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A characterization of involutes of a given curve in $\mathbb{E}^3$ via directional q-frame

MÜNEVVER YILDIRIM YILMAZ *[1] AND **ESRA ERDEM**[2]

[1,2] *Faculty of Science, Department of Mathematics, Firat University, Elazığ, Turkey.*

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Abstract. The orthogonal trajectories of the first tangents of the curve are called the involutes of α. In the present study, we obtain a characterization of involute curves of order k of the given curve α using directional q-frame. In virtue of the formulas, some results are obtained.

AMS Subject Classifications: Primary 53a04; Secondary 53C26.

Keywords: Frenet curve, Frenet frame, involute curve, directional q-frame.

Contents

1. Introduction

In differential geometry, there are many significant results and properties of curves. In the light of numerous studies authors introduce new works by using frame fields. The directional q- frame field is known as one of the frame field of the differential geometry. The q-frame has some useful advantages comparing to the other well-known frames Frenet and Bishop. One can define and calculate this frame even along a line $(\kappa = 0)$. Dede et al. offered the directional q-frame along a space curve to built a tubular surface. They obtained a parametric representation of a directional tubular surface using the q-frame $[1]$.

Involutes of a curve is another attractive research subject among geometers. The idea of a string involute is due to C. Huygens (1658), who is also known as on optician. He discovered involutes trying to build a more accurate clock $[2]$. There are many brillant works on involutes of a given curve in different aspects. For instance, Frenet frame of involute-evolute couple in the space $\mathbb{E}^3$ were given in $[3]$. T. Soyfidan and M. A. Güngör studied a quaternionic curve Euclidean 4-space $\mathbb{E}^4$ and gave the on the quaternionic involute-evolute curves for quaternionic curve $[4]$. Another is As and Sarıoğlugil study's. They obtained on the Bishop curvatures of involute-evolute curve couple in $\mathbb{E}^3$ $[5]$.

In this paper, the characterization of involutes of the 1 st. and 2 nd. order of a curve are given and proved in .$\mathbb{E}^3$ by the help of directional q-frame.

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*[*] **Corresponding author.** Email address: **munyildirim@firat.edu.tr** (" Münevver YILDIRIM YILMAZ")

2. Preliminaries

There are a number of different adapted frames along a space curve, like the parallel transport frame [6, 7] and the Frenet frame [8] . The Frenet frame is the most well-known frame along a space curve. Let $\alpha(s)$ be a space curve with a non-vanishing second derivative. The Frenet frame is described as follows:

$$t = \frac{\alpha^{'}}{\|\alpha'\|}, \quad b = \frac{\alpha^{'} \wedge \alpha^{''}}{\|\alpha' \wedge \alpha''\|}, \quad n = b \wedge t$$

The curvature κ and the torsion τ are obtain by;

$$\kappa = \frac{\left\|\alpha^{'} \wedge \alpha^{''}\right\|}{\|\alpha'\|^3}, \quad \tau = \frac{\det\left(\alpha^{'}, \alpha^{''}, \alpha^{'''}\right)}{\|\alpha' \wedge \alpha''\|^2}$$

The well-known Frenet formulas are obtain by;

$$\begin{bmatrix} t^{'} \\ n^{'} \\ b^{'} \end{bmatrix} = \varphi \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}$$

where $\varphi = \left\|\alpha^{'}(s)\right\|$.

As an alternative to the Frenet frame they define a new adapted frame along a space curve, the q-frame [1] . Dede et al. defined the directional q-frame along a space curve [9] . The directional q-frame offers two key advantages over the Frenet Frame [10, 11] : a) it is well defined even if the curve has vanishing second derivative [12] , b) it avoid the redundant twist around the tangent.

The directional q-frame of a regular curve $\alpha(s)$ is obtained by;

$$t = \frac{\alpha^{'}}{\|\alpha'\|}, \quad n_q = \frac{t \wedge k}{\|t \wedge k\|}, \quad b_q = t \wedge n_q \tag{1}$$

where k is the projection vector.

