1}}{(t+\alpha)^{\mu}} u(s)ds + g(t+\alpha), \quad t \in [0,T], \tag{2.2}$$

by splitting of the interval we have

$$u(t+\alpha) = \frac{1}{(t+\alpha)^{\mu}} \int_0^{\alpha} s^{\mu-1} u(s)ds + \int_{\alpha}^{t+\alpha} \frac{s^{\mu-1}}{(t+\alpha)^{\mu}} u(s)ds + g(t+\alpha), \quad t \in [\alpha,T], \tag{2.3}$$

or, equivalently,

$$u(t+\alpha) = \frac{I_{\alpha}}{(t+\alpha)^{\mu}} + \int_0^{t} \frac{(s+\alpha)^{\mu-1}}{(t+\alpha)^{\mu}} u(s+\alpha)ds + g(t+\alpha), \tag{2.4}$$

where

$$I_{\alpha} = \int_0^{\alpha} s^{\mu-1} u(s)ds. \tag{2.5}$$

Since I_{α} is known exactly for a chosen the exact solution by using the solution formula then we can apply the numerical method in (2.4) and obtain the approximation.
Now, let us define a uniform grid X_h with stepsize $h = \frac{t}{n}$

$$X_h := \{t_i = ih + \alpha, \quad 0 \leq i \leq N\}.$$

Setting $t = nh$ in (2.4) we have

$$u(t_n) = \frac{I_{\alpha}}{t_n^{\mu}} + \frac{1}{t_n^{\mu}} \int_0^{nh} (s+\alpha)^{\mu-1} u(s+\alpha)ds + g(t_n). \tag{2.6}$$

In the Simpson's method, we approximates the integral on the right-hand side of (2.6) by considering each subinterval using:

$$u(s+\alpha) \approx \frac{1}{6}\left[u(t_{j+1})(s-jh) + 4u\left(\frac{t_j+t_{j+1}}{2}\right)(t_{j+1}-t_j) + u(t_j)((j+1)h-s)\right], \tag{2.7}$$

on each subinterval $s \in [jh,(j+1)h]$. Defining the following

$$D_j^1 := \int_{jh}^{(j+1)h} (s+\alpha)^{\mu-1}(s-jh)ds$$
$$D_j^2 := \int_{jh}^{(j+1)h} (s+\alpha)^{\mu-1}ds$$
$$D_j^3 := \int_{jh}^{(j+1)h} (s+\alpha)^{\mu-1}((j+1)h-s)ds$$

which can be obtain analytically.
Hence the scheme:

$$u(t_n)_n^h = \frac{I_{\alpha}}{t_n^{\mu}} + \frac{h}{t_n^{\mu}} \sum_{j=0}^{n-1} \frac{(D_j^1 u_{j+1}^h + D_j^2 4u_{j/2}^h + D_j^3 u_j^h)}{6} + g(t_n), \quad n = 1,2,...,N. \tag{2.8}$$

2.3. Algorithm: Simpson's rule approach

Step1: Given $n=1$, $\epsilon=10^{-3}$, $t\in[0,T]$, $\mu\in(0,1]$, $\alpha>0$, $u(t)$, $g(t)$, I_α.
Step2: Set $h=\frac{t}{n}$
Step3: Compute

$$\begin{aligned} t_n &= nh+\alpha \\ t_n^\mu &= (nh+\alpha)^\mu \\ u_n^h &= \frac{I_\alpha}{t_n^\mu}+\frac{h}{t_n^\mu}\sum_{j=0}^{n-1}\frac{(D1_j u_{j+1}^h + D2_j u_{j/2}^h + D3_j u_j^h)}{6}+g(t_n) \end{aligned}$$

where
$D1_j := \frac{(jh+\alpha)^{\mu+1}-((j+1)h+\alpha)^{\mu+1}+h((j+1)h+\alpha)^\mu(\mu+1)}{(\mu+1)\mu}$,
$D2_j := \frac{h(((j+1)h+\alpha)^\mu-(jh+\alpha)^\mu)}{\mu}$,
$D3_j := \frac{((j+1)h+\alpha)^{\mu+1}-(jh+\alpha)^{\mu+1}-h(jh+\alpha)^\mu(\mu+1)}{(\mu+1)\mu}$,
$u_{j+1}^h := u\left((j+1)h+\alpha\right)$,
$u_{j/2}^h := 4u\left(\frac{(jh+\alpha)+((j+1)h+\alpha)}{2}\right)$
$u_j^h := u\left(jh+\alpha\right)$
If $|u(t)-u_n^h|\le 10^{-2}$ stop, else
Step4: set $n=n+1$ and go to Step3.

3. Error Bound of the Scheme in Simpson's Rule Approach

In this section we present the error bound for the convergence of the scheme.

