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The ω-compact topology on function spaces

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Abstract
The aim of this paper is to introduce a new topology called ω-open compact topology on the set of all real-valued continuous function on a Tychonoff space.

Keywords
Topological spaces, ω-open sets, ω-compact topology.

AMS Subject Classification
54C10, 54C35, 54D65, 54E18, 54E35.

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1. Introduction

A function space is an interesting example of a topological space. In past, many researchers studies different kind of function spaces by placing different topologies on the set of functions. This study is help to understand relationship between continuous function and a Baire measure on a Tychonoff space. The relationship between these two concepts can be clarified by the study of various locally convex topologies on spaces of continuous functions. The most interesting function space, C(X), which is set of all real-valued continuous function on a Tychonoff space X, have been studied by defining three topologies like point-open topology, compact-open topology and topology of uniform convergence. In this paper, our goal is to introduce and study another topology called ω-compact topology on $C(X)$ in term of ω-compact subset of X. We denote this topology by ω and the corresponding space by $C_{\omega}(X)$. Throughout this paper we use following conventions. All spaces in this paper considered is a Tychonoff space that is, a complete regular Housdroff space. If X and Y are spaces with same underlying set, then $X=Y$, $X\leq Y$, $X<Y$ indicate that X and Y have same topology, that the topology Y is finer or equal to the topology on X and that the topology on Y is strictly finer than the topology on X, respectively. The point-open topology, compact-open topology and uniform convergence topology on $C(X)$ is denoted by p, k and u, respectively and corresponding spaces are denoted by $C_p(X)$, $C_k(X)$ and $C_u(X)$, respectively. The $\mathbb{R}$ and $\mathbb{N}$ denote the space of real numbers and natural numbers respectively. The constant zero-function in $C(X)$ is denoted by 0 or sometimes 0_X. In this paper, we introduce and study ω-open-compact topology.

2. Preliminaries

For a subset A of a topological space (X,τ), we denote the closure of A and the interior of A by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 2.1. *A subset A of a topological space (X,τ) is said to be semiopen [1] if $A\subset \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a semiopen set is called a semiclosed set.*

Definition 2.2. *A subset A of a space X is called ω-closed [4] if $\mathrm{Cl}(A)\subset U$ whenever $A\subset U$ and U is semi-open in X. The complement of an ω-closed set is called an ω-open set. The family of all ω-open subsets of (X,τ) is denoted by $\omega(\tau)$. We set $\omega(X,x)=\{V\in\omega(\tau)|x\in V\}$ for $x\in X$.*

The union (resp. intersection) of all ω-open (resp. ω-closed) sets, each contained in (resp. containing) a set A in a space X is called the ω-interior (resp. ω-closure) of A and is denoted by $\omega\,\mathrm{Int}(A)$ (resp. $\omega\,\mathrm{Cl}(A)$) [5].

Note that the family of ω-open subsets of (X,τ) forms a topology [5].

Definition 2.3. *A function $f:(X,\tau)\to(Y,\sigma)$ is said to be ω-continuous [4] if $f^{-1}(V)$ is ω-open in X for every open set V of Y.*

Definition 2.4. *[5] A topological space (X,τ) is ω-compact if every ω- cover (a cover consisting of ω-open sets) of X has a finite subcover.*

Theorem 2.5. *[5] If (X,τ) is an ω-compact space, then it is compact.*

Theorem 2.6. *If a space (X,τ) is ω-irresolvable and ω-compact, then it is compact.*

Theorem 2.7. *If a mapping $f:(X,\tau)\to(Y,\sigma)$ is an ω-continuous surjective and A is ω-compact subset of X, then $f(A)$ is ω-compact in Y.*

