Malaya Journal of Matematik, Vol. 8, No. 1, 235-242, 2020

https://doi.org/10.26637/MJM0801/0040

# On certain subclass of normalized analytic function associated with Rusal differential operator

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**Abstract**

In this article the author discusses the two subclasses namely $\check{K}_p(A^n_\lambda;\gamma,\mu,m,\beta)$ and $K_p(A^m_\lambda;\gamma,\mu,m,\beta)$ of normalized analytic functions. With convex combination of Ruschwey and Al-Oboudi differential operator we derived Rusal differential operator. Two new subclasses $\check{K}_p(A^n_\lambda;\gamma,\mu,m,\beta)$ and $K_p(A^n_\lambda;\gamma,\mu,m,\beta)$ are studied with help of Rusal differential operator.Growththeorem, Closure theorem, Integral mean inequality, extreme point theorem, coefficient inequality, convolution and distortion theorem for given class are examined.

**Keywords**

Analytic function, Rusal differential operator.

**AMS Subject Classification**

11E45.

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**Article History**: Received **14** December **2019**; Accepted **29** February **2020**



## Contents



## 1. Introduction and Preliminaries

Let $N$ denotes subclass of all analytical functions in open unit disk $U=\{z:|z|<1\}$ normalized with conditions

$$f(0)=0, \quad f^{'}(0)=1$$

given by

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \tag{1.1}$$

Ruscheweyh in [3] has introduced following differential operator.

$R^n: N\rightarrow N$ defined by

$$\begin{aligned} R^n(f(z)) &= \frac{z}{(1-z)^{n+1}} * f(z), \quad n\in\mathbb{N}\cup\{0\} \\ &= z+\sum_{k=2}^{\infty} {}^{n+k-1}C_n a_k z^k \quad (z\in U) \end{aligned} \tag{1.2}$$

Where $*$ is hadmard product defined in (7.1).

We note that $R^0 f(z)=f(z), R^{'}f(z)=zf^{'}(z)$

[6] has used following definition 1.1 and 1.2

**Definition 1.1.** *A function $f$ in $N$ is said to be in $C(\alpha)$, if and only if*

$$\Re\{f^{'}(z)\}>\alpha \quad (z\in U \ \&\ 0\le\alpha<1) \tag{1.3}$$

**Definition 1.2.** *A function $f$ in $N$ is said to be in $CS^*(\alpha)$ if and only if*

$$\Re\{\frac{f^{'}(z)}{z}\}>\alpha \quad (z\in U \ \&\ 0\le\alpha<1) \tag{1.4}$$

*We write the classes $C(0)=C$, $CS^*(0)=CS^*$.*

**Definition 1.3.** *For two functions $f$ and $g$ analytic in $U$, we say that the function $f(z)$ is subordinate to $g(z)$ in $U$ and write*

$$f(z)>g(z) \quad (z\in U) \tag{1.5}$$

*If there exit Schwarz function $w(z)$, analytical in $U$ with $w(0)=0$ and $|w(z)|<1$ such that*

$$f(z)=g(w(z)) \quad (z\in U) \tag{1.6}$$

**Definition 1.4.** *For $f\in N$, [1] has introduced following differential operator, known as Al-Oboudi differential operator.*
*$D^n: N\to N$ defined by*

$$D_0 f(z)=f(z) \tag{1.7}$$

$$D_1 f(z)=(1-)f(z)+zf^{'}(z)=Df(z) \quad \geq 0 \tag{1.8}$$

$$D_n f(z)=D(D^{n-1}f(z)) \tag{1.9}$$

*From (1.10) and (1.5) we have*

$$D^n(f(z))=z+\sum_{k=2}^{\infty}[1+(k-1)\partial]^n a_k z^k \quad (z\in U) \tag{1.10}$$

*We will make use of definition of subordination between analytic functions [2] in our further investigation.*

J.E. Littlewood has introduced following subordination theorem which we stated as lemma.

