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On Aboodh transform for fractional differential operator

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Abstract

In this paper we introduce Aboodh transform for the different fractional differential operators. Using this transform solution of initial value fractional order differential equation is given and finally, we discuss some illustrative examples on the above results.

Keywords

Aboodh Transform, Riemann-Liouville and Caputo fractional derivative, Mittag-Leffler Function, Fractional differential equations.

AMS Subject Classification

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1. Introduction

Like classical theory of calculus, fractional calculus also has so many importance in different field like Statistics,Physics, Engineering, namely in control engineering [11], Electromagnetism [8], Fluid Mechanics [9], Diffusion Theory [10] and so on. From the last some decades fractional differential equations become most useful tool to solve many problems in this fields. So, we must develop the domain of solvable set of fractional differential equation.On the parallel line integral transform is the convenient way for solving differential equation of fractional order. Now a day there are so many methods has developed, using integral transforms like Laplace [12], Sumudu [13], Kamal [15], Shehutransform [14], Z-transform [16] are used to solve Fractional Differential Equation.

The motivation behind this study is to develop thenewway of solution for fractional differential equation with the help of Aboodh Transform. We generalize the definition of Aboodh Transform of differential operators from integer order to fractional order and use it, to solve problems of fractional differential Equations.

2. Preliminaries

Definition 2.1. *Mittag-Leffler Function [1, 2] is the generalization of exponential function denoted by $E_\alpha(z)$ (for one parameter), $E_{\alpha,\beta}(z)$ (for two parameter) defined as,*

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \quad \alpha\in R^+,\ z\in C \tag{2.1}$$

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \quad \alpha,\beta\in R^+,\ z\in C \tag{2.2}$$

Definition 2.2. *The Riemann-Liouville [1, 2] fractional derivative of function $f:(0,\infty)\to R$, for $\alpha\in R^+$ is defined as,*

$$\begin{aligned} D^\alpha f(t) &= f^{(n)}(t), \quad for\ \alpha=n\in N \\ &= \frac{d^\alpha}{dt^\alpha}\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}ds, \end{aligned} \tag{2.3}$$

for $\alpha \neq n \in R - N$. Where, $\Gamma(z)$ is Euler Gamma function [?] which is generalization of factorial function from set of integers to the set of complex numbers, Defined as,

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, z \in C, \text{ with } Re(z) > 0.$$

Definition 2.3. *The Caputo fractional [1, 2] derivative of function $f : (0,\infty) \to R$ is defined as,*

$$\begin{aligned} D^{\alpha} f(t) &= f^{(n)}(t), \quad \text{for } \alpha = n \in N \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \end{aligned} \tag{2.4}$$

for $\alpha \neq n \in R - N$.

2.1 Definition and properties of Aboodh transform:

The new integral transform name as Aboodh transform [3] defined on set S of functions,

$$S = \{f(t) | \exists M, k_1, k_2 > 0, |f(t)| < Me^{-vt}\}$$

Where M is finite number and k_1, k_2 may finite or infinite.

The Aboodh transform is denoted by $\mathscr{A}[p(t)]$.

Defined as, the integral equation of the form,

$$\mathscr{A}[p(t)] = q(v) = \frac{1}{v} \int_0^{\infty} p(t) e^{-vt} dt, \ t \geq 0, k_1 \leq v \leq k_2 \tag{2.5}$$

Definition 2.4. *Inverse Aboodh transform [5, 7] of function $q(v)$ denoted by $\mathscr{A}^{-1}[q(V)]$*

Defined as,

$$\mathscr{A}^{-1}[q(V)] = p(t) = \frac{1}{2\pi i} \int_{u-i\infty}^{u+i\infty} v e^{vt} q(v) dv \tag{2.6}$$

2.2 Aboodh Transform of some fractional operators:

In this section we prove Aboodh Transform of different fractional differential operators.

Theorem 2.5. *If Aboodh transform of function $p(t)$ is $q(v)$ then Aboodh transform of Riemann-Liouville fractional derivative of order $\alpha > 0$ is,*

$$\mathscr{A}[D^{\alpha} p(t)] = v^{\alpha} q(v) - \sum_{k=1}^{n} \left[v^{k-2} D^{\alpha-k} p(t) \right]_{t=0} \tag{2.7}$$

Where, $n - 1 < \alpha \leq n, n \in N$.