3. On ω-compact Topology on C(X)

In this section, we define three new topologies on C(X) with the help of ω-compact subset of X. First, consider a set for any ω-compact subset A of X and any open subset U of R as $(A,U)=\{f\in C(X):f(A)\subseteq U\}$. Suppose that the collection of all ω-compact subset of X is denoted by $\omega(X)$. The ω-compact-open topology on $C(X)$ is generated by the subbase which is defined as $\{(A,U):A\in\omega(X), U \text{ is open in } R\}$. It can be easily verify that this subbasis actually generate the topology on $C(X)$, called ω-compact-open topology on $C(X)$ and a new space corresponding to $C(X)$ is denoted by $C_\omega(X)$. As we can verify that closure of any ω-compact set is again ω-compact and because $\mathrm{Cl}(f(\mathrm{Cl}(A)))=\mathrm{Cl}(f(A))$ for any $f\in C(X)$, so we can always consider closed ω-compact subsets of X in (A,U). Now we will define topology of uniform convergence on ω-compact sets. For each $A\in\omega(X)$ and $\varepsilon>0$, let $A_\varepsilon=\{(f,g)\in C(X)\times C(X):|f(x)-g(x)|<\varepsilon$ for all $x\in A\}$. It can be easily show that that the collection $\{A_\varepsilon:A\in\omega(X),\varepsilon>0\}$ is a base for some uniformity on $C(X)$. We denote a space corresponding to $C(X)$ with the topology induced by this uniformity by $C_{\omega,u}(X)$ and this topology is called the topology of uniform convergence on $\omega(X)$. For each $f\in C(X)$, $A\in\omega(X)$, and $\varepsilon>0$, let $B_A(f,\varepsilon)=\{g\in C(X):|f(x)-g(x)|<\varepsilon$ for all $x\in A\}$. If $f\in C(X)$, the collection $\{B_A(f,\varepsilon):A\in\omega(X),\varepsilon>0\}$ forms a neighborhood base at f in $C_{\omega,u}(X)$ and this collection forms a base for the topology of uniform convergence on $\omega(X)$. Here each set $B_A(f,\varepsilon)$ is open in $C_{\omega,u}(X)$. If $\omega(X)$ covers X, then $C_{\omega,u}(X)$ is a Tychonoff space. If suppose $\omega(X)=\{X\}$, then we get a topology of uniform convergence and it is denoted by $C_u(X)$. We can verify that for any $\omega(X)$, $C_u(X)\le C_{\omega,u}(X)$. Now for each $A\in\omega(X)$ define ω-seminorm ω_A on $C(X)$ by $\omega_A(f)=min\{1,sup\{|f(x)|:x\in A\}\}$. Also for each $A\in\omega(X)$ and $\varepsilon>0$, let $U_{A,\varepsilon}=\{f\in C(X):\omega_A(f)<\varepsilon\}$. Let $\mathscr{U}=\{U_{A,\varepsilon}:A\in\omega(X),\varepsilon>0\}$. It can be easily shown that for each $f\in C(X)$, $f+\mathscr{U}=\{f+U:U\in\mathscr{U}\}$ forms a neighborhood base at f We say that this topology is generated by the collection of ω-seminorms $\{\omega_A:A\in\omega(X)\}$. Now we will establish a relation between above two defined topologies as $C_\omega(X)$ and $C_{\omega,u}(X)$.

Theorem 3.1. *For any space X, the ω-compact-open topology on $C(X)$ is same as the topology of uniform convergence on the ω-compact subsets of X, that is, $C_\omega(X)=C_{\omega,u}(X)$.*