Theorem 3.1. *Consider (1.1) with $0<\mu\le 1$ and $u\in C^1[0,T]$. Let $\alpha\neq 0$ be fixed in the equivalent (2.4) and assume the integral I_α is known exactly for a chosen particular solution (corresponding to a certain value of the parameter c_0). Then the approximate solution obtained by the product Simpson's method converges with order 3 to the particular exact solution.*

Proof. The solution u of the exact solution satisfies

$$u(t_n)_n^h = \frac{I_\alpha}{t_n^\mu}+\frac{h}{t_n^\mu}\sum_{j=0}^{n-1}\frac{(D_j^1 u_{j+1}^h + D_j^2 4u_{j/2}^h + D_j^3 u_j^h)}{6}+g(t_n)+\eta(h,t_n), \quad n\ge 1, \tag{3.1}$$

where $\eta(h,t_n)$ is the consistency error given by

$$\eta(h,t_n)=\int_0^{t_n}\frac{s^{\mu-1}}{t_n^\mu}u(s)ds-\frac{h}{t_n^\mu}\sum_{j=0}^{n-1}\frac{(D_j^1 u_{j+1}^h + D_j^2 4u_{j/2}^h + D_j^3 u_j^h)}{6}, \tag{3.2}$$

but the exact solution is

$$u(t_n)=\frac{I_\alpha}{t_n^\mu}+\frac{1}{t_n^\mu}\int_\alpha^T s^{\mu-1}u(s)ds+g(t_n). \tag{3.3}$$

Setting $e_n = u(t_n) - u(t_n)^h$ for $n \geq 1$ and by utilizing (3.3) and (3.1) this gives

$$
\begin{aligned}
e_n &= \frac{1}{t_n^\mu}\int_\alpha^T s^{\mu-1}u(s)ds - \frac{h}{t_n^\mu}\sum_{j=0}^{n-1}\frac{(D_j^1 u_{j+1}^h + D_j^2 4u_{j/2}^h + D_j^3 u_j^h)}{6} + \eta(h,t_n)\\
&= \frac{1}{t_n^\mu}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} s^{\mu-1}u(t_j)ds - \frac{h^2}{t_n^\mu}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} s^{\mu-1}\left(\frac{u(t_{j+1})^h + u(t_{j/2})^h + u(t_j)^h}{6}\right)ds\\
&+ \eta(h,t_n)\\
&= \frac{h^2}{t_n^\mu}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\left(u(t_j) - Su(t_j)^h\right)s^{\mu-1}ds + \eta(h,t_n)
\end{aligned}
$$

let $Su(t_j)^h = \frac{u(t_{j+1})^h + u(t_{j/2})^h + u(t_j)^h}{6}$ and defining $e_j^s := Se_j = \left(u(t_j) - Su(t_j)^h\right)$ yield

$$
e_n = \frac{h^2}{t_n^\mu}\sum_{j=0}^{n-1} e_j^s \int_{t_j}^{t_{j+1}} s^{\mu-1}ds + \eta(h,t_n), \quad n \geq 1, \tag{3.4}
$$

but

$$
\begin{aligned}
\frac{1}{t_n^\mu}\int_{t_j}^{t_{j+1}} s^{\mu-1}ds &\leq \frac{t_j^{\mu-1}}{t_n^\mu}\int_{t_j}^{t_{j+1}} ds\\
&= h\frac{t_j^{\mu-1}}{t_n^\mu}\\
&\leq h\left(\frac{t_j}{t_n}\right)^\mu \frac{1}{t_j}\\
&\leq \frac{h}{\alpha}.
\end{aligned} \tag{3.5}
$$

Since $\alpha \neq 0$ and for $\alpha > 0$ choose $\alpha \leq t_j\left(\frac{t_n}{t_j}\right)^\mu$. By utilizing (3.5) in (3.4) we have

$$
e_n \leq \frac{h^3}{\alpha}\sum_{j=0}^{n-1} e_j^s + \eta(h,t_n), \quad n \geq 1. \tag{3.6}
$$

Taking the modulus in (3.6) we have

$$
|e_n| \leq \frac{h^3}{\alpha}\sum_{j=0}^{n-1}\left|e_j^s\right| + |\eta(h,t_n)|, \quad n \geq 1 \tag{3.7}
$$