The varitation equations of the directional q-frame is obtained by;

$$\begin{bmatrix} t^{'} \\ n_q^{'} \\ b_q^{'} \end{bmatrix} = \left\|\alpha^{'}\right\| \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \tag{2}$$

where the q-curvatures are expressed as follows:

$$k_1 = \frac{\left\langle t^{'}, n_q \right\rangle}{\|\alpha'\|}, \quad k_2 = \frac{\left\langle t^{'}, b_q \right\rangle}{\|\alpha'\|}, \quad k_3 = -\frac{\left\langle n_q, b_q^{'} \right\rangle}{\|\alpha'\|}. \tag{3}$$

[9] .

3. Involutes of order 1 st. and order 2 nd. in $\mathbb{E}^3$ according to projection vector

As is well known q-frame is defined by the help of the projection vector k. For simplicity firstly we have choosen the projection vector $k = (0; 0; 1)$. For the cases t and k are parallel, the projection vector can be chosen as

$k = (0;1;0)$, $k = (1;0;0)$ (see [9]). This part we classified the q-frame into three types: z axis directional q-frames identified with the projection vector $k = (0;0;1)$ (see Theorem 3.1 and 3.2), y axis directional q-frames identified with the projection vector $k = (0;1;0)$ (see Theorem 3.3 and 3.4) and x axis directional q-frames identified with the projection vector $k = (1;0;0)$ (see Theorem 3.5 and 3.6).

Definition 3.1. Let $\alpha = \alpha(s)$ be a regular generic curve in $\mathbb{E}^n$ given with the arclength parameter s (i.e., $\left\|\alpha'(s)\right\| = 1$). Then the curves which are orthogonal to the system of k-dimensional osculating hyperplanes of α, are called the involutes of order k [13] of the curve α. For simplicity, we call the involutes of order 1, simply the involutes of the given curve [14].

The theorems below are given by taking $k = (0;0;1)$.

Theorem 3.1. Let $\alpha = \alpha(s)$ be a regular curve in $\mathbb{E}^3$ and any curve $\overline{\alpha}(s)$ be first order involute of $\alpha(s)$. Then q-curvatures $\overline{k_1}, \overline{k_2}$ and $\overline{k_3}$ of the involute $\overline{\alpha}$ of the curve α are obtain by

$$\overline{k_1} = -\sqrt{k_1^2 + k_2^2}, \quad \overline{k_2} = \frac{\left[k_1' k_2 - k_2' k_1\right] - \left\|\alpha'\right\| k_3 \left[k_1^2 + k_2^2\right]}{\left\|\alpha'\right\| \left[k_1^2 + k_2^2\right]},$$
$$\overline{k_3} = 0$$

Proof:

$$s \to \alpha(s) \to \sum_{i=1}^{3} \alpha_i(s) e_i \quad (1 \leq i \leq 3)$$

by using statement we obtain that

$$\overline{\alpha}(s) = \alpha(s) + \lambda(s) t(s)$$

we by using (2), differentiate this equation respect to s, we obtain

$$\overline{\alpha}'(s) = \alpha'(s) + \lambda'(s) t(s) + \lambda(s) \left\|\alpha'\right\| \left[k_1 n_q + k_2 b_q\right]$$

Since

$$\left\langle \overline{\alpha}'(s), t(s) \right\rangle = 0$$

and

$$\alpha'(s) = \left\|\alpha'\right\| t(s)$$

we write

$$\lambda(s) = c - \|\alpha\|$$

So, we get

$$\overline{\alpha}'(s) = \alpha'(s) - \left\|\alpha'\right\| t(s) + (c - \|\alpha\|) \left\|\alpha'\right\| \left[k_1 n_q + k_2 b_q\right]$$
$$= (c - \|\alpha\|) \left\|\alpha'\right\| \left[k_1 n_q + k_2 b_q\right] \tag{4}$$