Proof. Suppose that (A,U) is subbasis open set in $C_\omega(X)$ and $f\in(A,U)$. Since $f(A)$ is compact, there exist a_1, a_2, . . . a_n in $f(A)$ such that $f(A)\subseteq\bigcup_{i=1}^{\infty}(a_i-\varepsilon_i,a_i+\varepsilon_i)\subseteq\bigcup_{i=1}^{\infty}(a_i-2\varepsilon_i,a_i+2\varepsilon_i)\subseteq U$. If we consider $g\in B_A(f,\varepsilon)$ and $x\in A$, then this show that $|f(x)-g(x)|<\varepsilon$ and there exist i such that $|f(x)-a_i|<\varepsilon_i$. Hence $|g(x)-a_i|<2\varepsilon_i$ and this show that $g(x)\subseteq U$ for all $x\in A$. So $g(A)\subseteq U$, that is, $g\in(A,U)$. Hence we can say that $B_A(f,\varepsilon)\subseteq(A,U)$. Let $W=\bigcup_{i=1}^{\infty}(A_i,U_i)$ be a basic neighborhood of f in $C_\omega(X)$. Then there exists $\varepsilon_1,...,\varepsilon_i$ such that $f\in B_{A_i}(f,\varepsilon_i)\subseteq(A_i,U_i)M$. Suppose that $A=\bigcup_{i=1}^{\infty}A_i$ and choose $\varepsilon=min_{1\le i\le n}\{\varepsilon_i\}$. Then $f\in B_A(f,\varepsilon)\subseteq W=\bigcap_{i=1}^{\infty}(A_i,U_i)=(A,U)$. Finally as we supposed $f\in(A,U)$ and we proved that $f\in B_A(f,\varepsilon)$, we can say that $C_\omega(X)\le C_{\omega,u}(X)$. Now let $B_A(f,\varepsilon)$ be a basic neighborhood of f in $C_{\omega,u}(X)$. Since $f(A)$ is compact, there exist a_1, a_2, . . . , a_n in $f(A)$ such that $f(A)\subseteq\bigcup_{i=1}^{\infty}(a_i-\frac{\varepsilon}{4},a_i+\frac{\varepsilon}{4})$. Define $W_i=(a_i-\frac{\varepsilon}{2},a_i+\frac{\varepsilon}{2})$ and $A_i=\mathrm{Cl}_A(A\cap f^{-1}(a_i-\frac{\varepsilon}{4},a_i+\frac{\varepsilon}{4})$. Here each A_i will be ω-compact. It is clear that $f\in\bigcap_{i=1}^{\infty}(A_i,W_i)\subseteq B_A(f,\varepsilon)$. Let $g\in\bigcap_{i=1}^{\infty}(A_i,W_i)$ and $x\in A$. Then there exists i such that $x\in A_i$ and $f(x)\in(a_i-\frac{\varepsilon}{4},a_i+\frac{\varepsilon}{4})$. Since $g(x)\in(a_i-\frac{\varepsilon}{2},a_i+\frac{\varepsilon}{2})$, $|f(x)-g(x)|<\varepsilon$. Hence $g\in B_A(f,\varepsilon)$ and finally we can say that $C_{\omega,u}(X)\le C_\omega(X)$. □

Theorem 3.2. *For any space X, the family $\{(A,U):A\in\omega(\tau)\}$, where U is a bounded open interval in $\mathbb{R}$ forms a subbase for $C_\omega(X)$.*

Theorem 3.3. *For any space X, ω-compact-open topology is finer than compact-open topology that is $C_k(X)<C_\omega(X)$.*

Proof. Let (A_k,U) and (A,U) be the subbasis elements for compact-open topology and ω-compact-open topology respectively. Where A_k and A are compact and ω-compact subsets of X, respectively. Let $f\in(A,U)$. Then by definition, $f(A)$ contained in open subset U of $\mathbb{R}$. Since f is continuous, $f(A)$ is compact in U. This show that $f\in(A_k,U)$. Therefore, $(A,U)\subset(A_k,U)$. Hence $C_k(X)<C_\omega(X)$. □

For the space X, we can easily prove that topology of uniform convergence on $C(X)$ is finer than the ω-compact-open topology that is $C_\omega(X)<C_u(X)$. Hence we can say that

Theorem 3.4. *For any Tyhconoff space, $C_k(X)\le C_\omega(X)\le C_u(X)$.*

Theorem 3.5. *Every closed ω-compact subset of X is compact if and only if $C_k(X)=C_\omega(X)$.*

Proof. For any subset A of X $B_{\mathrm{Cl}(A)}(f,\varepsilon)\subseteq B_A(f,\varepsilon)$. So if every closed ω-compact subset of X is compact, then we can say that $C_\omega(X)\le C_k(X)$. Hence in this case we can say that