We use this lemma to prove integral mean inequality given in theorem 6.1

**Lemma 1.5.** *Let $f$ and $g$ analytic in unit disc and suppose $g<f$, then for $0<t<\infty$*

$$\int_0^{2\pi}|g(re^{i\phi}))|^t d\theta \leq \int_0^{2\pi}|f(re^{i\phi}))|^t d\theta \tag{1.11}$$

$$(0\leq r<1,\ t>0)$$

*Strict equality holds for $0\leq r<1$ unless $f$ is constant or $w(z)=\alpha z, |\alpha|=1$.*

## 2. Rusal Differential Operator, Classes $K_p(A^n_\lambda;\gamma,\mu,m,\beta)$ and $\check{K}_p(A^n_\lambda;\gamma,\mu,m,\beta)$

We formed the Rusal differential operator by making convex combination of Ruschwey & Al-Oboudi differential operators discussed in (1.2) and (1.10) respectively. We also introduced New subclasses $K_p(A^n_\lambda;\gamma,\mu,m,\beta)$ and $\check{K}_p(A^n_\lambda;\gamma,\mu,m,\beta)$ which are generalization of $K(\gamma,\mu,m,\beta)$ and $\check{K}(\gamma,\mu,m,\beta)$ respectively [7].

**Definition 2.1.** *Let $n\in N\cup\{0\},\lambda\geq 0$, $A^n_\lambda: N\to N$ defined by*

$$A^n_\lambda(f(Z))=(1-\lambda)D^n f(Z)+\lambda R^n f(z) \tag{2.1}$$

*On simplifying, we observed that*

$$A^n_\lambda(f(z))=z+\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_n C)a_k z^k. \tag{2.2}$$

*If $n=0$,*

$$A^0_\lambda f(z)=f(z) \tag{2.3}$$

**Definition 2.2.**

$$\begin{aligned}&Kp(A^m_\lambda;\gamma,\mu,m,\beta)\\&=\{f\in N:|\frac{1}{p\gamma}((p\text{–}u)\frac{A^m_\lambda f}{z}+u(A^m_\lambda f)^{'}-p)|<\beta\}\end{aligned} \tag{2.4}$$

*Where $z\in U$, $\gamma\in C|\{0\}, p\in\mathbb{R}^+$, $0<\beta\leq 1 \quad 0<\mu\leq p$, $m\in N\cup\{0\}$, $A^m_\lambda f$ is defined in (2.1).*

We illustrate the subclass $Kp(A^m_\lambda;\gamma,\mu,m,\beta)$ with following example.

**Example 2.3.** *If $f(z)=z$, then for $\gamma=1,\mu=p,m=0$, $0<\beta\leq 1$, show that $f(z)\in K_p(A^m_\lambda;\gamma,\mu,m,\beta)$*
*For $\gamma=1,\mu=p,m=0$, $0<\beta\leq 1$,*

$$\begin{aligned}&|\frac{1}{p\gamma}((p\text{–}u)\frac{A^m_\lambda f}{z}+u(A^m_\lambda f)^{'}-p)|\\&=|\frac{1}{p1}((p\text{–}p)\frac{A^0_\lambda f}{z}+p(A^0_\lambda f)^{'}-p)|\\&=|(A^0_\lambda f)^{'}-1|\\&=|(z)^{'}-1|\\&=|1-1|\\&<\beta.\end{aligned}$$

*Hence,*

$$f(z)\in Kp(A^0_\lambda;1,p,0,\beta)$$

**Definition 2.4.** *Let $\check{K}_p(A^n_\lambda;\gamma,\mu,m,\beta)$ be the subclass of $N$ which satisfies inequality*

$$\sum_{k=2}^{\infty}(p+(k-1)\mu)$$
$$([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_n C)|a_k|<p|\gamma|\beta \tag{2.5}$$

**Remark 2.5.**

$$Kp(A^m_\lambda;1,1,0,\beta)\subseteq C(1-\beta)$$

**Remark 2.6.**

$$Kp(A^m_\lambda;1,1,0,\beta)\subseteq CS^*(1-\beta)$$

## 3. Coefficient inequality, growth and distortion theorems, closure theorems

Our first theorem gives sufficient condition for normalized analytic functions to be in $Kp(A^m_\lambda;\gamma,\mu,m,\beta)$

**Theorem 3.1.** *Let $f(z)\in N$ satisfy*

$$\sum_{k=2}^{\infty}(p+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)$$
$$+\lambda^{n+k-1}_n C|a_k|<p|\gamma|\beta \tag{3.1}$$

$\gamma\in C|\{0\}, 0<\beta\leq 1 \quad 0<\mu\leq p, m\in N\cup\{0\}$ *Then* $f\in Kp(A^m_\lambda;\gamma,\mu,m,\beta)$.