Using norm of the equation (4), we get

$$\left\|\overline{\alpha}'(s)\right\| = (c - \|\alpha\|) \sqrt{k_1^2 + k_2^2} \left\|\alpha'\right\| \tag{5}$$

and by using the equations (1), (4) and (5), we get

$$\bar{t}(s) = \frac{\left[k_1 n_q + k_2 b_q\right]}{\sqrt{k_1^2 + k_2^2}} \tag{6}$$

if we have chosen the projection vector $k = (0; 0; 1)$

$$\bar{t} \wedge k = \frac{k_1 t}{\sqrt{k_1^2 + k_2^2}} \tag{7}$$

Hence, by taking norm of equation (7), we get

$$\left\| \bar{t} \wedge k \right\| = \sqrt{\frac{k_1^2}{\left(\sqrt{k_1^2 + k_2^2}\right)^2}} \tag{8}$$

Moreover, using the equations (1), (7) and (8), we have

$$\overline{n_q}\,(s) = t \tag{9}$$

In addition, using the equations (6),and (9)

$$\bar{t} \wedge \overline{n_q} = \frac{k_2 n_q - k_1 b_q}{\sqrt{k_1^2 + k_2^2}} \tag{10}$$

Therefore, from (1) and (10) , we get

$$\overline{b_q}\,(s) = \frac{k_2 n_q - k_1 b_q}{\sqrt{k_1^2 + k_2^2}} \tag{11}$$

Consequently, by using the equations (3) , we obtain

$$\overline{k_1} = -\sqrt{k_1^2 + k_2^2} \tag{12}$$

$$\overline{k_2} = \frac{\left[k_1^{'} k_2 - k_2^{'} k_1\right] - \left\|\alpha^{'}\right\| k_3 \left[k_1^2 + k_2^2\right]}{\left\|\alpha^{'}\right\| \left[k_1^2 + k_2^2\right]} \tag{13}$$

$$\overline{k_3} = 0 \tag{14}$$

This completes the proof.

Theorem 3.2. Let $\alpha = \alpha\,(s)$ be a regular curve in $\mathbb{E}^3$ and any curve $\overline{\alpha}\,(s)$ be second order involute of $\alpha\,(s)$. Then q-curvatures $\overline{k_1}, \overline{k_2}$ and $\overline{k_3}$ of the involute $\overline{\alpha}$ of the curve α are vanishes.

$$\overline{k_1} = 0, \quad \overline{k_2} = 0, \quad \overline{k_3} = 0$$

Proof:

$$s \to \alpha\,(s) \to \sum_{i=1}^{3} \alpha_i\,(s)\, e_i \quad (1 \leq i \leq 3)$$

by using statement we obtain that

$$\overline{\alpha}\,(s) = \alpha\,(s) + \lambda_1\,(s)\, t\,(s) + \lambda_2\,(s)\, n_q\,(s)$$

we by using (2) , differentiate this equation respect to s, we obtain

$$\begin{aligned} \overline{\alpha}^{'}\,(s) &= \alpha^{'}\,(s) + \lambda_1^{'}\,(s)\, t\,(s) + \lambda_1\,(s) \left\|\alpha^{'}\right\| [k_1 n_q + k_2 b_q] \\ &+ \lambda_2^{'}\,(s)\, n_q\,(s) - \lambda_2\,(s) \left\|\alpha^{'}\right\| k_1 t + \lambda_2\,(s) \left\|\alpha^{'}\right\| k_3 b_q \end{aligned}$$

Since

$$\langle \overline{\alpha}^{'}(s), t(s) \rangle = 0, \quad \langle \overline{\alpha}^{'}(s), n_q(s) \rangle = 0$$

and

$$\alpha^{'}(s) = \left\| \alpha^{'} \right\| t(s)$$

So, we get

$$\overline{\alpha}^{'}(s) = \left\| \alpha^{'} \right\| [\lambda_1 k_2 + \lambda_2 k_3] \, b_q$$

if we take

$$\lambda_1 k_2 = \theta(s), \quad \lambda_2 k_3 = \varphi(s)$$

we obtain

$$\overline{\alpha}^{'}(s) = \left\| \alpha^{'} \right\| [\theta(s) + \varphi(s)] \, b_q \tag{15}$$