$C_k(X) = C_\omega(X)$. Conversely, let $C_k(X) = C_\omega(X)$ and suppose that A be any closed ω-compact subset of X. So set $B_A(0,1)$ as open lies in $C_k(X)$. This show that there exists a compact subset K of X and $\varepsilon > 0$ such that $B_K(0,\varepsilon) \subseteq B_A(0,1)$. If possible, let $x \in A \backslash K$. Then this implies that there exists a continuous mapping $g : A \to [0,1]$ such that $g(x) = 1$ and $g(y) = 0$ for all $y \in K$. Now here $g \in B_K(0,\varepsilon) \backslash B_A(0,1)$ which gives a contradiction. Hence $A \subseteq K$ and consequently A is compact. □

Theorem 3.6. *Every space X is ω-compact if, and only if $C_\omega(X) = C_u(X)$.*

Proof. Let X be a ω-compact space. So, by definition it is possible for each $f \in C(X)$ and $\varepsilon > 0$ there exists a set $B_X(f,\varepsilon)$ which is basic open in $C_\omega(X)$. Hence $C_\omega(X) = C_u(X)$. Let us suppose $C_\omega(X) = C_u(X)$. Because a set $B_X(0,1)$ is a basic neighborhoods of a constant zero-mapping 0 in $C_u(X)$, then there exists an ω-compact subset A of X and $\varepsilon > 0$ such that $B_A(0,\varepsilon) \subseteq B_X(0,1)$. With the help of complete regularity of X it can be shown that $X = \mathrm{Cl}(A)$. But the closure of an ω-compact set is also ω-compact. Hence X is ω-compact. □

Theorem 3.7. *Every closed countably compact subset of X is compact if and only if $C_k(X) = C_\omega(X)$, where X is any normal Housdorff space.*

Further we would like to find conditions when some spaces for which every closed ω-compact subset is compact. Following definitions will help to find required conditions.

Definition 3.8. *A space X is said to be isocompact if every closed countably compact subset of X is compact.*

Definition 3.9. *A space X is said to ω-isocompact if every closed ω-compact subset of X is compact.*

Theorem 3.10. *Every space X is ω-isocompact if, and only if $C_\omega(X) = C_k(X)$.*

Theorem 3.11. *The space X is isocompact if and only $C_\omega(X) = C_k(X)$, where X is a normal space.*

Definition 3.12. *Let $f : (X,\tau) \to (Y,\sigma)$ be a continuous mapping. Then a mapping $f^\star : C(Y) \to C(X)$ is said to be the induced mapping of f if $f^\star(g) = g \circ f$ for all g in $C(Y)$.*

Definition 3.13. *Any mapping $f : (X,\tau) \to (Y,\sigma)$ is said to be an almost onto mapping if $f(X)$ is dense in Y, where X is any nonempty set and Y is any topological space.*

In this work, we will study nature of the induced mapping on $C(Y)$ to $C(X)$, when both are equipped with the ω-compact-open topology.

Theorem 3.14. *If $f : (X,\tau) \to (Y,\sigma)$ be a continuous mapping, then the induced mapping of f as $f^\star : C_\omega(Y) \to C_\omega(X)$ is continuous.*

Proof. Let $g \in C_\omega(Y)$ and $B_A(f^\star(g),\varepsilon)$ be a basic neighborhood of $f^\star(g)$ in $C_\omega(X)$. Then this show that $f^\star(B_A(g,\varepsilon)) \subseteq B_A(f^\star(g),\varepsilon)$ and consequently, $f^\star$ is continuous. □

Theorem 3.15. *If $f : (X,\tau) \to (Y,\sigma)$ be a continuous mapping, then the function $f^\star : C_\omega(Y) \to C_\omega(X)$ is one-to-one if, and only if f is almost onto.*