*Proof.* Assume (3.1) is valid for $f(z) \in N$ and $\gamma(\gamma \in C|\{0\})$, $\beta(0<\beta \leq 1)$, $\mu(0<\mu \leq |p|), m \in N \cup \{0\}$, Using (1.10) we have

$$((p-u)\frac{A_\lambda^m f}{z} + \mu(A_\lambda^m f)' - p)$$
$$= \frac{(p-u)}{z}[Z + \sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda) + \lambda\binom{n+k-1}{n}C)a_k z^k.]$$
$$+ \mu[1 + \sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda) + \lambda\binom{n+k-1}{n}C)k a_k z^{k-1}] - p$$
$$= \sum_{k=2}^{\infty}(p+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda) + \lambda\binom{n+k-1}{n}C)|a_k||z^{k-1}|$$

Therefore

$$|((p-\mu)\frac{A_\lambda^m f}{z} + \mu(A_\lambda^m f)' - p)|$$
$$= \sum_{k=2}^{\infty}(p+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda) + \lambda\binom{n+k-1}{n}C)|a_k| + |z^{k-1}|$$
$$\leq \sum_{k=2}^{\infty}(p+(k-1)\mu)([1+(k-1)\partial]^n (1-\lambda) + \lambda\binom{n+k-1}{n}C)|a_k|$$
$$\leq p|\gamma|\beta$$

Hence

$$|\frac{1}{p\gamma}((p-u)\frac{A_\lambda^m f}{z} + u(A_\lambda^m f)' - p)| < \beta$$

Thus $f(z) \in Kp(A_\lambda^m; \gamma, \mu, m, \beta)$. □

**Corollary 3.2.** $\check{K}p(A_\lambda^m; \gamma, \mu, m, \beta) \subseteq Kp(A_\lambda^m; \gamma, \mu, m, \beta)$

The following example illustrate that converse of the above corollary need not be true.

**Example 3.3.** *If* $f(z) = z + \frac{z^2}{2}$ *and* $|p| \geq 1$. *Taking* $\gamma = 1, \mu = p, m = 0, \beta = 1$, *we will get*

$$|\frac{1}{p\gamma}((p-u)\frac{A_\lambda^m f}{z} + u(A_\lambda^m f)' - p)|$$
$$= |\frac{1}{p_1}((p-p)\frac{A_\lambda^0 f}{z} + p(A_\lambda^0 f)' - p)|$$
$$= |(A_\lambda^0 f)' - 1|$$
$$= |(z + \frac{z^2}{2})' - 1|$$
$$= |1 + 2\frac{z}{2} - 1|$$
$$= |z| < 1.$$

*Therefore,* $f(z) \in Kp(A_\lambda^0; 1, p, 0, 1)$. *But*

$$\sum_{k=2}^{\infty}(p+(k-1)p)([1+(k-1)\partial]^0(1-\lambda) + \lambda\binom{0+k-1}{n}C)|a_k|$$
$$= \sum_{k=2}^{\infty} kp(1-\lambda+\lambda)|a_k|$$
$$= 2p.\frac{1}{2}$$
$$= p \not< p|\gamma|\beta.$$

*Therefore,*

$$f(z) \notin \check{K}p(A_\lambda^0; 1, p, 0, 1)$$

*Hence,*

$$f(z) \in Kp(A_\lambda^0; 1, p, 0, 1)$$

*but*

$$f(z) \notin \check{K}p(A_\lambda^0; 1, p, 0, 1).$$

Our next theorem gives coefficient inequalities for $f(z)$ belonging to class $\check{K}p(A_\lambda^m; \gamma, \mu, m, \beta)$.

**Theorem 3.4.** *(***Coefficient inequality***)*
*If* $f(z) \in \check{K}p(A_\lambda^m; \gamma, \mu, m, \beta)$ *then*

$$|a_k| \leq \frac{p|\gamma|\beta}{(p+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda) + \lambda_n^{n+k-1}C)} \quad k \geq 2$$

*Proof.* Given that $f(z) \in \check{K}p(A_\lambda^m; \gamma, \mu, m, \beta)$
Therefore

$$\sum_{k=2}^{\infty}(p+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda) + \lambda_n^{n+k-1}C)|a_k| \leq p|\gamma|\beta$$
$$(p+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda) + \lambda_n^{n+k-1}C)|a_k| \leq p|\gamma|\beta$$

$$|a_k| \leq \frac{p|\gamma|\beta}{(p+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda) + \lambda_n^{n+k-1}C)}$$

