Malaya Journal of Matematik, Vol. 8, No. 1, 206-215, 2020
https://doi.org/10.26637/MJM0801/0035

Edge magic and bimagic harmonious labeling of ladder graphs

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Abstract
A graph $G=(V,E)$ with p vertices and q edges is said to be edge magic harmonious if there exists a bijection $f:V\cup E\rightarrow\{1,2,3,\ldots,p+q\}$ such that for each edge xy in $E(G)$, the value of $[(f(x)+f(y))(\text{mod } q)+f(xy)]$ is equal to the constant k, called magic constant. A bijection $f:V\cup E\rightarrow\{1,2,3,\ldots,p+q\}$ is called an edge bimagic harmonious labeling if $[(f(x)+f(y))(\text{mod } q)+f(xy)]=k_1$ or k_2 for each edge xy in $E(G)$, where k_1 and k_2 are two distinct magic constants. A graph G is said to be edge bimagic harmonious, if it admits an edge bimagic harmonious labeling. Here we prove that the ladder, double ladder are edge bimagic harmonious graphs and circular ladder, triangular ladder are edge magic and bimagic harmonious graphs.

Keywords
Graph , Bijection, Harmonious, Magic labeling, Bimagic labeling, Ladder, Circular ladder, Triangular ladder, Double ladder.

AMS Subject Classification
05C78.

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Article History: Received **24** November **2019**; Accepted **14** January **2020**

Contents

1. Introduction

All graphs in this paper are finite and undirected with p vertices and q edges without loops or parallel edges. The graph labeling was introduced by Rosa in 1960 [7]. Magic labeling was introduced by Sedlacek [9]. In 1970 Kotzig and Rosa [8] defined a magic valuation of a graph. In 1996, Ringel and Llado [3] called this labeling as edge magic. Edge bimagic labeling of graphs was introduced by Babujee [1] in 2004. Harmonious labeling naturally arose in the study by Graham and Sloane [4]. Dushyant Tanna [2] introduce some harmonious labeling techniques. For more annotations, we utilize dynamic survey of graph labeling by Gallian [7]. Here, we introduce the concept of edge magic and bimagic harmonious labeling of graphs and proved that ladder, double ladder are edge bimagic harmonious graphs and circular ladder, triangular ladder are edge magic and bimagic harmonious graphs.

Definition 1.1. *[6] The Cartesian product graph $G_1\times G_2$ of two graphs G_1 and G_2 is defined to be the graph whose vertex set is $V_1\times V_2$, that is every vertex of $G_1\times G_2$ is an ordered pair (u,v), where $u\in V_1$ and $v\in V_2$ and two distinct vertices (u,v) and (x,y) are adjacent in $G_1\times G_2$ if either $u=x$ and $vy\in E(G_2)$ or $v=y$ and $ux\in E(G_1)$. $P_n\times K_2$ is called a ladder.*

Definition 1.2. *[5] A circular ladder graph is defined as the cartesian product $C_n\times K_2$ where K_2 is the complete graph on two vertices and C_n is the cycle graph on n vertices.*

Definition 1.3. *[6] A triangular ladder $TL_n, n\geq 2$, is a graph obtained from the ladder $P_n\times K_2$ by adding the edges u_iv_{i+1} for $1\leq i\leq n-1$.*

Definition 1.4. *[6] The double ladder L_n is the graph $P_n\times P_3$ with vertex set $V=\{u_i/1\leqslant i\leqslant n\}\cup\{v_i/1\leqslant i\leqslant n\}$*

$\cup \ \{w_i/1 \leqslant i \leqslant n\}$ and the edge set $E=\{u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1}/1 \leqslant i \leqslant n-1\} \cup \{u_iv_i/1 \leqslant i \leqslant n\} \cup \{v_iw_i/1 \leqslant i \leqslant n\}$.

2. Main Results

Here, we introduce the concept of edge magic and bimagic harmonious labeling of graphs and proved that ladder, double ladder are edge bimagic harmonious graphs and circular ladder, triangular ladder are edge magic and bimagic harmonious graphs.

Definition 2.1. *A graph $G=(V,E)$ with p vertices and q edges is said to be edge magic harmonious if there exists a bijection $f: V(G) \cup E(G) \rightarrow \{1,2,3,\cdots,p+q\}$ such that for each edge xy in $E(G)$, the value of $[(f(x)+f(y))(modq)+f(xy)] = k$ where k is a constant.*

Definition 2.2. *A bijection $f: V(G) \cup E(G) \rightarrow \{1,2,3,\cdots,p+q\}$ is called an edge bimagic harmonious labeling if $[(f(x)+f(y))(modq)+f(xy)] = k_1$ or k_2 for each edge xy in $E(G)$, where k_1 and k_2 are two distinct magic constants. A graph G is said to be edge bimagic harmonious, if it admits an edge bimagic harmonious labeling.*

Definition 2.3. *An edge magic harmonious labeling is said to be super edge magic harmonious labeling if the graph G has the additional property that the vertex labels are 1 to $|V|$.*

Definition 2.4. *An edge bimagic harmonious labeling is said to be super edge bimagic harmonious labeling if the graph G has the additional property that the vertex labels are 1 to $|V|$.*

Example 2.5. *An edge magic and bimagic harmonious labeling of $K_{1,7}$ is given in figure 1 and figure 2.*

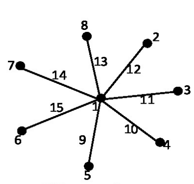

Figure 1
$K_{1,7}$ with k = 15.

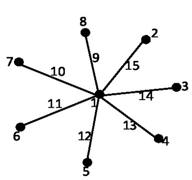

Figure 2
$K_{1,7}$ with k_1 = 11 and k_2 = 18.

Theorem 2.6. *The ladder L_n admits an edge bimagic harmonious labeling for all $n>2$.*

Proof. Let $V(L_n) = \{u_i, v_i/1 \leq i \leq n\}$ and $E(L_n)=\{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{u_iv_i/1 \leq i \leq n-1\}$. Then the graph L_n has $2n$ vertices and $3n-2$ edges.