Proof. Let g_1 and g_2 in $C(Y)$ with $f^\star(g_1) = f^\star(g_2)$ and let y in $f(X)$. Then for some $x \in X$, $y = f(x)$ and $g_1(y) = g_1(f(x)) = f^\star(g_1)(x) = f^\star(g_2)(x) = g_2(f(x)) = g_2(y)$. Since $f(X)$ is dense in Y, then $g_1 = g_2$. Conversely, let there exists a y in $Y \backslash \omega\mathrm{Cl}(f(X))$ and p in $C[0,1]$ be a path in $\mathbb{R}$ so that $p(0) \neq p(1)$. Now the continuous function mapping $\omega\mathrm{Cl}(f(X))$ onto $\{0\}$ and y to 1 has an extension φ in $C(Y,[0,1])$. If $g = p \circ \varphi$ and c is the constant mapping taking Y onto $\{p(0)\}$, then for each x in X, $g(f(x)) = p(0) = c(f(x))$. But then $f^\star(g) = f^\star(c)$, so that $f^\star$ is not one-to-one. □

Theorem 3.16. *If $f : (X,\tau) \to (Y,\sigma)$ be a continuous mapping, then if $f^\star : C_\omega(Y) \to C_\omega(X)$ is almost onto, then f is one-to-one.*

Proof. Suppose that x_1 and x_2 be two distinct elements of X. So, we have some h in $C(X)$ such that $h(x_1) \neq h(x_2)$, and also there exist disjoint neighborhoods U and V of $h(x_l)$ and $h(x_2)$ in $\mathbb{R}$. Suppose that $S = [x_l,U] \cap [x_2,V]$, which is a neighborhood of h in $C_\omega(X)$. Then since $f^\star$ is almost onto, there is some g in $C(Y)$ with $f^\star(g) \in S$. This show that $g(f(x_l))$ and $g(f(x_2))$ in U and V, respectively, so that $f(x_1) \neq f(x_2)$. □

Definition 3.17. *A continuous mapping $f : (X,\tau) \to (Y,\sigma)$ is called ω-covering if given any ω-compact subset A in Y, there exists an ω-compact subset C in X such that $A \subseteq \mathrm{Cl}(f(C))$.*

Theorem 3.18. *Suppose that $f : (X,\tau) \to (Y,\sigma)$ be a continuous mapping. If $f^\star : C_\omega(Y) \to C_\omega(X)$ is an embedding, then f is an ω-covering map.*

Proof. Suppose that A is ω-compact subset of Y. Then $f^\star(B_A(0_Y,1))$ is an open neighborhood of the zero function 0_X in $f^\star(C_\omega(Y))$. Now consider an ω-compact subset C of X such that $0_X \in B_C(0_X,\varepsilon) \cap f^\star(C_\omega(Y)) \subseteq f^\star(B_A(0_Y,1))$. Now will prove that $A \subset \mathrm{Cl}(f(C))$. If it is possible, suppose y in $A \backslash \mathrm{Cl}(f(C))$. This show that there exists a continuous function $g : Y \to [0,1]$ such that $g(y) = 1$ and $g(\mathrm{Cl}(f(C))) = 0$, by definition. Since $g(f(C)) = 0$ and $f^\star(g) \in B_C(0_X,\varepsilon) \cap f^\star(C_\omega(Y)) \subseteq f^\star(B_A(0_Y,1))$. Since $g^\star$ is an injective mapping, $g \in B_A(0_Y,1)$. But $y \in A$ show that $|g(y)| < 1$. This is contradiction and hence $A \subseteq \mathrm{Cl}(f(C))$. By definition, we can say that f is ω-covering. □

The converse of above result not hold good but next theorem show that under some condition converse of above can establish.