□

Now we prove the following theorem

**Theorem 3.5.** *(***Growth theorem***)*
*Let function* $f(z)$ *defined by* (1.1) *be in class* $\check{K}p(A_\lambda^m; \gamma, \mu, m, \beta)$, *then*

$$|z| - \frac{p|\gamma|\beta}{[p+\mu]((1+\partial)^n(1-\lambda) + \lambda_n^{n+1}C)}|z|^2$$
$$\leq |f(z)|$$
$$\leq |z| + \frac{p|\gamma|\beta}{[p+\mu]((1+\partial)^n(1-\lambda) + \lambda_n^{n+1}C)}|z|^2 \quad (3.2)$$

*Equality is attained for function $f(z)$ given by*

$$f(z)=z+\frac{p|\gamma|\beta}{[p+\mu]((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}z^2$$

*Proof.*

$$[p+\mu]((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)\sum_{k=2}^{\infty}|a_k|$$
$$\leq\sum_{k=2}^{\infty}(p+(k-1)\mu)([1+(k-1)\partial]^n$$
$$(1-\lambda)+\lambda_n^{n+k-1}C)|a_k|$$
$$\leq p|\gamma|\beta$$

Therefore

$$\sum_{k=2}^{\infty}|a_k|\leq\frac{p|\gamma|\beta}{[p+\mu]((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}\tag{3.3}$$

Also $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$ and using (3.3)

$$|f(z)|\leq|z|+\sum_{k=2}^{\infty}|a_k||z^k|$$
$$\leq|z|+|z|^2\sum_{k=2}^{\infty}|a_k|$$
$$\leq|z|+|z|^2\frac{p|\gamma|\beta}{[p+\mu]((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}.\tag{3.4}$$

Similarly

$$|f(z)|\geq|z|+\sum_{k=2}^{\infty}|a_k||z^k|$$
$$\geq|z|-|z|^2\sum_{k=2}^{\infty}|a_k|$$
$$\geq|z|-|z|^2\frac{p|\gamma|\beta}{[p+\mu]((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}\tag{3.5}$$

Using (3.4) and (3.5)

$$|z|-\frac{p|\gamma|\beta}{[p+\mu]((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}|z|^2$$
$$\leq|f(z)|$$
$$\leq|z|+\frac{p|\gamma|\beta}{[p+\mu]((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}|z|^2.$$

Hence

$$|z|-\frac{p|\gamma|\beta}{[p+\mu]((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}|z|^2$$
$$\leq|f(z)|$$
$$\leq|z|+\frac{p|\gamma|\beta}{[p+\mu]((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}|z|^2$$

Equality is attained for function $f(z)$ given by

$$f(z)=z+\frac{p|\gamma|\beta}{[p+\mu]((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}z^2$$

□

**Theorem 3.6.** *(***Distortion theorem)**
*Let function $f(z)$ defined by (1.1) be in class $\check{K}p(A_\lambda^m;\gamma,\mu,m,\beta)$, then*

$$1-\frac{2p|\gamma|\beta}{u((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}|z|$$
$$\leq|f'(z)|$$
$$\leq1+\frac{2p|\gamma|\beta}{u((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}|z|$$

*Equality attained for the function $f(z)$ given by*

$$f(z)=z+\frac{p|\gamma|\beta}{u((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}z^2$$

*Proof.* Let $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$ and $f'(z)=1+\sum_{k=2}^{\infty}ka_kz^{k-1}$

$$|f'(z)|\leq1+\sum_{k=2}^{\infty}k|a_k|\left|z^{k-1}\right|$$
$$\leq1+|z|\sum_{k=2}^{\infty}k|a_k|\tag{3.6}$$

But
$\sum_{k=2}^{\infty}(p+(k-1)\mu([1+(k-1)\partial]^n(1-\lambda)+\lambda_n^{n+k-1}C)$
$|a_k|\leq p|\gamma|\beta$
Also

$$2p+(k-2)\mu\geq0$$
$$2p+k\mu-2\mu\geq0$$
$$2p+2k\mu-2\mu\geq k\mu$$
$$\frac{k\mu}{2}\leq p+(k-1)\mu$$