Proof. Suppose first that Aboodh transform of function $p(t)$ is $q(v)$ and Riemann-Liouville fractional derivative of order $\alpha > 0$ is,

$$D^{\alpha} p(t) = \frac{d^{\alpha}}{dt^{\alpha}} \left[\frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{p(s)}{(t-s)^{\alpha-n+1}} ds \right]$$

by Applying Laplace Transform on both to this equation we get,

$$L[D^{\alpha} p(t)] = v^{\alpha} L[p(v)] - \sum_{k=0}^{n-1} \left[v^{k} D^{\alpha-k-1} p(t) \right]_{t=0}$$

$$L[D^{\alpha} p(t)] = v^{\alpha} L[p(v)] - \sum_{k=1}^{n} \left[v^{k-1} D^{\alpha-k} p(t) \right]_{t=0}$$

Using Aboodh-Laplace Duality, $L[p(t)] = v\mathscr{A} p(t)$ [6] we get,

$$v\mathscr{A}[D^{\alpha} p(t)] = v^{\alpha} v\mathscr{A}[p(v)] - \sum_{k=1}^{n} \left[v^{k-1} D^{\alpha-k} p(t) \right]_{t=0}$$

$$\mathscr{A}[D^{\alpha} p(t)] = v^{\alpha} \mathscr{A}[p(v)] - \sum_{k=1}^{n} \left[v^{k-1} D^{\alpha-k} p(t) \right]_{t=0}$$

$$\mathscr{A}[D^{\alpha} p(t)] = v^{\alpha} q(v) - \sum_{k=1}^{n} \left[v^{k-2} D^{\alpha-k} p(t) \right]_{t=0} \tag{2.8}$$

This is the required proof. □

Theorem 2.6. *If Aboodh transform of function $p(t)$ is $q(v)$ then Aboodh transform of Caputo fractional derivative of order $\alpha > 0$is,*

$$\mathscr{A}[D^{\alpha} p(t)] = v^{\alpha} q(v) - \sum_{k=1}^{n} \left[v^{\alpha-k-1} D^{k-1} p(t) \right]_{t=0} \tag{2.9}$$

Where, $n - 1 < \alpha \leq n, n \in N$.

Proof. Given that Aboodh transform of function $p(t)$ is $q(v)$ and Caputo fractional derivative of order $\alpha > 0$ is,

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds$$

by Applying Laplace transform on both to this equation we get,

$$L[D^{\alpha} p(t)] = v^{\alpha} L[p(v)] - \sum_{k=0}^{n-1} \left[v^{\alpha-k-1} D^{k} p(t) \right]_{t=0}$$

$$L[D^{\alpha} p(t)] = v^{\alpha} L[p(v)] - \sum_{k=1}^{n} \left[v^{\alpha-k} D^{k-1} p(t) \right]_{t=0}$$

Using Aboodh-Laplace Duality [6],

$$v\mathscr{A}[D^{\alpha} p(t)] = v^{\alpha} v\mathscr{A}[p(v)] - \sum_{k=1}^{n} \left[v^{\alpha-k} D^{k-1} p(t) \right]_{t=0}$$

$$\mathscr{A}[D^{\alpha} p(t)] = v^{\alpha} \mathscr{A}[p(v)] - \sum_{k=1}^{n} \left[v^{\alpha-k-1} D^{k-1} p(t) \right]_{t=0}$$

$$\mathscr{A}[D^{\alpha} p(t)] = v^{\alpha} q(v) - \sum_{k=1}^{n} \left[v^{\alpha-k-1} D^{k-1} p(t) \right]_{t=0} \tag{2.10}$$

This is the required proof. □

3. Applications of Aboodh Transform for solving Fractional Differential Equation:

3.1 Solve fractional order differential equation of the form,

$$D^{\alpha}(p(t)) = p(t) \text{ With } p(0) = p_0, \text{ Where } \alpha \in R^{+}$$

Taking Aboodh transform on both side we get,

$$\mathscr{A}\{D^{\alpha}(p(t))\} = \mathscr{A}[p(t)]$$

$$v^{\alpha}q(v) - \sum_{k=1}^{n} v^{\alpha-k-1}D^{k-1}p(t) = q(v)$$

$$v^{\alpha}q(v) - \sum_{k=1}^{n} v^{\alpha}v^{-k-1}D^{k-1}p(t) = q(v)$$

$$v^{\alpha}q(v) - q(v) = \sum_{k=1}^{n} v^{\alpha}v^{-k-1}D^{k-1}p(t)$$

$$(v^{\alpha}-1)q(v) = \sum_{k=1}^{n} v^{\alpha}v^{-k-1}D^{k-1}p(t)$$

$$q(v) = \frac{v^{\alpha}}{(v^{\alpha}-1)}\sum_{k=1}^{n} v^{-k-1}D^{k-1}p(t)$$

By taking inverse Aboodh Transform on both side we get,

$$\mathscr{A}^{-1}\{q(v)\} = \mathscr{A}^{-1}\{\frac{v^{\alpha}}{(v^{\alpha}-1)}\sum_{k=1}^{n} v^{-k-1}D^{k-1}p(t)\}$$

$$p(t) = \mathscr{A}^{-1}\{\frac{v^{\alpha}}{(v^{\alpha}-1)}\sum_{k=1}^{n} v^{-k-1}D^{k-1}p(t)\} \tag{3.1}$$