Case 1: n is odd

Define a bijection $f: V \cup E \rightarrow \{1,2,3,\cdots,5n-2\}$ such that

$$f(u_i) = i, 1 \leq i \leq n$$
$$f(v_1) = 2n$$
$$f(v_i) = n+i-1,\ 2 \leq i \leq n$$
$$f(u_iu_{i+1}) = 5n-2i-3,\ 1 \leq i \leq n-1$$
$$f(v_1v_2) = 4n - 3$$
$$f(v_iv_{i+1}) = 3n-2i-1\ , 2 \leq i \leq \frac{n-3}{2}$$
$$f(v_iv_{i+1}) = 6n-2i-3,\ \frac{n-1}{2} \leq i \leq n-1$$
$$f(u_1v_1) = 3n-3$$
$$f(u_iv_i) = 4n-2i-1\ ,\ 2 \leq i \leq n-1$$
$$f(u_nv_n) = 5n-3$$

For the edges u_iu_{i+1}, $1 \leq i \leq n-1$.

$$[(f(u_i)+f(u_{i+1})) \bmod (q) + f(u_iu_{i+1})]$$
$$= [(i+i+1) \bmod (3n-2) + 5n-2i-3]$$
$$= [(2i+1)+5n-2i-3] = 5n-2 = k_1 \text{ (say)}$$

For the edge v_1v_2

$$[(f(v_1)+f(v_2)) \bmod (q) + f(v_1v_2)]$$
$$= [(2n+n+1) \bmod (3n-2) + 4n-3]$$
$$= [3+4n-3] = 4n = k_2 \text{ (say)}$$

For the edges $v_iv_{i+1}, 2 \leq i \leq \frac{n-3}{2}$

$$[(f(v_i)+f(v_{i+1})) \bmod (q) + f(v_iv_{i+1})]$$
$$= [((n+i-1)+(n+i)) \bmod (3n-2) + 3n-2i-1]$$
$$= [(2n+2i-1)+3n-2i-1]$$
$$= 5n-2 = k_1$$

For the edges v_iv_{i+1}, $\frac{n-1}{2} \leq i \leq n-1$

$$[(f(v_i)+f(v_{i+1})) \bmod (q) + f(v_iv_{i+1})]$$
$$= [((n+i-1)+(n+i)) \bmod (3n-2) + 6n-2i-3]$$
$$= [(2i-n+1)+6n-2i-3]$$
$$= 5n-2 = k_1$$

For the edge u_1v_1

$$[(f(u_1)+f(v_1)) \bmod (q) + f(u_1v_1)]$$
$$= [(1+2n) \bmod (3n-2) + 3n-3]$$
$$= [(2n+1)+3n-3] = 5n-2 = k_1$$

For the edges $u_iv_i, 2 \leq i \leq n-1$

$$[(f(u_i)+f(v_i)) \bmod (q) + f(u_iv_i)]$$
$$= [(i+(n+i-1)) \bmod (3n-2) + 4n-2i-1]$$
$$= [(n+2i-1)+4n-2i-1] = 5n-2 = k_1$$

For the edge $u_n v_n$

$$[(f(u_n)+f(v_n)) \bmod (q)+f(u_n v_n)]$$
$$=[(n+2n-1) \bmod (3n-2)+5n-3]$$
$$=[1+5n-3]=5n-2=k_1$$

Here, the edge labels are distinct and there exist two magic constants for each edge $xy \in E$, $[(f(x)+f(y))(\bmod q)+f(xy)]$ yields any one of the magic constants $k_1=5n-2$ and $k_2=4n$. Therefore, the ladder L_n admits an edge bimagic harmonious labeling for odd $n>2$.

Case 2: n is even
Define a bijection $f: V\cup E \to \{1,2,3,\cdots,5n-2\}$ such that

$$f(u_i)=i,\ 1\le i\le n-1$$
$$f(v_i)=n+i,\ 1\le i\le n-1$$

$$f(u_i u_{i+1})=5n-2i-4,\ 1\le i\le n-1$$
$$f(v_i v_{i+1})=3n-2i-3,\ 1\le i\le \frac{n-4}{2}$$
$$f(v_i v_{i+1})=6n-2i-5,\ \frac{n-2}{2}\le i\le n-1$$
$$f(u_i v_i)=4n-2i-3,\ 1\le i\le \frac{n}{2}$$
$$f(u_i v_i)=4n-2i-2,\ \frac{n+2}{2}\le i\le n-2$$
$$f(u_i v_i)=7n-2i-4,\ n-1\le i\le n$$

For the edges $u_i u_{i+1}, 1\le i\le n-1$

$$[(f(u_i)+f(u_{i+1})) \bmod (q)+f(u_i u_{i+1})]$$
$$=[(i+i+1) \bmod (3n-2)+5n-2i-4]$$
$$=[(2i+1)+5n-2i-4]=5n-3=k_1\,(\text{say})$$

For the edges $v_i v_{i+1}, 1\le i\le \frac{n-4}{2}$

$$[(f(v_i)+f(v_{i+1})) \bmod (q)+f(v_i v_{i+1})]$$
$$=[((n+i)+(n+i+1)) \bmod (3n-2)+3n-2i-3]$$
$$=[(2n+2i+1)+3n-2i-3]$$
$$=5n-2=k_2\,(\text{say})$$

For the edges $v_i v_{i+1}, \frac{\mathrm{n}-2}{2}\le \mathrm{i}\le \mathrm{n}-1$

$$[(f(v_i)+f(v_{i+1})) \bmod (q)+f(v_i v_{i+1})]$$

$$=[((n+i)+(n+i+1)) \bmod (3n-2)+6n-2i-5]$$
$$=[(2i-n+3)+6n-2i-5]=5n-2=k_2$$

For the edges $u_i v_i,\ 1\le i\le \frac{n}{2}$

$$[(f(u_i)+f(v_i)) \bmod (q)+f(u_i v_i)]$$
$$=[(i+(n+i)) \bmod (3n-2)+4n-2i-3]$$
$$=[(n+2i)+4n-2i-3]=5n-3=k_1$$

For the edges $u_i v_i,\ \frac{\mathrm{n}+2}{2}\le \mathrm{i}\le \mathrm{n}-2$

$$[(f(u_i)+f(v_i)) \bmod (q)+f(u_i v_i)]$$
$$=[(i+(n+i)) \bmod (3n-2)+4n-2i-2]$$
$$=[(n+2i)+4n-2i-2]=5n-2=k_2$$

For the edge $u_i v_i,\ n-1\le i\le n$

$$[(f(u_i)+f(v_i)) \bmod (q)+f(u_i v_i)]$$
$$=[(i+(n+i)) \bmod (3n-2)+7n-2i-4]$$
$$=[(2i-2n+2)+7n-2i-4]=5n-2=k_2$$

Here, the edge labels are distinct and there exist two magic constants for each edge $xy\in E$, $[(f(x)+f(y))(mod q)+f(xy)]$ yields any one of the magic constants $k_1=5n-3$ and $k_2=5n-2$. Therefore, the ladder L_n admits an edge bimagic harmonious labeling for even $n>2$. From cases (1) and (2), ladder L_n admits an edge bimagic harmonious labeling for all $n>2$. □

Corollary 2.7. *The ladder L_n admits a super edge bimagic harmonious labeling for all $n>2$.*

Proof. We proven that the ladder L_n admits an edge bimagic harmonious labeling for all $n>2$. The labeling given in the proof of Theorem 2.6, the vertices get labels $1,2,3,\ldots,2n$. Since the ladder graph has $2n$ vertices and the $2n$ vertices have labels $1,2,3,\ldots,2n$ for odd and even $n>2$, the ladder graph L_n is a super edge bimagic harmonious for all $n>2$. □

Example 2.8. *Bimagic harmonious labeling of L_9 and L_{10} are given in figure 3 and figure 4.*

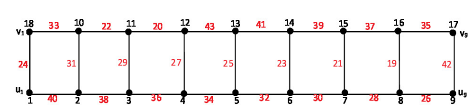

Figure 3. ladder L_9 with $k_1=43$ and $k_2=36$.