Similarly

$$(p+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)_n^{n=k-1}C)$$
$$\geq\frac{k\mu}{2}((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)$$
$$\sum_{k=2}^{\infty}\frac{k\mu}{2}((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)|a_k|$$
$$\leq\sum_{k=2}^{\infty}(p+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)_n^{n=k-1}C)|a_k|$$
$$\leq p|\gamma|\beta$$
$$\sum_{k=2}^{\infty}k|a_k|\leq\frac{2p|\gamma|\beta}{u(1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C}$$

from (3.6)

$$\left|f'(z)\right| \leq 1+\frac{2p|\gamma|\beta}{u(1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C}|z| \tag{3.7}$$

Similarly

$$\begin{aligned}\left|f'(z)\right| &\geq 1-\sum_{k=2}^{\infty}k|a_k||z^{k-1}| \\ &\geq 1-|z|\sum_{k=2}^{\infty}k|a_k| \\ &\geq 1-\frac{2p|\gamma|\beta}{u((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}|z| \end{aligned}\tag{3.8}$$

(3.7) and (3.8) implies that

$$\begin{aligned}&1-\frac{2p|\gamma|\beta}{u((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}|z| \\ &\leq \left|f'(z)\right| \\ &\leq 1+\frac{2p|\gamma|\beta}{u((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}|z|\end{aligned}$$

Equality attained for the function $f(z)$ given by

$$f(z)=z+\frac{p|\gamma|\beta}{u((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}z^2.$$

□

## 4. Closure Theorem

In this theorem we prove that finite convex combination of the functions in the class $\check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta)$ is again belongs to $\check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta$.

**Theorem 4.1.** *Let*

$$f_j(z)=z+\sum_{k=2}^{\infty}a_{k,j}z^k,$$

$$f_j(z)\in\check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta),$$

*then for* $g(z)=\sum_{k=1}^{l}c_jf_j(z)$.

$$g(z)\in\check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta),$$

*where* $\sum_{k=1}^{l}c_j=1$.

*Proof.* Let

$$f_j(z)=z+\sum_{k=2}^{\infty}a_{k,j}z^k$$

with $f_j(z)\in\check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta)$.

$$\sum_{k=2}^{\infty}[p+\mu(k-1)]([1+(k-1)\partial]^n(1-\lambda)_n^{n+k-1}C)|a_{k,j}| \leq p|\gamma|\beta$$

$$\begin{aligned}g(z)&=\sum_{k=1}^{l}c_jf_j(z)\\ &=\sum_{k=1}^{l}c_j(z+\sum_{k=2}^{\infty}a_{k,j}z^k)\\ &=z+\sum_{k=1}^{l}c_j\sum_{k=2}^{\infty}a_{k,j}z^k\\ &=z+\sum_{k=2}^{\infty}z^k\sum_{k=1}^{l}c_ja_{k,j}\\ &=z+\sum_{k=2}^{\infty}e_kz^k \quad \textit{where}\;\; e_k=\sum_{j=1}^{l}c_ja_{k,j}\end{aligned}$$

Claim:$g(z)\in\check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta)$

$$\begin{aligned}&\sum_{k=2}^{\infty}[p+\mu(k-1)]([1+(k-1)\partial]^n(1-\lambda)_n^{n+k-1}C)|e_k|\\ &=\sum_{k=2}^{\infty}[p+\mu(k-1)]([1+(k-1)\partial]^n\\ &\quad(1-\lambda)_n^{n+k-1}C)\left|\sum_{j=1}^{l}c_ja_{k,j}\right|\\ &\leq\sum_{j=1}^{l}(c_j\sum_{k=2}^{\infty}[p+\mu(k-1)]([1+(k-1)\partial]^n\\ &\quad(1-\lambda)_n^{n+k-1}C)|a_{k,j}|)\\ &\leq\sum_{j=1}^{l}c_jp|\gamma|\beta\\ &\leq p|\gamma|\beta.\end{aligned}$$

Therefore

$$g(z)\in\check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta).$$

□

## 5. Extreme Point Theorem

In this section we will prove the extreme point theorem. We also find the extreme points for the subclass $\check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta)$.