Example 3.1. *Solve*

$$D^{\frac{1}{2}}(p(t)) = t, \text{ With } p(0) = p_0, \tag{3.2}$$

By taking Abooth transform on both sides we get,

$$v^{\frac{1}{2}}q(v) - [v^{\frac{1}{2}-2}p(0)] = \frac{1}{v^3}$$

$$v^{\frac{1}{2}}q(v) - v^{\frac{1}{2}-2}p_0 = \frac{1}{v^3}$$

$$v^{\frac{1}{2}}q(v) = \frac{1}{v^3} + p_0v^{\frac{1}{2}-2}$$

$$q(v) = \frac{v^{-\frac{1}{2}}}{v^3} + p_0v^{\frac{1}{2}-2}v^{-\frac{1}{2}}$$

$$q(v) = \frac{1}{v^{3+\frac{1}{2}}} + p_0v^{-2}$$

$$q(v) = \frac{1}{v^{2+\frac{3}{2}}} + \frac{p_0}{v^2}$$

By taking inverse Aboodh transform on both side we get,

$$\mathscr{A}^{-1}\{q(v)\} = \mathscr{A}^{-1}\{\frac{1}{v^{2+\frac{3}{2}}} + \frac{p_0}{v^2}\}$$

$$p(t) = \frac{t^{\frac{3}{2}}}{(\frac{3}{2})!} + p_0$$

$$p(t) = \frac{t^{\frac{3}{2}}}{\frac{3}{4}\sqrt{\pi}} + p_0$$

$$p(t) = \frac{4t^{\frac{3}{2}}}{3\sqrt{\pi}} + p_0$$

is the solution of initial value fractional differential equation (3.2).

3.2 Solve fractional order differential equation of the form

$$D^{\alpha}p(t) + p(t) = \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)}t^{1-\alpha} + t^2 - t \tag{3.3}$$

With $p^{(\alpha)}(0) = p\alpha = 0$, Where $\alpha \in R^{+}$ Taking Aboodh transform on both side we get,

$$v^{\alpha}q(v) - \sum_{k=1}^{n} v^{\alpha-k-1}D^{k-1}p(t) + q(v)$$

$$= \frac{2}{\Gamma(3-\alpha)}\frac{\Gamma(3-\alpha)}{v^{2-\alpha-2}} - \frac{1}{\Gamma(2-\alpha)}\frac{\Gamma(2-\alpha)}{v^{1-\alpha+2}} + \frac{\Gamma(2)}{v^4} - \frac{1}{v^3}$$

$$v^{\alpha}q(v) - v^{\alpha}\sum_{k=1}^{n} v^{-k-1}D^{k-1}p(0) + q(v)$$

$$= \frac{2}{v^{-\alpha}} - \frac{1}{v^{3-\alpha}} + \frac{2}{v^4} - \frac{1}{v^3}$$

$$v^{\alpha}q(v) + q(v) = \frac{2}{v^{-\alpha}} - \frac{1}{v^{3-\alpha}} + \frac{2}{v^4} - \frac{1}{v^3}$$

$$(v^{\alpha}+1)q(v) = 2(v^{\alpha}+1)\frac{1}{v^4} - (v^{\alpha}+1)\frac{1}{v^3}$$

$$q(v) = \frac{2}{v^4} - \frac{1}{v^3}$$

By taking inverse Aboodh transform on both side we get,

$$\mathscr{A}^{-1}\{q(v)\} = \mathscr{A}^{-1}\{\frac{2}{v^4} - \frac{1}{v^3}\}$$

$$p(t) = \frac{2t^2}{2!} - t$$

Finally gives,

$$p(t) = t^2 - t$$

is the solution of initial value fractional differential equation (3.5).

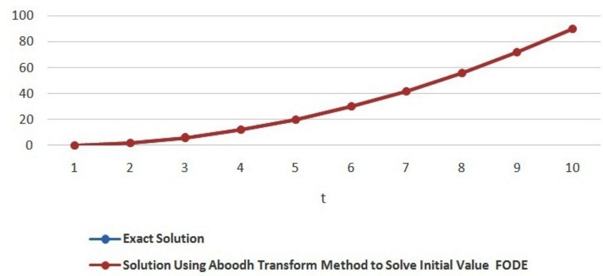

Fig.1. Comparison of the exact solution of fractional order initial value differential equation & solutions using proposed work for different values of t.