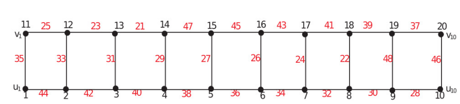

Figure 4. ladder L_{10} with $k_1=47$ and $k_2=48$.

Theorem 2.9. *The circular ladder CL_n admits an edge magic harmonious labeling for odd n.*

Proof. Let $V(CL_n) = \{u_i, v_i / 1 \le i \le n\}$ and $E(CL_n) = \{u_i u_{i+1}, v_i v_{i+1} / 1 \le i \le n-1\} \cup \{u_i v_i / 1 \le i \le n\} \cup \{u_1 u_n, v_1 v_n\}$. Then the graph CL_n has $2n$ vertices and $3n$ edges.
Define a bijection $f : V \cup E \to \{1,\ 2,\ 3, \ldots, 5n\}$ such that

$$f(u_i) = i,\ 1 \le \mathrm{i} \le \mathrm{n}$$
$$f(v_1) = 2n$$
$$f(v_i) = n + i - 1,\ 2 \le \mathrm{i} \le \mathrm{n}$$

$$f(u_i u_{i+1}) = 5n - 2i - 1,\ 1 \le \mathrm{i} \le \mathrm{n} - 1$$
$$f(u_n u_1) = 4n - 1$$
$$f(v_1 v_2) = 5n - 1$$
$$f(v_i v_{i+1}) = 3n - 2i + 1,\ 2 \le \mathrm{i} \le \frac{\mathrm{n}-1}{2}$$
$$f(v_i v_{i+1}) = 6n - 2i + 1,\ \frac{\mathrm{n}+1}{2} \le \mathrm{i} \le \mathrm{n} - 1$$
$$f(v_n v_1) = 4n + 1$$
$$f(u_1 v_1) = 3n - 1$$
$$f(u_i v_i) = 4n - 2i + 1,\ 2 \le \mathrm{i} \le \mathrm{n}$$

For the edges $u_i u_{i+1},\ 1 \le i \le n-1$

$$\begin{aligned}&[(f(u_i) + f(u_{i+1})) \bmod (q) + f(u_i u_{i+1})]\\ &= [(i + i + 1) \bmod (3n) + 5n - 2i - 1]\\ &= [(2i+1) + 5n - 2i - 1] = 5n = k\,(say)\end{aligned}$$

For the edge $u_n u_1$

$$\begin{aligned}&[(f(u_n) + f(u_1)) \bmod (q) + f(u_n u_1)]\\ &= [(n+1) \bmod (3n) + 4n - 1]\\ &= [(n+1) + 4n - 1] = 5n = k\end{aligned}$$

For the edge $v_1 v_2$

$$\begin{aligned}&[(f(v_1) + f(v_2)) \bmod (q) + f(v_1 v_2)]\\ &= [(2n + n + 1) \bmod (3n) + 5n - 1]\\ &= [1 + 5n - 1] = 5n = k\end{aligned}$$

For the edges $v_i v_{i+1},\ 2 \le \mathrm{i} \le \frac{\mathrm{n}-1}{2}$

$$\begin{aligned}&[(f(v_i) + f(v_{i+1})) \bmod (q) + f(v_i v_{i+1})]\\ &= [((n + i - 1) + (n + i))\\ &\qquad \bmod (3n) + 3n - 2i + 1]\\ &= [(2n + 2i - 1) + 3n - 2i + 1] = 5n = k\end{aligned}$$

For the edges $v_i v_{i+1},\ \frac{\mathrm{n}+1}{2} \le \mathrm{i} \le \mathrm{n} - 1$

$$\begin{aligned}&[(f(v_i) + f(v_{i+1})) \bmod (q) + f(v_i v_{i+1})]\\ &= [((n + i - 1) + (n + i))\\ &\qquad \bmod (3n) + 6n - 2i + 1]\\ &= [(2i - n - 1) + 6n - 2i + 1] = 5n = k\end{aligned}$$

For the edge $v_n v_1$

$$\begin{aligned}&[(f(v_n) + f(v_1)) \bmod (q) + f(v_n v_1)]\\ &= [((2n-1) + 2n) \bmod (3n) + 4n + 1]\\ &= [(n-1) + 4n + 1] = 5n = k\end{aligned}$$

For the edge $u_1 v_1$

$$\begin{aligned}&[(f(u_1) + f(v_1)) \bmod (q) + f(u_1 v_1)]\\ &= [(1 + 2n) \bmod (3n) + 3n - 1]\\ &= [(2n+1) + 3n - 1] = 5n = k\end{aligned}$$

For the edges $u_i v_i$, $2 \le \mathrm{i} \le \mathrm{n}$

$$\begin{aligned}&[(f(u_i) + f(v_i)) \bmod (q) + f(u_i v_i)]\\ &= [(i + (n + i - 1)) \bmod (3n) + 4n - 2i + 1]\\ &= [(n + 2i - 1) + 4n - 2i + 1] = 5n = k\end{aligned}$$

Here, the edge labels are distinct and there exist a magic constant for each edge $xy \in E$, $[(f(x) + f(y))(mod\ q) + f(xy)]$ yields the magic constant $k = 5n$. Therefore, the circular ladder CL_n admits an edge magic harmonious labeling for odd n. □

Corollary 2.10. *The circular ladder CL_n admits a super edge magic harmonious labeling for odd n.*

Proof. We proven that the circular ladder CL_n admits an edge magic harmonious labeling for odd n. The labeling given in the proof of Theorem 2.9, the vertices get labels $1, 2, 3, \ldots, 2n$. Since the circular ladder graph has $2n$ vertices and the $2n$ vertices have labels $1, 2, 3, \ldots, 2n$ for odd n, the circular ladder graph CL_n is a super edge magic harmonious for odd n. □

Example 2.11. *Magic harmonious labeling of CL_{11} is given in figure 5.*

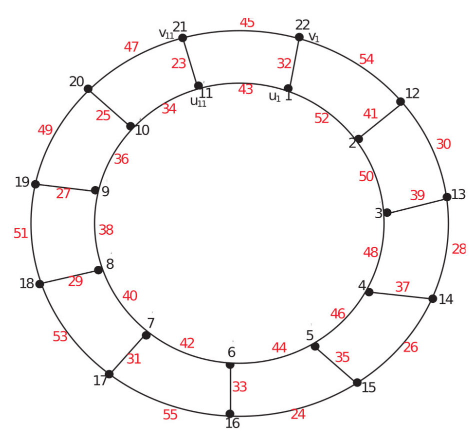

Figure 5. Circular ladder CL_{11} with $k = 55$.