**Remark 5.1.** *For* $\gamma\in C|0, 0<\beta\leq 1, 0\leq\mu\leq p$, $m\in N\cup 0$ *the following functions are in class* $\check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta)$

$$f_1(z)=z+\frac{p\beta|\gamma|}{(p+\mu)((1+\partial)^n(1-\lambda)+\lambda_n^{n+1}C)}z^2$$

$$f_2(z)=z+\frac{p\beta|\gamma|}{(p+2\mu)((1+2\partial)^n(1-\lambda)+\lambda_n^{n+2}C)}z^3$$

$$\begin{aligned}f_3(z)=z&+\frac{z^2}{(p+\mu)((1+\partial)^n(1-\lambda)+\lambda_n^{n+2}C)}\\ &+\frac{p|\gamma|\beta-1}{(p+2\mu)((1+2\partial)^n(1-\lambda)+\lambda_n^{n+2}C)}z^3\end{aligned}$$

$$(z\in U)$$

**Theorem 5.2.** *Let $f_1 z = z$ and $k \geq 2$*

$$f_k(z) = z + \frac{p|\gamma|\beta}{(p+(k-1)\mu)((1+(k-1)\partial)^n(1-\lambda)+\lambda_n^{n+2}C)} z^k \tag{5.1}$$

*Then $f(z) \in \check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$$

*Where $\lambda_k \geq 0$ and*

$$\sum_{k=1}^{\infty} \lambda_k = 1.$$

*Proof.* Suppose that

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} \lambda_k f_k(z) \\ &= \lambda_1 f_1(z) + \sum_{k=2}^{\infty} \lambda_k (z + \\ &\frac{p|\gamma|\beta}{(p+(k-1)\mu)((1+(k-1)\partial)^n(1-\lambda)+\lambda_n^{n+2}C)} z^k) \\ &= (1-\sum_{k=2}^{\infty}\lambda_k)(z+ \\ &\frac{p|\gamma|\beta}{(p+(k-1)\mu)((1+(k-1)\partial)^n(1-\lambda)+\lambda_n^{n+2}C)} z^k) \\ &= z + \sum_{k=2}^{\infty} \lambda_k z^k \\ &\frac{p|\gamma|\beta}{(p+(k-1)\mu)((1+(k-1)\partial)^n(1-\lambda)+\lambda_n^{n+2}C)} \\ &= z + \sum_{k=2}^{\infty} a_k z^k \end{aligned}$$

where

$$a_k = \frac{p|\gamma|\beta}{(p+(k-1)\mu)((1+(k-1)\partial)^n(1-\lambda)+\lambda_n^{n+2}C)} \lambda_k$$

Claim: $f(z) \in \check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta)$

$$\begin{aligned} &\sum_{k=2}^{\infty} [p+(k-1)\mu]([1+(k-1)\partial]^n(1-\lambda)+\lambda_n^{n+k-1}C)|a_k| \\ &= \sum_{k=2}^{\infty} [p+(k-1)\mu]([1+(k-1)\partial]^n(1-\lambda)+\lambda_n^{n+k-1}C) \\ &\frac{p|\gamma|\beta}{(p+(k-1)\mu)((1+(k-1)\partial)^n(1-\lambda)+\lambda_n^{n+2}C)} \lambda_k \\ &= p|\gamma|\beta \sum_{k=2}^{\infty} \lambda_k \\ &= p|\gamma|\beta(1-\lambda_1) \\ &\leq p|\gamma|\beta \end{aligned}$$

From theorem (2.1). If $f(z) \in \check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta)$.
Conversely suppose that $f(z) \in \check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta)$
Setting

$$\lambda_k = \frac{(p+(k-1)\mu)((1+(k-1)\partial)^n(1-\lambda)+\lambda_n^{n+2}C)}{p|\gamma|\beta} a_k$$