Theorem 3.19. *If a continuous mapping $f : (X,\tau) \to (Y,\sigma)$ is ω-covering, then $f^\star : C_\omega(Y) \to C_\omega(X)$ is an embedding of f, where every ω-compact subset of Y is closed.*

Proof. For each a in X, there is possibility to find an ω-compact subset C of X such that $\{a\} \subseteq f(C)$. So f is onto. Hence $f^\star$ is one-to-one. Now our aim is to show that $f^\star : C_\omega \to f^\star(C_\omega(Y))$ is an open map. Let $B_A(g,\varepsilon)$ be a basic open set in $C_\omega(Y)$, where A is any ω-compact subset in Y and $\varepsilon > 0$. Suppose that $h \in f^\star(B_A(g,\varepsilon))$. Then there exists $h_1 \in B_A(g,\varepsilon)$ such that $f^\star(h_1) = h$. Since $B_A(g,\varepsilon)$ is open in $C_\omega(Y)$, then there exists an ω-compact subset D in Y such that $B_D(h_1,\delta) \subseteq B_A(g,\varepsilon)$, where $\delta > 0$. As f is ω-covering, there exists an ω-compact set C in X such that $D \subseteq f(C)$. Now we will show that $B_C(h,\delta) \cap f^\star(C_\omega(Y)) \subseteq f^\star B_D(h_1,\delta)$. Let us choose $l \in C(Y)$ such that $f^\star(l) \in B_C(h,\delta) \cap f^\star(C_\omega(Y))$. Since $D = f(C)$ for all $d \in D$, there exists $c \in f(C)$ such that $d = f(c)$. Since $f^\star(l) \in B_C(h,\delta)$ and $|l(d) - h1(d)| = |l(f(c)) - h_1(f(c))| = |f^\star(l)(c) - f^\star(h_1)(c)| = |f^\star(l)(c) - h(c)| < \delta$. So we can say that $l \in B_D(h_1,\delta)$. This show that $f^\star(l) \in f^\star(B_D(h_1,\delta))$. Hence $B_C(h,\delta) \cap f^\star(C_\omega(Y)) \subseteq f^\star(B_D(h_1,\delta)) \subseteq f^\star(B_A,(g,\varepsilon))$. Consequently, $f^\star(B_A(g,\varepsilon))$ is open in $f^\star(C_\omega(Y))$. □

Definition 3.20. *A space (X,τ) is said to be submetrizable if there is a topology $\tau^\star$ that can be defined on X such that $(X,\tau^\star)$ is a metrizable space and $\tau^\star \subset \tau$. In other words, (X,τ) is said to be submitriziable if there exists a continuous injective mapping $f : (X,\tau) \to (Y,d)$, where (Y,d) is a metric space and X is a completely regular Housdroff space.*

Definition 3.21. *The diagonal of a space X is the subset of its square $X \times X$, that is, defined by $\Delta = \{(x,x) : x \in X\}$.*

Definition 3.22. *A G_δ-set is a set which can be written as a countable intersection of open set of a space X. If Δ is a G_δ-set in $X \times X$, the space X is said to have a G_δ-diagonal.*

The properties of G_δ-set are as: In metrizable spaces, every closed set is a G_δ-set, the intersection of countably many G_δ-sets is a G_δ-set, and the union of finitely many G_δ-sets is a G_δ-set, a metric space has G_δ-diagonal and all compact subsets, countably compact subsets and the singletons are G_δ-sets in a submetrizable space. Every metrizable space has a zero-set diagonal consequently, every submetrizable space has also a zero-set-diagonal.

Definition 3.23. *A space X is called an E_0-space if every point in the space is a G_δ-set. The submetrizable spaces are E_0-spaces.*

Definition 3.24. *A completely regular Hausdorff space X is said to be σ-ω-compact if there exists a sequence $\{A_n\}$ of ω-compact sets in X such that $X = \bigcup_{n=1}^{\infty} A_n$. A space X is said to be almost σ-ω-compact if it has a dense σ-ω-compact subset.*