Theorem 2.12. *The circular ladder CL_n admits an edge bimagic harmonious labeling for all n.*

Proof. Let $V(CL_n) = \{u_i, v_i / 1 \le i \le n\}$ and $E(CL_n) = \{u_i u_{i+1}, v_i v_{i+1} / 1 \le i \le n-1\} \cup \{u_i v_i / 1 \le i \le n\} \cup \{u_1 u_n, v_1 v_n\}$. Then the graph CL_n has $2n$ vertices and $3n$ edges.

Case 1: n is odd

Define a bijection $f : V \cup E \to \{1, 2, 3, \ldots, 5n\}$ such that

$$f(u_i) = i, 1 \le i \le n$$
$$f(v_1) = 2n$$
$$f(v_i) = n + i - 1, 2 \le i \le n$$
$$f(u_i u_{i+1}) = 3n - 2i + 1, 1 \le i \le \frac{n-1}{2}$$
$$f(u_i u_{i+1}) = 6n - 2i + 1, \frac{n+1}{2} \le i \le n-1$$
$$f(u_n u_1) = 2n + 1$$
$$f(v_1 v_2) = 3n + 1$$
$$f(v_i v_{i+1}) = 4n - 2i + 3, 2 \le i \le n-1$$
$$f(v_n v_1) = 2n + 3$$
$$f(u_1 v_1) = 4n + 1$$
$$f(u_i v_i) = 5n - 2i + 3, 2 \le i \le n$$

For the edges $u_i u_{i+1}, 1 \le i \le \frac{n-1}{2}$

$$[(f(u_i) + f(u_{i+1})) \bmod (q) + f(u_i u_{i+1})]$$
$$= [(i + i + 1) \bmod (3n) + 3n - 2i + 1]$$
$$= [(2i + 1) + 3n - 2i + 1] = 3n + 2 = k_1 \, (say)$$

For the edges $u_i u_{i+1}, \frac{n+1}{2} \le i \le n-1$

$$[(f(u_i) + f(u_{i+1})) \bmod (q) + f(u_i u_{i+1})]$$
$$= [(i + i + 1) \bmod (3n) + 6n - 2i + 1]$$
$$= [(2i + 1) + 6n - 2i + 1] = 6n + 2 = k_2 \, (say)$$

For the edge $u_n \, u_1$

$$[(f(u_n) + f(u_1)) \bmod (q) + f(u_n u_1)]$$
$$= [(n + 1) \bmod (3n) + 2n + 1]$$
$$= [(n + 1) + 2n + 1] = 3n + 2 = k_1 \, (say)$$

For the edge $v_1 \, v_2$

$$[(f(v_1) + f(v_2)) \bmod (q) + f(v_1 v_2)]$$
$$= [(2n + n + 1) \bmod (3n) + 3n + 1]$$
$$= [1 + 3n + 1] = 3n + 2 = k_1$$

For the edges $v_i v_{i+1}, 2 \le i \le \frac{n-1}{2}$

$$[(f(v_i) + f(v_{i+1})) \bmod (q) + f(v_i v_{i+1})]$$
$$= [((n + i - 1) + (n + i)) \bmod (3n) + 4n - 2i + 3]$$
$$= [(2n + 2i - 1) + 4n - 2i + 3]$$
$$= 6n + 2 = k_2$$

For the edges $v_i v_{i+1}, \frac{n+1}{2} \le i \le n-1$

$$[(f(v_i) + f(v_{i+1})) \bmod (q) + f(v_i v_{i+1})]$$
$$= [((n + i - 1) + (n + i)) \bmod (3n) + 4n - 2i + 3]$$
$$= [(2i - n - 1) + 4n - 2i + 3]$$
$$= 3n + 2 = k_1$$

For the edge $v_n \, v_1$

$$[(f(v_n) + f(v_1)) \bmod (q) + f(v_n v_1)]$$
$$= [((2n - 1) + 2n) \bmod (3n) + 2n + 3]$$
$$= [(n - 1) + 2n + 3] = 3n + 2 = k_1$$

For the edge $u_1 \, v_1$

$$[(f(u_1) + f(v_1)) \bmod (q) + f(u_1 v_1)]$$
$$= [(1 + 2n) \bmod (3n) + 4n + 1]$$
$$= [(2n + 1) + 4n + 1] = 6n + 2 = k_2$$

For the edges $u_i v_i, 2 \le i \le n$

$$[(f(u_i) + f(v_i)) \bmod (q) + f(u_i v_i)]$$
$$= [(i + (n + i - 1)) \bmod (3n) + 5n - 2i + 3]$$
$$= [(2i + n - 1) + 5n - 2i + 3] = 6n + 2 = k_2$$

Here, the edge labels are distinct and there exist two magic constants for each edge $xy \in E$, $[(f(x) + f(y))(\bmod q) + f(xy)]$ yields any one of the magic constant $k_1 = 3n + 2$ and $k_2 = 6n + 2$. Therefore, the circular ladder CL_n admits an edge bimagic harmonious labeling for odd n.

Case 2: n is even

Define a bijection $f : V \cup E \to \{1, 2, 3, \ldots, 5n\}$ such that

$$f(u_i) = i, 1 \le i \le n$$
$$f(v_i) = n + i, 1 \le i \le n$$
$$f(u_i u_{i+1}) = 5n - 2i - 1, 1 \le i \le \frac{n}{2}$$
$$f(u_i u_{i+1}) = 5n - 2i - 2, \frac{n+2}{2} \le i \le n-1$$
$$f(u_n u_1) = 4n - 2$$
$$f(v_i v_{i+1}) = 3n - 2i - 1, 1 \le i \le \frac{n-2}{2}$$
$$f\left(v_{\frac{n}{2}} v_{\frac{n+2}{2}}\right) = 5n - 1$$
$$f(v_i v_{i+1}) = 6n - 2i - 2, \frac{n+2}{2} \le i \le n-1$$
$$f(v_n v_1) = 5n - 2$$
$$f(u_i v_i) = 4n - 2i - 1, 1 \le i \le \frac{n}{2}$$
$$f(u_i v_i) = 4n - 2i, \frac{n+2}{2} \le i \le n-1$$
$$f(u_n v_n) = 5n$$