3.3 Solve fractional order differential equation of the form,

$D^{\alpha}p(t)+Dp(t)=p(t)$ With $p^{(\alpha)}(0)=p_{\alpha}$, Where $\alpha\in R^{+}$
Taking Aboodh transform on both side we get,

$$v^{\alpha}q(v)-\sum_{k=1}^{n}v^{\alpha-k-1}D^{k-1}p(t)+vq(v)-v^{-1}p(0)=q(v)$$

$$v^{\alpha}q(v)-v^{\alpha}\sum_{k=1}^{n}v^{-k-1}D^{k-1}p(t)+vq(v)-v^{-1}p_0=q(v)$$

$$v^{\alpha}q(v)+vq(v)-q(v)=v^{\alpha}\sum_{k=1}^{n}v^{-k-1}D^{k-1}p(t)+v^{-1}p_0$$

$$(v^{\alpha}+v-1)q(v)=v^{\alpha}\sum_{k=1}^{n}v^{-k-1}D^{k-1}p(t)+v^{-1}p_0$$

$$q(v)=\frac{1}{(v^{\alpha}+v-1)}\{v^{\alpha}\sum_{k=1}^{n}v^{-k-1}D^{k-1}p(t)+v^{-1}p_0\}$$

By taking inverse Aboodh transform on both side we get,

$$\mathscr{A}^{-1}\{q(v)\}=\mathscr{A}^{-1}\{\frac{1}{(v^{\alpha}+v-1)}$$
$$[v^{\alpha}\sum_{k=1}^{n}v^{-k-1}D^{k-1}p(t)+v^{-1}p_0]\}$$

$$p(t)=\mathscr{A}^{-1}\{\frac{1}{(v^{\alpha}+v-1)}[v^{\alpha}\sum_{k=1}^{n}v^{-k-1}D^{k-1}p(t)+v^{-1}p_0]\} \tag{3.4}$$

Example 3.2. *Solve $D^{\frac{3}{2}}p(t)+Dp(t)=t$, With initial conditions,*

$$\begin{aligned}p(0)&=p_0=0,\\ Dp(0)&=p_1=\frac{1}{v}\end{aligned} \tag{3.5}$$

By taking Abooth transform on both sides we get,

$$v^{\frac{3}{2}}q(v)-\sum_{k=1}^{2}v^{\frac{3}{2}-k-1}D^{k-1}p(t)+vq(v)-v^{-1}p(0)=\frac{1}{v^3}$$

$$v^{\frac{3}{2}}q(v)-v^{\frac{3}{2}-2}p(0)-v^{\frac{3}{2}-3}Dp(0)+vq(v)-v^{-1}p(0)=\frac{1}{v^3}$$

$$v^{\frac{3}{2}}q(v)-v^{\frac{-1}{2}}p(0)-v^{\frac{-3}{2}}Dp(0)+vq(v)-v^{-1}p(0)=\frac{1}{v^3}$$

$$v^{\frac{3}{2}}q(v)-v^{\frac{-1}{2}}p_0-v^{\frac{-3}{2}}p_1+vq(v)-v^{-1}p_0=\frac{1}{v^3}$$

$$v^{\frac{3}{2}}q(v)+vq(v)=v^{\frac{-1}{2}}p_0+v^{\frac{-3}{2}}p_1v^{-1}p_0+\frac{1}{v^3}$$

$$(v^{\frac{3}{2}}+v)q(v)=v^{\frac{-1}{2}}p_0+v^{\frac{-3}{2}}p_1v^{-1}p_0+\frac{1}{v^3}$$

$$(v^{\frac{1}{2}}+1)q(v)=v^{\frac{-3}{2}}p_0+v^{\frac{-5}{2}}p_1v^{-2}p_0+\frac{1}{v^4}$$

$$(v^{\frac{1}{2}}+1)q(v)=v^{\frac{-5}{2}}v^{-1}+\frac{1}{v^4}$$

$$(v^{\frac{1}{2}}+1)q(v)=v^{\frac{-5}{2}-1}+\frac{1}{v^4}$$

$$(v^{\frac{1}{2}}+1)q(v)=v^{\frac{1}{2}-4}+\frac{1}{v^4}$$

$$(v^{\frac{1}{2}}+1)q(v)=(v^{\frac{1}{2}}+1)\frac{1}{v^4}$$

$$q(v)=\frac{1}{v^4}$$

Then by taking inverse Aboodh Transform on both side we get,

$$\mathscr{A}^{-1}\{q(v)\}=\mathscr{A}^{-1}\{\frac{1}{v^4}\}$$

$$p(t)=\frac{t^2}{2!}$$

is the solution of initial value fractional differential equation (3.5).

4. Conclusion

From this study it is conclude that with the help of Aboodh transform we can easily find the solutions of initial value fractional differential equations of any order $\alpha\in R^{+}$. This method is applying for both the types, Homogeneous or Non-homogeneous differential equations of fractional order in the given forms. From the graph it is clearly observe that by using Aboodh transform we get exact analytic solutions of Fractional Differential equation.

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