Theorem 3.25. *If X is any space, then following are equivalent:*

1. *$C_\omega(X)$ is submetriziable.*
2. *Every ω-compact subsets of $C_\omega(X)$ is a G_δ-set in $C_\omega(X)$.*
3. *Every countable compact subsets of $C_\omega(X)$ is a G_δ-set in $C_\omega(X)$.*
4. *Every compact subsets of $C_\omega(X)$ is a G_δ-set in $C_\omega(X)$.*
5. *$C_\omega(X)$ is a E_0-space.*
6. *X is an almost σ-ω-compact set.*
7. *$C_\omega(X)$ has zero set-diagonal.*
8. *$C_\omega(X)$ has G_δ-diagonal.*

Proof. By above definitions and properties we can show following implications: $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$.
Here we are going to show that $(5) \Rightarrow (6)$. As given $C_\omega(X)$ is an E_0-space so by definition any constant zero function 0 defined on X will be G_δ-set. Let $\{0\} = \bigcup_{i=1}^{\infty} B_{A_n}(0,\varepsilon_n)$, where $A_n \in \omega(X)$ and $\varepsilon_n > 0$. Here we are going to show that $X = \mathrm{Cl}(\bigcup_{i=1}^{\infty} A_n)$. Let x_0 be a arbitrary element in $X \backslash l(\bigcup_{i=1}^{\infty} A_n)$. Then there exists a continuous mapping $f : X \to [0,1]$ such that $f(x) = 0$ for all x in $\bigcup_{i=1}^{\infty} A_n$ and by property of X $f(x_0) = 1$. Since $f(x) = 0$ for all x in A_n and $f \in B_{A_n}(0,\varepsilon_n)$ for all n show that $f \in \bigcup_{i=1}^{\infty} B_{A_n}(0,\varepsilon_n) = \{0\}$. So $f(x) = 0$ for all $x \in X$. But $f(x_0) = 1$. This show a contradiction. Hence X is almost σ-ω-compact. □

Theorem 3.26. *Let X be an almost σ-ω-compact space and $\mathscr{P}$ is subset of $C_\omega(X)$. Then following are equivalent:*

1. *$\mathscr{P}$ is compact.*
2. *$\mathscr{P}$ is sequentially compact.*
3. *$\mathscr{P}$ is countably compact.*
4. *$\mathscr{P}$ is ω-compact.*

Proof. It is easy to prove that $(2) \Rightarrow (3) \Rightarrow (4)$.
From the above Theorem $C_\omega(X)$ is submetriziable and so $\mathscr{P}$ will be also. An ω-submetriziable space is metriziable hence it will be compact also. As we know that for a metriziable space all the above form of compactness are coincides. Hence we can say that $(1) \Rightarrow (2)$ and also $(4) \Rightarrow (1)$. □

Definition 3.27. *A subset S of a space X is said to have countable character if there is a sequence $\{W_n : n \in \mathbb{N}\}$ of open subsets in X such that $S \subseteq W_n$ for each n, and if W is any open set containing S, then $W_n \subseteq W$ for some n.*

Definition 3.28. *A space X is said to be of (pointwise) countable type if each (point) compact set is contained in a compact set having countable character.*

Definition 3.29. *A space X is said to be a q-space if for every $x \in X$ has a sequence $\{U_i\}$ of neighborhoods satisfying the condition: If $\{x_i\}$ is an infinite sequence of points in X such that $x_i \in U_i$ for each i, then $\{x_i\}$ has an accumulation point in X.*

Definition 3.30. *A space X is said to be a M-space if it can be mapped onto a metric space by a quasi-perfect mapping (a continuous closed mapping in which inverse images of points are countably compact). This space is stronger than a q-space.*

Definition 3.31. *A space X is said to be a hemi-ω-compact space if there exists a sequence of ω-compact sets $\{A_n\}$ in X such that for any ω-compact subset A of X, $A \subseteq A_n$ holds for some $n \in \mathbb{N}$.*

Theorem 3.32. *For any space X, following are equivalent:*

1. *$C_\omega(X)$ is metrizable.*
2. *$C_\omega(X)$ is of first countable.*
3. *$C_\omega(X)$ is of countable type.*
4. *$C_\omega(X)$ is of pointwise countable type.*
5. *$C_\omega(X)$ has a dense subspace of pointwise countable type.*
6. *$C_\omega(X)$ is a M-space.*
7. *$C_\omega(X)$ is a q-space.*
8. *X is a hemi-ω-compact space.*