Using norm of the equation (15), we get

$$\left\| \overline{\alpha}^{'}(s) \right\| = \sqrt{\| \alpha' \| \, [\theta(s) + \varphi(s)]^2} \tag{16}$$

and by using the equations (1), (15) and (16), we attain

$$\bar{t}(s) = \frac{\left\| \alpha^{'} \right\| [\theta(s) + \varphi(s)] \, b_q}{\sqrt{\| \alpha' \| \, [\theta(s) + \varphi(s)]^2}} = b_q \tag{17}$$

if we have chosen the projection vector $k = (0; 0; 1)$

$$\bar{t} \wedge k = 0 \tag{18}$$

Hence, by taking norm of equation (18), we get

$$\left\| \bar{t} \wedge k \right\| = 0 \tag{19}$$

Moreover, using the equations (1), (18) and (19), we have

$$\overline{n_q}(s) = 0 \tag{20}$$

In addition, using the equations (17),and (20)

$$\bar{t} \wedge \overline{n_q} = 0 \tag{21}$$

Therefore, from (1) and (21) , we get

$$\overline{b_q}(s) = 0 \tag{22}$$

Consequently, by using the equations (3) , we obtain

$$\overline{k_1} = 0 \tag{23}$$

$$\overline{k_2} = 0 \tag{24}$$

$$\overline{k_3} = 0 \tag{25}$$

This completes the proof.

The theorems below are given by taking $k=(0;1;0)$.

Theorem 3.3. Let $\alpha=\alpha(s)$ be a regular curve in $\mathbb{E}^3$ and any curve $\overline{\alpha}(s)$ be first order involute of $\alpha(s)$. Then q-curvatures $\overline{k_1},\overline{k_2}$ and $\overline{k_3}$ of the involute $\overline{\alpha}$ of the curve α are obtain by

$$\overline{k_1}=\sqrt{k_1^2+k_2^2},\quad \overline{k_2}=\frac{\left[k_2^{'}k_1-k_2k_1^{'}\right]+\left\|\alpha^{'}\right\|k_3\left[k_1^2+k_2^2\right]}{\|\alpha'\|\left[k_1^2+k_2^2\right]},$$
$$\overline{k_3}=0$$

Proof:

$$s\to\alpha(s)\to\sum_{i=1}^{3}\alpha_i(s)\,e_i\quad(1\leq i\leq 3)$$

by using statement we obtain that

$$\overline{\alpha}(s)=\alpha(s)+\lambda(s)\,t(s)$$

we by using (2), differentiate this equation respect to s, we obtain

$$\overline{\alpha}^{'}(s)=\alpha^{'}(s)+\lambda^{'}(s)\,t(s)+\lambda(s)\left\|\alpha^{'}\right\|\left[k_1n_q+k_2b_q\right]$$

Since

$$\left\langle\overline{\alpha}^{'}(s),t(s)\right\rangle=0$$

and

$$\alpha^{'}(s)=\left\|\alpha^{'}\right\|t(s)$$

we write

$$\lambda(s)=c-\|\alpha\|$$

So, we get

$$\overline{\alpha}^{'}(s)=\alpha^{'}(s)-\left\|\alpha^{'}\right\|t(s)+(c-\|\alpha\|)\left\|\alpha^{'}\right\|\left[k_1n_q+k_2b_q\right]$$

$$=(c-\|\alpha\|)\left\|\alpha^{'}\right\|\left[k_1n_q+k_2b_q\right]\tag{26}$$

Using norm of the equation (26), we get

$$\left\|\overline{\alpha}^{'}(s)\right\|=(c-\|\alpha\|)\sqrt{k_1^2+k_2^2}\left\|\alpha^{'}\right\|\tag{27}$$

and by using the equations (1), (26) and (27), we get

$$\bar{t}(s)=\frac{\left[k_1n_q+k_2b_q\right]}{\sqrt{k_1^2+k_2^2}}\tag{28}$$

if we have chosen the projection vector $k=(0;1;0)$

$$\bar{t}\wedge k=\frac{-k_2t}{\sqrt{k_1^2+k_2^2}}\tag{29}$$

Hence, by taking norm of equation (29), we get

$$\left\|\bar{t}\wedge k\right\|=\sqrt{\frac{k_2^2}{\left(\sqrt{k_1^2+k_2^2}\right)^2}}\tag{30}$$