On the other hand from equation (3.2) we have

$$
\begin{aligned}
|\eta(h,t_n)| &= \left|\int_0^{t_n}\frac{s^{\mu-1}}{t_n^\mu}u(s)ds - \frac{h}{t_n^\mu}\sum_{j=0}^{n-1}\frac{(D_j^1 u_{j+1}^h + D_j^2 4u_{j/2}^h + D_j^3 u_j^h)}{6}\right|\\
&= \left|\frac{1}{t_n^\mu}\sum_{j=0}^{n-1} D_j u(t_j) - \frac{h^2}{t_n^\mu}\sum_{j=0}^{n-1} D_j\left(\frac{u(t_{j+1})^h + u(t_{j/2})^h + u(t_j)^h}{6}\right)\right|\\
&= \left|\frac{h^2}{t_n^\mu}\sum_{j=0}^{n-1} D_j\left(u(s) - Su(t_j)^h\right)\right|
\end{aligned}
$$

but

$$D_j := \int_{t_j}^{t_{j+1}} s^{\mu-1} ds$$

Therefore,

$$|\eta(h,t_n)| \leq \frac{h^2}{t_n^\mu} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} s^{\mu-1} \left|\left(u(s) - Su(t_j)^h\right)\right| ds \tag{3.8}$$

by applying the mean value theorem in (3.8), we have

$$|\eta(h,t_n)| \leq \frac{h^3}{t_n^\mu} \max_{s\in[\alpha,T]} |u'(s)| \int_\alpha^{t_n} s^{\mu-1} ds \tag{3.9}$$

Defining $M(\alpha) := \max_{s\in[\alpha,T]} |u'(s)|$

$$\begin{aligned} |\eta(h,t_n)| &\leq \frac{M(\alpha)h^3}{t_n^\mu} \int_\alpha^{t_n} s^{\mu-1} ds \\ &= \frac{M(\alpha)h^3}{\mu}\left(\frac{t_n^\mu - \alpha^\mu}{t_n^\mu}\right) \\ &= \left(1 - \frac{\alpha^\mu}{t_n^\mu}\right)\frac{M(\alpha)h^3}{\mu} \end{aligned}$$

we obtained the following bound

$$|\eta(h,t_n)| \leq \left(1 - \frac{\alpha^\mu}{t_n^\mu}\right)\frac{M(\alpha)h^3}{\mu} \tag{3.10}$$

substitute (3.10) into (3.7) we have

$$|e_n| \leq \left(1 - \frac{\alpha^\mu}{t_n^\mu}\right)\frac{M(\alpha)h^3}{\mu} + \frac{h^3}{\alpha}\sum_{j=0}^{n-1}|e_j^s| \tag{3.11}$$

by applying the special Gronwall lemma for the discrete in (3.11) we have

$$|e_n| \leq \left(1 - \frac{\alpha^\mu}{t_n^\mu}\right)\frac{M(\alpha)h^3}{\mu}\prod_{j=0}^{n-1}\left(1 + \frac{n-1}{\alpha}\right)$$

we obtained the error bound as

$$|e_n| \leq \left(1 - \frac{\alpha^\mu}{t_n^\mu}\right)\frac{M(\alpha)h^3}{\mu}\exp\left(\frac{T-1}{\alpha}\right)$$

Hence, a third order convergence follows.

4. Main Results

In this section we tested the scheme using Maple13 version 10 with the stoping rule as $|u_n^h - u(t)| \leq 10^{-3}$.

Problem 4.1. *Given $g(t) = 1 + t + t^2$ and $0 < \mu \leq 1$ in (1.1), then using (2.1) we obtained the general form of its family of solutions:*

$$u(t) = c_0 t^{1-\mu} + \frac{\mu}{\mu-1} + \frac{\mu+1}{\mu}t + \frac{\mu+2}{\mu+1}t^2 \tag{4.1}$$

where c_0 is an arbitrary constant. The exact solution (4.1) when $t = 1.02$ is compared with numerical solution (2.8) and errors are presented in Table 1

Table 1: The results obtained by the numerical scheme (2.8) on problem1.

n	u_n^h Eqn (2.8)	$\lvert u(t) - u_n^h \rvert$
80	2.4543	$6.664E-1$
82	2.5122	$6.085E-1$
84	2.5707	$5.500E-1$
86	2.6299	$4.908E-1$
88	2.6899	$4.308E-1$
90	2.7506	$3.701E-1$
92	2.1190	$3.088E-1$
94	2.8741	$2.466E-1$
96	2.9369	$1.838E-1$
98	3.0004	$1.203E-1$
100	3.0647	$5.600E-2$
102	3.1297	$9.000E-3$

Table (1) shows that the numerical results of problem 4.1 obtained from scheme (2.8) which has exact solution of $u(t) = 3.1207$ and the best result is obtained when $n = 102$ with corresponding to an error of $9.000E-3$.