For the edges $u_iu_{i+1}, 1 \le i \le \frac{n}{2}$

$$\begin{aligned}&[(f(u_i)+f(u_{i+1})) \bmod (q)+f(u_iu_{i+1})]\\&=[(i+i+1) \bmod (3n)+5n-2i-1]\\&=[(2i+1)+5n-2i-1]=5n=k_1\,(say)\end{aligned}$$

For the edges $u_iu_{i+1}, \frac{n+2}{2} \le i \le n-1$

$$\begin{aligned}&[(f(u_i)+f(u_{i+1})) \bmod (q)+f(u_iu_{i+1})]\\&=[(i+i+1) \bmod (3n)+5n-2i-2]\\&=[(2i+1)+5n-2i-2]=5n-1=k_2\,(say)\end{aligned}$$

For the edge $u_n u_1$

$$\begin{aligned}&[(f(u_n)+f(u_1)) \bmod (q)+f(u_nu_1)]\\&=[(n+1) \bmod (3n)+4n-2]\\&=[(n+1)+4n-2]=5n-1=k_2\end{aligned}$$

For the edges $v_iv_{i+1}, 1 \le i \le \frac{n-2}{2}$

$$\begin{aligned}&[(f(v_i)+f(v_{i+1})) \bmod (q)+f(v_iv_{i+1})]\\&=[((n+i)+(n+i+1))\\&\qquad \bmod (3n)+3n-2i-1]\\&=[(2n+2i+1)+3n-2i-1]=5n=k_1\end{aligned}$$

For the edge $v_{\frac{n}{2}}v_{\frac{n+2}{2}}$

$$\begin{aligned}&\left[\left(f\left(v_{\frac{n}{2}}\right)+f\left(v_{\frac{n+2}{2}}\right)\right) \bmod (q)+f\left(v_{\frac{n}{2}}v_{\frac{n+2}{2}}\right)\right]\\&=\left[\left(\left(n+\frac{n}{2}\right)+\left(n+\frac{n+2}{2}\right)\right)\right.\\&\qquad \left. \bmod (3n)+5n-1\right]\\&=[1+5n-1]\\&=5n=k_1\end{aligned}$$

For the edges $v_iv_{i+1}, \frac{n+2}{2} \le i \le n-1$

$$\begin{aligned}&[(f(v_i)+f(v_{i+1})) \bmod (q)+f(v_iv_{i+1})]\\&=[((i+i+1)) \bmod (3n)+6n-2i-2]\\&=[(2i-n+1)+6n-2i-2]\\&=5n-1=k_2\end{aligned}$$

For the edge $v_n\, v_1$,

$$\begin{aligned}&[(f(v_n)+f(v_1)) \bmod (q)+f(v_nv_1)]\\&=[((2n)+n+1) \bmod (3n)+5n-2]\\&=[1+5n-2]=5n-1=k_2\end{aligned}$$

For the edges $u_iv_i, 1 \le i \le \frac{n}{2}$

$$\begin{aligned}&[(f(u_i)+f(v_i)) \bmod (q)+f(u_iv_i)]\\&=[(i+(n+i)) \bmod (3n)+4n-2i-1]\\&=[(2i+n)+4n-2i-1]=5n-1=k_2\end{aligned}$$

For the edges $u_iv_i, \frac{n+2}{2} \le i \le n-1$

$$\begin{aligned}&[(f(u_i)+f(v_i)) \bmod (q)+f(u_iv_i)]\\&=[(i+(n+i)) \bmod (3n)+4n-2i]\\&=[(2i+n)+4n-2i]=5n=k_1\end{aligned}$$

For the edge $u_n\, v_n$

$$\begin{aligned}&[(f(u_n)+f(v_n)) \bmod (q)+f(u_nv_n)]\\&=[(n+2n) \bmod (3n)+5n]\\&=[0+5n]\\&=5n=k_1\end{aligned}$$

Here, the edge labels are distinct and there exist two magic constants for each edge $xy \in E, [(f(x)+f(y))(\bmod q)+f(xy)]$ yields any one of the magic constant $k_1=5n$ and $k_2=5n-1$. Therefore, the circular ladder CL_n admits an edge bimagic harmonious labeling for even n.

From cases (1) and (2), circular ladder CL_n admits an edge bimagic harmonious labeling for all n. □

Corollary 2.13. *The circular ladder CL_n admits a super edge bimagic harmonious labeling for all n.*

Proof. We proven that the circular ladder CL_n admits an edge bimagic harmonious labeling for all n. The labeling given in the proof of Theorem 2.12, the vertices get labels $1,2,3,\dots,2n$. Since the circular ladder graph has 2n vertices and the $2n$ vertices have labels $1,2,3,\dots,2n$ for odd and even n, the circular ladder graph CL_n is a super edge bimagic harmonious for all n. □

Example 2.14. *Bimagic harmonious labeling of CL_{11} and CL_{12} are given in figure 6 and figure 7.*

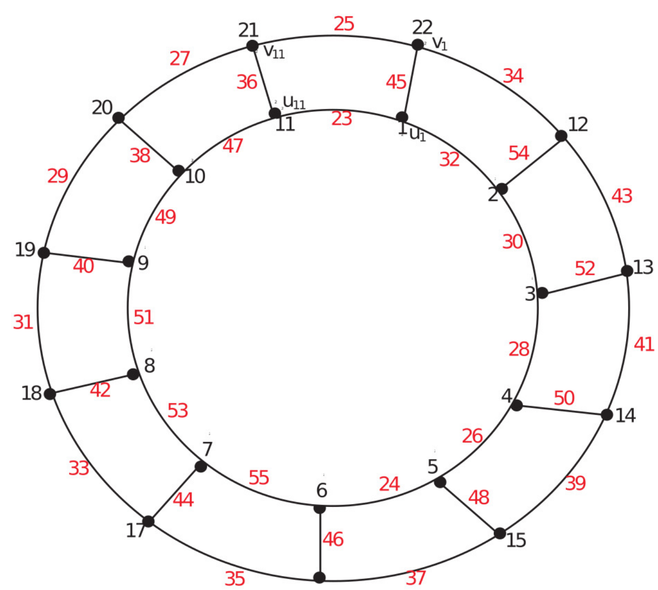

Figure 6. Circular ladder CL_{11} with $k_1=35$ and $k_2=68$.