Moreover, using the equations (1), (29) and (30), we have

$$\overline{n_q}\,(s) = -t \tag{31}$$

In addition, using the equations (28),and (31)

$$\bar{t} \wedge \overline{n_q} = \frac{-k_2 n_q + k_1 b_q}{\sqrt{k_1^2 + k_2^2}} \tag{32}$$

Therefore, from (1) and (32) , we get

$$\overline{b_q}\,(s) = \frac{-k_2 n_q + k_1 b_q}{\sqrt{k_1^2 + k_2^2}} \tag{33}$$

Consequently, by using the equations (3) , we obtain

$$\overline{k_1} = \sqrt{k_1^2 + k_2^2} \tag{34}$$

$$\overline{k_2} = \frac{\left[k_2^{'} k_1 - k_2 k_1^{'}\right] + \left\|\alpha^{'}\right\| k_3 \left[k_1^2 + k_2^2\right]}{\|\alpha^{'}\| \left[k_1^2 + k_2^2\right]} \tag{35}$$

$$\overline{k_3} = 0 \tag{36}$$

This completes the proof.

Theorem 3.4. Let $\alpha = \alpha\,(s)$ be a regular curve in $\mathbb{E}^3$ and any curve $\overline{\alpha}\,(s)$ be second order involute of $\alpha\,(s)$. Then q-curvatures $\overline{k_1}, \overline{k_2}$ and $\overline{k_3}$ of the involute $\overline{\alpha}$ of the curve α are obtain by

$$\overline{k_1} = k_2, \quad \overline{k_2} = k_3, \quad \overline{k_3} = k_1$$

Proof:

$$s \to \alpha\,(s) \to \sum_{i=1}^{3} \alpha_i\,(s)\, e_i \quad (1 \leq i \leq 3)$$

by using statement we obtain that

$$\overline{\alpha}\,(s) = \alpha\,(s) + \lambda_1\,(s)\, t\,(s) + \lambda_2\,(s)\, n_q\,(s)$$

we by using (2) , differentiate this equation respect to s, we obtain

$$\begin{aligned} \overline{\alpha}^{'}\,(s) = \alpha^{'}\,(s) + \lambda_1^{'}\,(s)\, t\,(s) + \lambda_1\,(s) \left\|\alpha^{'}\right\| [k_1 n_q + k_2 b_q] \\ + \lambda_2^{'}\,(s)\, n_q\,(s) - \lambda_2\,(s) \left\|\alpha^{'}\right\| k_1 t + \lambda_2\,(s) \left\|\alpha^{'}\right\| k_3 b_q \end{aligned}$$

Since

$$\langle \overline{\alpha}^{'}\,(s)\,, t\,(s)\rangle = 0, \quad \langle \overline{\alpha}^{'}\,(s)\,, n_q\,(s)\rangle = 0$$

and

$$\alpha^{'}\,(s) = \left\|\alpha^{'}\right\| t\,(s)$$

So, we get

$$\overline{\alpha}^{'}\,(s) = \left\|\alpha^{'}\right\| [\lambda_1 k_2 + \lambda_2 k_3]\, b_q$$

if we take

$$\lambda_1 k_2 = \theta\,(s)\,, \quad \lambda_2 k_3 = \varphi\,(s)$$

we obtain

$$\overline{\alpha}^{'}(s)=\left\|\alpha^{'}\right\|\left[\theta(s)+\varphi(s)\right]b_q \tag{37}$$