Proof. From the above discussion, we can directly show that $(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (7)$, $(1) \Rightarrow (6) \Rightarrow (7)$ and also $(1) \Rightarrow (2) \Rightarrow (7)$. We will prove $(4) \Rightarrow (5)$. As we know that if D is any dense subset of a space X and A is a compact subset of D, then A has countable character in D if, and only if A is of countable character in X. Since $C_\omega(X)$ is a locally convex space, it is homogenous. So, with the help of this result and above discussion we can say that $(4) \Rightarrow (5)$. Next, we will prove $(7) \Rightarrow (8)$. Let $C_\omega(X)$ be a q-space So, there exists a sequence $\{V_n : n \in \mathbb{N}\}$ of neighborhoods of the zero-function 0 in $C_\omega(X)$ such that $f_n \in V_n$ for each n, then the set $\{f_n : n \in \mathbb{N}\}$ has a cluster point in $C_\omega(X)$. Now for each n, there exists a closed ω-compact subset A_n of X and $\varepsilon_n > 0$ such that $0 \in B_{A_n}(0, \varepsilon_n)$. Let A be ω-compact subset of X. Suppose that A is not subset of A_n for any $\mathbb{N}$. Then for each $n \in \mathbb{N}$, there exists $a_n \in A \backslash A_n$. Hence for each $n \in \mathbb{N}$ there exists a continuous function $f_n : X \to [0,1]$ such that $f_n(a_n) = n$ and $f_n(x) = 0$ for all $x \in A_n$. Hence it is clear that $f_n \in B_{A_n}(0, \varepsilon_n)$. But the sequence $\{f_n\}_{n \in \mathbb{N}}$ does not have a cluster point in $C_\omega(X)$. If possible, let that this sequence has a cluster point $f \in C_\omega(X)$. Then for each $k \in \mathbb{N}$, there exists a positive integer $n_k > k$ such that $f_{n_k} \in B_A(f, 1)$. So, for all $k \in \mathbb{N}$, $f(a_{n_k}) > f_{n_k}(a_{n_k} - 1) = n_k - 1 \geq k$. This show that f is unbounded on the ω-compact set A. Hence the sequence f_n cannot have a cluster point in $C_\omega(X)$ and consequently, $C_\omega(X)$ fails to be a q-space. Hence X must be a hemi-ω-compact space. Finally, we will show that $(8) \Rightarrow (1)$. As we know that if the topology of a locally convex Hausdroff space is generated by a countable family of seminorms, then it is metrizable. Now the locally convex topology on $C(X)$ generated by the countable family of seminorms $\{p_{A_n} : n \in \mathbb{N}\}$ is metrizable and weaker than the ω-compact-open topology. However, since for each ω-compact set A in X, there exists A_n such that $A \subseteq A_n$, the locally convex topology generated by the family of seminorms $\{p_A : A \in \omega(X)\}$, that is the ω-compact-open topology, is weaker than the topology generated by the family of seminorms $\{p_A : n \in \mathbb{N}\}$. Hence $C_\omega(X)$ is metrizable. □

Theorem 3.33. *For any space X, the following are equivalent:*

1. *$C_\omega(X)$ is separable.*
2. *$C_p(X)$ is separable, where p is point-open topology.*
3. *$C_k(X)$ is separable.*
4. *X has a weaker separable metriziable topology.*

Proof. From the Corollary 4.2.2 in [2] that (2), (3) and (4) are equivalent. Also, since $C_p \subseteq C_\omega(X)$, so $(1) \Rightarrow (2)$.
We will prove $(4) \Rightarrow (1)$. As we know that if X has a weaker separable metrizable topology, then X is real compact. Hence X is ω-isocompact. Consequently, $C_\omega(X) = C_k(X)$. Since $(4) \Rightarrow (3)$, $C_\omega(X)$ is separable. □

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