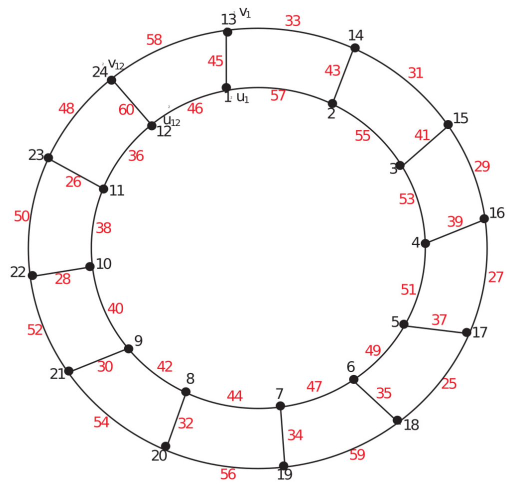

Figure 7. Circular ladder CL_{12} with $k_1 = 60$ and $k_2 = 59$.

Theorem 2.15. *The triangular ladder TL_n admits an edge magic harmonious labeling for all n.*

Proof. Let $V(TL_n) = \{u_i, v_i/1 \leq i \leq n\}$ and $E(TL_n) = \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{u_iv_i/1 \leq i \leq n\} \cup \{u_iv_{i+1}/1 \leq i \leq n-1\}$. Then the graph TL_n has $2n$ vertices and $4n-3$ edges.
Define a bijection $f : V \cup E \to \{1, 2, 3, \ldots, 6n-3\}$ such that

$$f(u_i) = 2i-1,\ 1 \leq i \leq n$$
$$f(v_i) = 2i,\ 1 \leq i \leq n$$
$$f(u_iu_{i+1}) = 6n-4i-3,\ 1 \leq i \leq n-1$$
$$f(v_iv_{i+1}) = 6n-4i-5,\ 1 \leq i \leq n-2$$
$$f(v_{n-1}v_n) = 6n-4$$
$$f(u_iv_i) = 6n-4i-2,\ 1 \leq i \leq n-1$$
$$f(u_nv_n) = 6n-5$$
$$f(u_iv_{i+1}) = 6n-4i-4,\ 1 \leq i \leq n-2$$
$$f(u_{n-1}v_n) = 6n-3$$

For the edges $u_iu_{i+1}, 1 \leq i \leq n-1$

$$[(f(u_i) + f(u_{i+1})) \bmod (q) + f(u_iu_{i+1})]$$
$$= [((2i-1)+(2i+1)) \bmod (4n-3) + 6n-4i-3]$$
$$= [4i+6n-4i-3] = 6n-3 = k(say)$$

For the edges $v_iv_{i+1}, 1 \leq i \leq n-2$

$$[(f(v_i) + f(v_{i+1})) \bmod (q) + f(v_iv_{i+1})]$$
$$= [((2i)+(2i+2)) \bmod (4n-3) + 6n-4i-5]$$
$$= [(4i+2)+6n-4i-5] = 6n-3 = k$$

For the edge $v_{n-1}v_n$

$$[(f(v_{n-1}) + f(v_n)) \bmod (q) + f(v_{n-1}v_n)]$$
$$= [((2n-2)+2n) \bmod (4n-3) + 6n-4]$$
$$= [1+6n-4] = 6n-3 = k$$

For the edges $u_iv_i, 1 \leq i \leq n-1$

$$[(f(u_i) + f(v_i)) \bmod (q) + f(u_iv_i)]$$
$$= [((2i-1)+2i) \bmod (4n-3) + 6n-4i-2]$$
$$= [(4i-1)+6n-4i-2] = 6n-3 = k$$

For the edge u_nv_n

$$[(f(u_n) + f(v_n)) \bmod (q) + f(u_nv_n)]$$
$$= [((2n-1)+2n) \bmod (4n-3) + 6n-5]$$
$$= [2+6n-5] = 6n-3 = k$$

For the edges $u_iv_{i+1}, 1 \leq i \leq n-2$

$$[(f(u_i) + f(v_{i+1})) \bmod (q) + f(u_iv_{i+1})]$$
$$= [((2i-1)+(2i+2)) \bmod (4n-3) + 6n-4i-4]$$
$$= [(4i+1)+6n-4i-4] = 6n-3 = k$$

For the edge $u_{n-1}v_n$

$$[(f(u_{n-1}) + f(v_n)) \bmod (q) + f(u_{n-1}v_n)]$$
$$= [((2n-3)+(2n)) \bmod (4n-3) + 6n-3]$$
$$= [0+6n-3] = 6n-3 = k$$

Here, the edge labels are distinct and there exist a magic constant for each edge $xy \in E$, $[(f(x)+f(y))(\bmod\ q) + f(xy)]$ yields the magic constant $k = 6n-3$. Therefore, the triangular ladder TL_n admits an edge magic harmonious labeling for all n. □

Corollary 2.16. *The triangular ladder TL_n admits a super edge magic harmonious labeling for all n.*

Proof. We proven that the triangular ladder TL_n admits an edge magic harmonious labeling for all n. The labeling given in the proof of Theorem 2.15, the vertices get labels $1, 2, 3, \ldots, 2n$. Since the triangular ladder graph has $2n$ vertices and the $2n$ vertices have labels $1, 2, 3, \ldots, 2n$ for odd and even n, the triangular ladder graph TL_n is a super edge magic harmonious for all n. □

Example 2.17. *Magic harmonious labeling of TL_{10} is given in figure 8.*

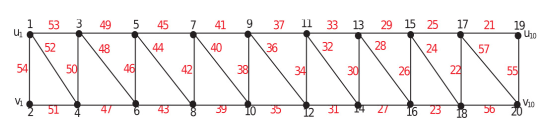

Figure 8. Triangular ladder TL_{10} with $k = 57$.

Theorem 2.18. *The triangular ladder TL_n admits an edge bimagic harmonious labeling for all n.*

Proof. Let $V(TL_n) = \{u_i, v_i/1 \leq i \leq n\}$ and $E(TL_n) = \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{u_iv_i/1 \leq i \leq n\} \cup \{u_iv_{i+1}/1 \leq i \leq n-1\}$. Then the graph TL_n has $2n$ vertices and $4n-3$ edges.
Define a bijection $f : V \cup E \rightarrow \{1, 2, 3, \ldots, 6n-3\}$ such that

$$f(u_i) = 2i-1,\ 1 \leq i \leq n$$
$$f(v_i) = 2i,\ 1 \leq i \leq n$$
$$f(u_iu_{i+1}) = 6n-4i,\ 1 \leq i \leq n-1$$
$$f(v_iv_{i+1}) = 6n-4i-2,\ 1 \leq i \leq n-2$$
$$f(v_{n-1}v_n) = 2n+2$$
$$f(u_iv_i) = 6n-4i+1,\ 1 \leq i \leq n-1$$
$$f(u_nv_n) = 2n+1$$
$$f(u_iv_{i+1}) = 6n-4i-1,\ 1 \leq i \leq n-2$$
$$f(u_{n-1}v_n) = 2n+3$$