Using norm of the equation (37), we get

$$\left\|\overline{\alpha}^{'}(s)\right\|=\sqrt{\|\alpha'\|\left[\theta(s)+\varphi(s)\right]^2} \tag{38}$$

and by using the equations (1), (37) and (38), we attain

$$\bar{t}(s)=\frac{\left\|\alpha^{'}\right\|\left[\theta(s)+\varphi(s)\right]b_q}{\sqrt{\|\alpha'\|\left[\theta(s)+\varphi(s)\right]^2}}=b_q \tag{39}$$

if we have chosen the projection vector $k=(0;1;0)$

$$\bar{t}\wedge k=-t \tag{40}$$

Hence, by taking norm of equation (40), we get

$$\left\|\bar{t}\wedge k\right\|=1 \tag{41}$$

Moreover, using the equations (1), (40) and (41), we have

$$\overline{n_q}(s)=-t \tag{42}$$

In addition, using the equations (39),and (42)

$$\bar{t}\wedge\overline{n_q}=-n_q \tag{43}$$

Therefore, from (1) and (43) , we get

$$\overline{b_q}(s)=-n_q \tag{44}$$

Consequently, by using the equations (3) , we obtain

$$\overline{k_1}=k_2 \tag{45}$$

$$\overline{k_2}=k_3 \tag{46}$$

$$\overline{k_3}=k_1 \tag{47}$$

This completes the proof.

The theorems below are given by taking $k=(1;0;0)$.

Theorem 3.5. Let $\alpha=\alpha(s)$ be a regular curve in $\mathbb{E}^3$ and any curve $\overline{\alpha}(s)$ be first order involute of $\alpha(s)$. Then q-curvatures $\overline{k_1},\overline{k_2}$ and $\overline{k_3}$ of the involute $\overline{\alpha}$ of the curve α are obtain by

$$\overline{k_1}=\frac{\left[k_1^{'}k_2-k_1k_2^{'}\right]-\left\|\alpha^{'}\right\|k_3\left[k_1^2+k_2^2\right]}{\|\alpha'\|\left[k_1^2+k_2^2\right]},\quad \overline{k_2}=\sqrt{k_1^2+k_2^2},$$
$$\overline{k_3}=0$$

Proof:

$$s\to\alpha(s)\to\sum_{i=1}^{3}\alpha_i(s)e_i\quad(1\leq i\leq 3)$$

by using statement we obtain that

$$\overline{\alpha}(s) = \alpha(s) + \lambda(s)\, t(s)$$

we by using (2), differentiate this equation respect to s, we obtain

$$\overline{\alpha}'(s) = \alpha'(s) + \lambda'(s)\, t(s) + \lambda(s) \left\|\alpha'\right\| [k_1 n_q + k_2 b_q]$$

Since

$$\langle \overline{\alpha}'(s), t(s) \rangle = 0$$

and

$$\alpha'(s) = \left\|\alpha'\right\| t(s)$$

we write

$$\lambda(s) = c - \|\alpha\|$$

So, we get

$$\begin{aligned} \overline{\alpha}'(s) &= \alpha'(s) - \left\|\alpha'\right\| t(s) + (c - \|\alpha\|) \left\|\alpha'\right\| [k_1 n_q + k_2 b_q] \\ &= (c - \|\alpha\|) \left\|\alpha'\right\| [k_1 n_q + k_2 b_q] \end{aligned} \tag{48}$$

Using norm of the equation (48), we get

$$\left\|\overline{\alpha}'(s)\right\| = (c - \|\alpha\|) \sqrt{k_1^2 + k_2^2} \left\|\alpha'\right\| \tag{49}$$

and by using the equations (1), (48) and (49), we get

$$\bar{t}(s) = \frac{[k_1 n_q + k_2 b_q]}{\sqrt{k_1^2 + k_2^2}} \tag{50}$$

if we have chosen the projection vector $k = (1; 0; 0)$

$$\bar{t} \wedge k = \frac{[k_2 n_q - k_1 b_q]}{\sqrt{k_1^2 + k_2^2}} \tag{51}$$

Hence, by taking norm of equation (51), we get

$$\left\|\bar{t} \wedge k\right\| = \sqrt{\frac{k_1^2}{\left(\sqrt{k_1^2 + k_2^2}\right)^2}} \tag{52}$$

Moreover, using the equations (1), (51) and (52), we have

$$\overline{n_q}(s) = \frac{[k_2 n_q - k_1 b_q]}{\sqrt{k_1^2 + k_2^2}} \tag{53}$$