and

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k.$$

$$\begin{aligned} &\sum_{k=1}^{\infty} \lambda_k f_k(z) = \lambda_1 f_1(z) + \sum_{k=2}^{\infty} \lambda_k f_k(z) \\ &= (1-\sum_{k=2}^{\infty}\lambda_k) z + \sum_{k=2}^{\infty} \lambda_k (z \\ &+ \frac{p|\gamma|\beta}{(p+(k-1)\mu)((1+(k-1)\partial)^n(1-\lambda)+\lambda_n^{n+2}C)} z^k) \\ &= z + \sum_{k=2}^{\infty} \frac{p|\gamma|\beta}{(p+(k-1)\mu)((1+(k-1)\partial)^n(1-\lambda)+\lambda_n^{n+2}C)} z^k) \\ &= z + \sum_{k=2}^{\infty} \frac{(p+(k-1)\mu)((1+(k-1)\partial)^n(1-\lambda)+\lambda_n^{n+2}C) a_k}{p|\gamma|\beta} \\ &\frac{p|\gamma|\beta}{(p+(k-1)\mu)((1+(k-1)\partial)^n(1-\lambda)+\lambda_n^{n+2}C)} z_k \\ &= z + \sum_{k=2}^{\infty} a_k z^k \\ &= f(z). \end{aligned}$$

Hence

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z).$$

□

**Corollary 5.3.** *The extreme points of the $\check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta)$ are the functions $f_1(z) = z$ and*

$$f_k(z) = z + \frac{p|\gamma|\beta}{(p+(k-1)\mu)((1+(k-1)\partial)^n(1-\lambda)+\lambda_n^{n+k-1}C)} z_k, \quad k = 2,3,4,\ldots$$

## 6. Integral Mean Inequality

**Theorem 6.1.** *$f(z) \in \check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta)$ and suppose that*

$$\sum_{k=2}^{\infty} ([1+(k-1)\partial]^n(1-\lambda)+\lambda_n^{n+k-1}C)|a_k| \leq \frac{p|\gamma|\beta}{(p+(j-1)\mu)} \tag{6.1}$$

*Also let the function*

$$f_j(z) = z + \frac{p|\gamma|\beta}{(p+(j-1)\mu)([1+(j-1)\partial]^n(1-\lambda)+\lambda_n^{n+k-1}C)} z^j \quad (j \geq 2)$$

*Consider the function $w(z)$ as given bellow*

$$w(z)^{j-1} = \frac{(p+(j-1)\mu)}{p|\gamma|\beta} \sum_{k=2}^{\infty} [(1-\lambda) [1+(k-1)\partial]^n + \lambda_n^{n+k-1} C] a_k z^{k-1}.$$

*Then for $z = re^{i\theta}$ with $0 < r < 1$.*

$$\int_0^{2\pi} \left|A_\lambda^n f(z)\right|^t d\theta \le \int_0^{2\pi} \left|A_\lambda^n f_j(z)\right|^t d\theta$$
$$(0 \le \lambda \le 1, t > 0)$$

*Where $A_\lambda^n$ is differential operator defined in (1.7).*

*Proof.* We have from definition (1.7)

$$A_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [(1-\lambda)[1+(k-1)\partial]^n + \lambda_n^{n+k-1} C] a_k z^k$$
$$D_\lambda^n f_j(z) = z + \frac{p|\gamma|\beta}{(p+(j-1)\mu)} z^j$$

For $z = re^{i\theta}$ with $0 < r < 1$ we have to show that

$$\int_0^{2\pi} \left| 1 + \sum_{\infty}^{k=2} [(1-\lambda)[1+(k-1)\partial]^n + \lambda_n^{n+k-1} C] a_k z^{k-1} \right|^t d\theta$$
$$\le \int_0^{2\pi} \left| 1 + \frac{p|\gamma|\beta}{(p+(j-1)\mu)} z^{j-1} \right|^t d\theta$$
$$(t > 0)$$

By applying Littlewoods sunordination theorem, it would sufficient to show that

$$1 + \sum_{\infty}^{k=2} [(1-\lambda)[1+(k-1)\partial]^n + \lambda_n^{n+k-1} C] a_k z^{k-1} \prec 1 + \frac{p|\gamma|\beta}{(1+(j-1)\mu)} z^{j-1}$$

That is
$$t(z) \prec h(z)$$

Where

$$t(z) = 1 + \sum_{k=2}^{\infty} [(1-\lambda)[1+(k-1)\partial]^n + \lambda_n^{n+k-1} C] a_k z^{k-1}$$

$$h(z) = 1 + \frac{p|\gamma|\beta}{(p+(j-1)\mu)} z^{j-1}$$

That is we want to show that $t(z) = h(w(z)), w(0) = 0$ and $|w(z)| \le 1$

$$\begin{aligned} &h(w(z)) \\ &= 1 + \frac{p|\gamma|\beta}{(p+(j-1)\mu)} w(z)^{j-1} \\ &= 1 + \frac{p|\gamma|\beta}{(p+(j-1)\mu)} \frac{(p+(j-1)\mu)}{p|\gamma|\beta} \\ &\quad \sum_{k=2}^{\infty} [(1-\lambda)[1+(k-1)\partial]^n + \lambda_n^{n+k-1} C] a_k z^{k-1} \\ &= 1 + \sum_{k=2}^{\infty} [(1-\lambda)[1+(k-1)\partial]^n + \lambda_n^{n+k-1} C] a_k z^{k-1} \\ &= t(z) \end{aligned}$$