For the edges $u_iu_{i+1}, 1 \leq i \leq n-1$

$$[(f(u_i) + f(u_{i+1})) \bmod (q) + f(u_iu_{i+1})]$$
$$= [((2i-1) + (2i+1)) \bmod (4n-3) + 6n-4i]$$
$$= [4i + 6n - 4i] = 6n = k_1\ (say)$$

For the edges $v_iv_{i+1}, 1 \leq i \leq n-2$

$$[(f(v_i) + f(v_{i+1})) \bmod (q) + f(v_iv_{i+1})]$$
$$= [((2i) + (2i+2)) \bmod (4n-3) + 6n-4i-2]$$
$$= [(4i+2) + 6n - 4i - 2] = 6n = k_1$$

For the edge $v_{n-1}v_n$

$$[(f(v_{n-1}) + f(v_n)) \bmod (q) + f(v_{n-1}v_n)]$$
$$= [((2n-2) + 2n) \bmod (4n-3) + 2n+2]$$
$$= [1 + 2n + 2] = 2n+3 = k_2\ (\text{say})$$

For the edges $u_iv_i, 1 \leq i \leq n-1$

$$[(f(u_i) + f(v_i)) \bmod (q) + f(u_iv_i)]$$
$$= [((2i-1) + (2i)) \bmod (4n-3) + (6n-4i+1)]$$
$$= [(4i-1) + 6n - 4i + 1]$$
$$= 6n = k_1$$

For the edge u_nv_n

$$[(f(u_n) + f(v_n)) \bmod (q) + f(u_nv_n)]$$
$$= [((2n-1) + (2n)) \bmod (4n-3) + (2n+1)]$$
$$= [2 + 2n + 1] = 2n+3 = k_2$$

For the edges $u_iv_{i+1}, 1 \leq i \leq n-2$

$$[(f(u_i) + f(v_{i+1})) \bmod (q) + f(u_iv_{i+1})]$$
$$= [((2i-1) + (2i+2)) \bmod (4n-3) + (6n-4i-1)]$$
$$= [(4i+1) + 6n - 4i - 1] = 6n = k_1$$

For the edge $u_{n-1}v_n$

$$[(f(u_{n-1}) + f(v_n)) \bmod (q) + f(u_{n-1}v_n)]$$
$$= [((2n-3) + 2n) \bmod (4n-3) + (2n+3)]$$
$$= [0 + 2n + 3] = 2n+3 = k_2$$

Here, the edge labels are distinct and there exist two magic constants for each edge $xy \in E$, $[(f(x) + f(y))(\bmod q) + f(xy)]$ yields any one of the magic constant $k_1 = 6n$ and $k_2 = 2n+3$. Therefore, the triangular ladder TL_n admits an edge bimagic harmonious labeling for all n. □

Corollary 2.19. *The triangular ladder TL_n admits a super edge bimagic harmonious labeling for all n.*

Proof. We proven that the triangular ladder TL_n admits an edge bimagic harmonious labeling for all n. The labeling given in the proof of Theorem 2.18, the vertices get labels $1, 2, 3, \ldots, 2n$. Since the triangular ladder graph has $2n$ vertices and the $2n$ vertices have labels $1, 2, 3, \ldots, 2n$ for odd and even n, the triangular ladder graph TL_n is a super edge bimagic harmonious for all n. □

Example 2.20. *Bimagic harmonious labeling of TL_{10} is given in figure 9.*

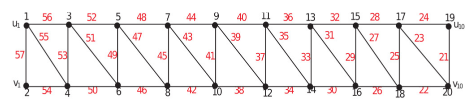

Figure 9. Triangular ladder TL_{10} with $k_1 = 60$ and $k_2 = 23$.

Theorem 2.21. *The double ladder $P_n \times P_3$ admits an edge bimagic harmonious labeling for odd n.*

Proof. Let $V(P_n \times P_3) = \{u_i, v_i, w_i/1 \leq i \leq n\}$ and $E(P_n \times P_3) = \{u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1}/1 \leq i \leq n-1\} \cup \{u_iv_i, v_iw_i/1 \leq i \leq n\}$. Then the graph $P_n \times P_3$ has $3n$ vertices and $5n-3$ edges.
Define a bijection $f : V \cup E \rightarrow \{1, 2, 3, \ldots, 8n-3\}$ such that

$$f(u_i) = i,\ 1 \leq i \leq n$$
$$f(v_1) = 2n$$
$$f(v_i) = n+i-1,\ 2 \leq i \leq n$$
$$f(w_i) = 2n+i,\ 1 \leq i \leq n$$
$$f(u_iu_{i+1}) = 8n-2i-5,\ 1 \leq i \leq n-1$$
$$f(v_1v_2) = 5n-5$$
$$f(v_iv_{i+1}) = 6n-2i-3,\ 2 \leq i \leq n-1$$
$$f(w_iw_{i+1}) = 4n-2i-4,\ 1 \leq i \leq \frac{n-5}{2}$$
$$f(w_iw_{i+1}) = 9n-2i-7,\ \frac{n-3}{2} \leq i \leq n-1$$
$$f(u_1v_1) = 6n-5$$
$$f(u_iv_i) = 7n-2i-3, 2 \leq i \leq n$$
$$f(v_1w_1) = 7n-1 \text{ for } n = 3$$
$$f(v_1w_1) = 4n-4 \text{ for } n > 3$$

$$f(v_iw_i) = 5n-2i-3,\ 2 \le i \le \frac{n-1}{2}$$
$$f(v_iw_i) = 5n-2i-2,\ \frac{n+1}{2} \le i \le n-2$$
$$f(v_iw_i) = 10n-2i-5,\ n-1 \le i \le n$$

For the edges $u_iu_{i+1}, 1 \le i \le n-1$,

$$\begin{aligned}&[(f(u_i)+f(u_{i+1})) \bmod (q) + f(u_iu_{i+1})]\\&= [(i+i+1) \bmod (5n-3)+8n-2i-5]\\&= [(2i+1)+8n-2i-5] = 8n-4 = k_1 \text{ (say)}\end{aligned}$$

For the edge v_1v_2

$$\begin{aligned}&[(f(v_1)+f(v_2)) \bmod (q) + f(v_1v_2)]\\&= [(2n+n+1) \bmod (5n-3)+5n-5]\\&= [(3n+1)+5n-5] = 8n-4 = k_1\end{aligned}$$

For the edges $v_iv_{i+1}, 2 \le i \le n-1$,

$$\begin{aligned}&[(f(v_i)+f(v_{i+1})) \bmod (q) + f(v_iv_{i+1})]\\&= [((n+i-1)+(n+i))\\&\qquad \bmod (5n-3)+6n-2i-3]\\&= [(2n+2i-1)+6n-2i-3]\\&= 8n-4 = k_1\end{aligned}$$