Problem 4.2. *Given $g(t) = 1 + t$ and $0 < \mu \leq 1$ in (1.1), then using (2.1) we obtained the general form of its family of solutions:*

$$u(t) = c_0 t^{1-\mu} + \frac{\mu}{\mu - 1} + \frac{\mu + 1}{\mu} t \tag{4.2}$$

where c_0 is an arbitrary constant. The exact solution (4.2) when $t = 1.02$ is compared with numerical solution (2.8) and errors are presented in Table (2)

Table 2: The results obtained by the numerical scheme (2.8) on problem2.

n	u_n^h Eqn (2.8)	$\lvert u(t) - u_n^h \rvert$
80	1.7946	$2.652E-1$
82	1.8195	$2.403E-1$
84	1.8442	$2.156E-1$
86	1.8689	$1.909E-1$
88	1.8934	$1.664E-1$
90	1.9179	$1.419E-1$
92	1.9423	$1.175E-1$
94	1.9666	$9.320E-2$
96	1.9908	$6.900E-2$
98	2.0149	$4.490E-2$
100	2.0389	$2.090E-2$
102	2.0663	$6.500E-3$

Table (2) shows that the numerical results of problem 4.2 obtained from scheme (2.8) which has exact solution of $u(t) = 2.0598$ and the best result is obtained when $n = 102$ with corresponding to an error of $6.50E - 3$. The error decreases when the number of iterations are increased. The results is an improvement when compared with the work of [5] which uses Euler's method with number of iterations up to 1600 corresponding to an error of $4.82E - 2$.

Problem 4.3. *Given $g(t) = 1 + t + t^3$ and $0 < \mu < 1$ in (1.1), then using (2.1) we obtained the general form of its family of solutions:*

$$u(t) = c_0 t^{1-\mu} + \frac{\mu}{\mu - 1} + \frac{\mu + 1}{\mu} t + \frac{\mu + 3}{\mu + 2} t^3 \tag{4.3}$$

where c_0 is an arbitrary constant. The exact solution (4.3) when $t = 1.02$ is compared with numerical solution (2.8) and errors are presented in Table (3)

Table 3: The results obtained by the numerical scheme (2.8) on problem3.

n	u_n^h Eqn (2.8)	$\lvert u(t) - u_n^h \rvert$
80	2.3278	$8.247E - 1$
82	2.3932	$7.593E - 1$
84	2.4604	$6.921E - 1$
86	2.5296	$6.229E - 1$
88	2.6008	$5.517E - 1$
90	2.6739	$4.786E - 1$
92	2.7493	$4.032E - 1$
94	2.8268	$3.257E - 1$
96	2.9065	$2.460E - 1$
98	2.9884	$1.641E - 1$
100	3.0727	$7.980E - 2$
102	3.1593	$6.800E - 3$

Table (3) shows that the numerical results of problem (4.3) obtained from scheme (2.8) which has exact solution of $u(t) = 3.1525$ and the best result is obtained when $n = 102$ with corresponding to an error of $6.800E-3$. The error decreases when the number of iterations are increased. The results is an improvement when compared with the work of [5] which uses Euler's method with number of iterations up to 1600 corresponding to an error of $4.82E - 2$.

4.1. The comparison of the numerical schemes

Here we presented the scheme (2.8) derived from Midpoint's rule when compared with Euler's method in [5].

Table 4: The comparison of scheme (2.8) and Euler's methods in [5] using errors of problem 1 and 2.

n	scheme (2.8) Errors1	scheme (2.8) Errors2	Euler's in [5] Errors
80	$5.759E-1$	$2.321E-1$	$3.919E-1$
82	$5.183E-1$	$2.072E-1$	$4.173E-1$
84	$4.599E-1$	$1.825E-1$	$4.423E-1$
86	$4.009E-1$	$1.578E-1$	$4.671E-1$
88	$3.412E-1$	$1.333E-1$	$4.817E-1$
90	$2.807E-1$	$1.089E-1$	$5.159E-1$
92	$2.196E-1$	$8.460E-2$	$5.400E-1$
94	$1.578E-1$	$6.030E-2$	$5.638E-1$
96	$9.520E-2$	$3.620E-2$	$5.874E-1$
98	$3.190E-2$	$1.210E-2$	$6.108E-1$
99	$1.000E-4$	$1.000E-4$	$6.224E-1$

Table (4) Shows that the errors obtained from the scheme (2.8) is an improvement when compared with the work of [5] which uses Euler's method, since the error decreases when the number of iterations are increased. This shows that the scheme obtained has a better result when compared with the Euler's method with number of iterations up to 1600 corresponding to an error of $4.82E-2$.

5. Conclusion

The function under the integrals were approximated base on the concepts of Simpson's rule. We used error bound estimates for the convergence of the scheme obtained. The numerical results were obtained by means of some examples so as to test the efficiency, accuracy and effectiveness of the new scheme derived. The new approach of the numerical scheme obtained from Simpson's rule was compared with exact solutions.

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