Therefore $(w(z)) = t(z)$ and $w(0) = 0$
Moreover, We prove that analytic function $|w(z)| < 1, z \in U$.

$$\begin{aligned} \left|w(z)^{j-1}\right| &= \left| \frac{(p+(j-1)\mu)}{p|\gamma|\beta} \sum_{k=2}^{\infty} [(1-\lambda)[1+(k-1)\partial]^n + \lambda_n^{n+k-1} C] a_k z^{k-1} \right| \\ &\le \frac{(p+(j-1)\mu)}{p|\gamma|\beta} \sum_{k=2}^{\infty} [(1-\lambda)[1+(k-1)\partial]^n + \lambda_n^{n+k-1} C] \, |a_k| \left|z^{k-1}\right| \\ &\le |z| \frac{(p+(j-1)\mu)}{p|\gamma|\beta} \sum_{k=2}^{\infty} [(1-\lambda)[1+(k-1)\partial]^n + \lambda_n^{n+k-1} C] \, |a_k| \\ &\le |z| < 1 \text{ by hypothesis (6.1)} \end{aligned}$$

Hence proved. □

## 7. Convolution Theorems

**Definition 7.1.** *If*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

*then hadmad product (Convolution) is defined as given bellow*

$$f * g = z + \sum_{k=2}^{\infty} (a_k b_k) z^k \tag{7.1}$$

*We now turn to convolution theorem, which gives that the class $\check{K}_p(A_\lambda^m; \gamma, \mu, m, \beta)$ is closed under convolution.*

**Theorem 7.2.** *Let $f, g \in \check{K}_p(A_m^\lambda; \gamma, \mu, m, \beta)$*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

*with $a_k \ge 0, b_k \ge 0$ and $(a_k b_k)^{\frac{1}{2}} < 1$. Then*

$$f * g \in \check{K}_p(A_\lambda^m; \gamma, \mu, m, \beta).$$

*Proof.* We have $f \in \check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta)$ if and only if

$$\sum_{k=2}^{\infty}[p+\mu(k-1)]([1+(k-1)\partial]^n(1-\lambda) + \lambda_n^{n+k-1}C)\,|a_k| \le p\,|\gamma|\,\beta$$

$g \in \check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta)$ if and only if

$$\sum_{k=2}^{\infty}[p+\mu(k-1)]([1+(k-1)\partial]^n(1-\lambda) + \lambda_n^{n+k-1}C)\,|b_k| \le p\,|\gamma|\,\beta$$

By Cauchy Schwarz inequality

$$\sum_{k=2}^{\infty}(t_k\,|a_k|\,t_k\,|b_k|)^{\frac{1}{2}} \quad \le (\sum_{k=2}^{\infty} t_k\,|a_k|)^{\frac{1}{2}}(\sum_{k=2}^{\infty} t_k\,|b_k|)^{\frac{1}{2}}$$

Where

$$t_k = (p+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda)+\lambda_n^{n+k-1}C)$$

$$\sum_{k=2}^{\infty}(p+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda) + \lambda_n^{n+k-1}C)\,|a_k b_k|^{\frac{1}{2}} \le p\,|\gamma|\,\beta \qquad (7.2)$$

By assumption

$$(a_k b_k)^{\frac{1}{2}} < 1$$

Then

$$a_k b_k < (a_k b_k)^{\frac{1}{2}} \qquad (7.3)$$

Thus from (7.2) and (7.3)

$$\sum_{k=2}^{\infty}(p+(k-1)\mu)([1+(k-1)\partial]^n(1-\lambda) + \lambda_n^{n+k-1}C)\,|a_k b_k| \le p\,|\gamma|\,\beta$$

Hence

$$f * g \in \check{K}_p(A_\lambda^m;\gamma,\mu,m,\beta).$$

□

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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