For the edges $w_iw_{i+1}, 1 \le i \le \frac{n-5}{2}$

$$\begin{aligned}&[(f(w_i)+f(w_{i+1})) \bmod (q) + f(w_iw_{i+1})]\\&= [((2n+i)+(2n+i+1))\\&\qquad \bmod (5n-3)+4n-2i-4]\\&= [(4n+2i+1)+4n-2i-4]\\&= 8n-3 = k_2 \text{ (say)}\end{aligned}$$

For the edges $w_iw_{i+1}, \frac{n-3}{2} \le i \le n-1$

$$\begin{aligned}&[(f(w_i)+f(w_{i+1})) \bmod (q) + f(w_iw_{i+1})]\\&= [((2n+i)+(2n+i+1))\\&\qquad \bmod (5n-3)+9n-2i-7]\\&= [(2i-n+4)+9n-2i-7]\\&= 8n-3 = k_2\end{aligned}$$

For the edge u_1v_1

$$\begin{aligned}&[(f(u_1)+f(v_1)) \bmod (q) + f(u_1v_1)]\\&= [(1+2n) \bmod (5n-3)+6n-5]\\&= [(2n+1)+6n-5] = 8n-4 = k_1\end{aligned}$$

For the edges $u_iv_i, 2 \le i \le n$

$$\begin{aligned}&[(f(u_i)+f(v_i)) \bmod (q) + f(u_iv_i)]\\&= [(i+(n+i-1))\\&\qquad \bmod (5n-3)+7n-2i-3]\\&= [(n+2i-1)+7n-2i-3]\\&= 8n-4 = k_1\end{aligned}$$

For the edge v_1w_1 for the graph $n=3$

$$\begin{aligned}&[(f(v_1)+f(w_1)) \bmod (q) + f(v_1w_1)]\\&= [((2n)+(2n+1)) \bmod (5n-3)+7n-1]\\&= [(n-2)+7n-1] = 8n-3 = k_2\end{aligned}$$

For the edge v_1w_1 for the graph $n>3$

$$\begin{aligned}&[(f(v_1)+f(w_1)) \bmod (q) + f(v_1w_1)]\\&= [((2n)+(2n+1))\\&\qquad \bmod (5n-3)+4n-4]\\&= [(4n+1)+4n-4]\\&= 8n-3 = k_2\end{aligned}$$

For the edges $v_iw_i, 2 \le i \le \frac{n-1}{2}$

$$\begin{aligned}&[(f(v_i)+f(w_i)) \bmod (q) + f(v_iw_i)]\\&= [((n+i-1)+(2n+i))\\&\qquad \bmod (5n-3)+5n-2i-3]\\&= [(3n+2i-1)+5n-2i-3]\\&= 8n-4 = k_1\end{aligned}$$

For the edges $v_iw_i, \frac{n+1}{2} \le i \le n-2$

$$\begin{aligned}&[(f(v_i)+f(w_i)) \bmod (q) + f(v_iw_i)]\\&= [((n+i-1)+(2n+i))\\&\qquad \bmod (5n-3)+5n-2i-2]\\&= [(3n+2i-1)+5n-2i-2]\\&= 8n-3 = k_2\end{aligned}$$

For the edges $v_iw_i, n-1 \le i \le n$

$$\begin{aligned}&[(f(v_i)+f(w_i)) \bmod (q) + f(v_iw_i)]\\&= [((n+i-1)+(2n+i))\\&\qquad \bmod (5n-3)+10n-2i-5]\\&= [(2i-2n+2)+10n-2i-5]\\&= 8n-3 = k_2\end{aligned}$$

Here, the edge labels are distinct and there exist two magic constants for each edge $xy \in E$, $[(f(x)+f(y))(\bmod\ q)+f(xy)]$ yields any one of the magic constant $k_1 = 8n-4$ and $k_2 = 8n-3$. Therefore, the double ladder $P_n \times P_3$ admits an edge bimagic harmonious labeling for odd n. □

Corollary 2.22. *The double ladder $P_n \times P_3$ admits a super edge bimagic harmonious labeling for odd n.*

Proof. We proven that the double ladder $P_n \times P_3$ admits an edge bimagic harmonious labeling for odd n. The labeling given in the proof of Theorem 2.21, the vertices get labels $1, 2, 3, \ldots, 3n$. Since the double ladder graph has $3n$ vertices and the $3n$ vertices have labels $1, 2, 3, \ldots, 3n$ for odd n, the double ladder graph $P_n \times P_3$ is a super edge bimagic harmonious for odd n. □

Example 2.23. *Bimagic harmonious labeling of $P_9 \times P_3$ is given in figure 10.*

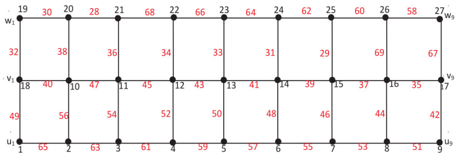

Figure 10. Double ladder $P_9 \times P_3$ with $k_1 = 68$ and $k_2 = 69$.

3. Conclusion

Here we proven that the ladder L_n, double ladder $P_n \times P_3$ are edge bimagic harmonious graphs and circular ladder CL_n, triangular ladder TL_n are edge magic and bimagic harmonious graphs.

References

[1] J. Baskar Babujee, On edge bimagic labeling, *Journal of Combinatorics Information & System Sciences*, 28(1 - 4)(2004), 239–244.

[2] Dushyant Tanna, Harmonious labeling of certain graphs, *International Journal of Advanced Engineering Research and Studies*, (2013), 46–68.

[3] H. Enomoto, Anna S. Llado, Tomoki Nakamigawa and Gerhard Ringel, On Super Edge Magic Graphs, *SUT Journal of Mathematics*, 34(1998), 105–109.

[4] R.L. Graham and N.J.A.Sloane, On additive bases and harmonious graphs, *SIAM Journal on Algebraic and Discrete Methods,* 1(1980), 382–404.

[5] C. Jayasekran and J. Little Flower, On edge trimagic labeling of umbrella, dumb bell, and circular ladder graphs, *Annals of Pure and Applied Mathematics*, 13(1)(2017), 73–87.

[6] C. Jayasekran and M. Regees and C. Davidraj, Edge trimagic labeling of some graphs, *International Journal for Combinatorial Graph Theory and Applications*, 6(2)(2013), 175–186.

[7] Joseph A. Gallian, A dynamic survey of graph labeling of some graphs, *The Electronic Journal of Combinatorics*, (2018), #DS6.

[8] A. Kotzig and A. Rosa, Magic Valuations of finite graphs, *Canad. Math. Bull.*, 13(1970), 415–416.

[9] J. Sedlacek, Theory of Graphs and its Applications, *Proc. Symposium Smolience*, (1963), 163–167.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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