In addition, using the equations (50),and (53)

$$\bar{t} \wedge \overline{n_q} = t \tag{54}$$

Therefore, from (1) and (54), we get

$$\overline{b_q}(s) = t \tag{55}$$

Consequently, by using the equations (3), we obtain

$$\overline{k_1} = \frac{\left[k_1' k_2 - k_1 k_2'\right] - \left\|\alpha'\right\| k_3 \left[k_1^2 + k_2^2\right]}{\|\alpha'\| \left[k_1^2 + k_2^2\right]} \tag{56}$$

$$\overline{k_2} = \sqrt{k_1^2 + k_2^2} \tag{57}$$

$$\overline{k_3} = 0 \tag{58}$$

This completes the proof

Theorem 3.6. Let $\alpha = \alpha(s)$ be a regular curve in $\mathbb{E}^3$ and any curve $\overline{\alpha}(s)$ be second order involute of $\alpha(s)$. Then q-curvatures $\overline{k_1}, \overline{k_2}$ and $\overline{k_3}$ of the involute $\overline{\alpha}$ of the curve α are obtain by

$$\overline{k_1} = -k_3, \quad \overline{k_2} = k_2, \quad \overline{k_3} = k_1$$

Proof:

$$s \to \alpha(s) \to \sum_{i=1}^{3} \alpha_i(s)\, e_i \quad (1 \leq i \leq 3)$$

by using statement we obtain that

$$\overline{\alpha}(s) = \alpha(s) + \lambda_1(s)\, t(s) + \lambda_2(s)\, n_q(s)$$

we by using (2), differentiate this equation respect to s, we obtain

$$\begin{aligned} \overline{\alpha}'(s) = \alpha'(s) + \lambda_1'(s)\, t(s) + \lambda_1(s) \left\|\alpha'\right\| [k_1 n_q + k_2 b_q] \\ + \lambda_2'(s)\, n_q(s) - \lambda_2(s) \left\|\alpha'\right\| k_1 t + \lambda_2(s) \left\|\alpha'\right\| k_3 b_q \end{aligned}$$

Since

$$\langle \overline{\alpha}'(s), t(s) \rangle = 0, \quad \langle \overline{\alpha}'(s), n_q(s) \rangle = 0$$

and

$$\alpha'(s) = \left\|\alpha'\right\| t(s)$$

So, we get

$$\overline{\alpha}'(s) = \left\|\alpha'\right\| [\lambda_1 k_2 + \lambda_2 k_3]\, b_q$$

if we take

$$\lambda_1 k_2 = \theta(s), \quad \lambda_2 k_3 = \varphi(s)$$

we obtain

$$\overline{\alpha}'(s) = \left\|\alpha'\right\| [\theta(s) + \varphi(s)]\, b_q \tag{59}$$

Using norm of the equation (59), we get

$$\left\|\overline{\alpha}'(s)\right\| = \sqrt{\|\alpha'\| \left[\theta(s) + \varphi(s)\right]^2} \tag{60}$$

and by using the equations (1), (59) and (60), we attain

$$\bar{t}(s) = \frac{\left\|\alpha'\right\| [\theta(s) + \varphi(s)]\, b_q}{\sqrt{\|\alpha'\| \left[\theta(s) + \varphi(s)\right]^2}} = b_q \tag{61}$$

if we have chosen the projection vector $k = (1; 0; 0)$

$$\bar{t} \wedge k = n_q \tag{62}$$

Hence, by taking norm of equation (62), we get

$$\left\|\bar{t} \wedge k\right\| = 1 \tag{63}$$

Moreover, using the equations (1), (62) and (63), we have

$$\overline{n_q}\,(s) = n_q \tag{64}$$

In addition, using the equations (61),and (64)

$$\bar{t} \wedge \overline{n_q} = -t \tag{65}$$

Therefore, from (1) and (65) , we get

$$\overline{b_q}\,(s) = -t \tag{66}$$

Consequently, by using the equations (3) , we obtain

$$\overline{k_1} = -k_3 \tag{67}$$

$$\overline{k_2} = k_2 \tag{68}$$

$$\overline{k_3} = k_1 \tag{69}$$

This completes the proof.

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