An application of a new coupled fixed point theorem on nonlinear integro-differential equations

Antony Santhi[1]*

Abstract
In this paper, we define the concept of the diameter for an orbit at a point, with respect to a context, which is entirely different from the one available in the literature. As a sequel, we prove the existence of a coupled fixed point for a mapping defined on a b-metric space, using a Meir-Keeler type of contractive condition. Finally, we give an application to prove the significance of the theory developed.

Keywords
Diameter, Coupled fixed point, Complete b-metric space, Integro-differential equation.

AMS Subject Classification
54H25, 47H10.

[1]*School of Mathematics, Madurai Kamaraj University, Madurai- 625 021,Tamil Nadu, India.*
Corresponding author: [1] santhi25.antony@gmail.com;

Contents

1. Introduction

The concept of fixed point theory emerged as an inevitable tool, to study the existence of solutions for various types of equations, as never before, after the development of Banach contraction principle. In 1930, Caccioppoli[6] gave a natural extension of the Banach contraction principle; Bryant[4] generalized the results of Caccioppoli. In 1969, Kannan[15] gave a characterization of a complete metric space involving a contractive condition. Caristi, Ciric, Matkowski and Rakotch are some others who studied and extended the theory further (see[7, 8, 17, 18]).

In relative to our work, a generalization of Banach contraction principle was given by Meir and Keeler[16]. In 1968, Browder[3] introduced a non-decreasing right continuous control function to prove a fixed point

theorem; Boyd and Wong[5] extended the result of Browder using a right upper semi continuous control function. Following them, the theory on contractive condition involving control function was studied by many others (see [11, 14, 19]). In 1980, Hegedus and Szilagyi[13] defined the concept diameter of an orbit and proved a fixed point theorem. Suzuki[22] gave a generalization of the fixed point theorem proved by Hegedus and Szilagyi.

The concept of a coupled fixed point and mixed monotone property of the mapping $F : X^2 \to X$, where X is a partially ordered metric space defined by Bhaskar et al.[12] in 2006. In 2012, Sintunavart et al.[20] proved the existence of a coupled fixed point for a nonlinear contraction mapping $F : X^2 \to X$ in a complete metric spaces excluding the premise that F has the mixed monotone property. In 2016, Su et al.[21] proved the existence of a multivariate fixed point for N-variable mappings. The concept of b-metric space was introduced by Bakhtin[2], to study the pattern matching problems, in the field of computer Sciences. Babu, Czerwik, Demmaa are some other who studied and extended the theory further see ([1, 9, 10]).

In our work, we first define the concept of the diameter for an orbit at a point, with respect to a context, that is totally heterogeneous to the one available in the literature. In follow, we prove a coupled fixed point theorem through a Meir-Keeler type of contractive

condition, in the context of b-metric space; conclusively we give an application to justify the significance of the theory developed.

2. Preliminaries

As usual, let $\mathbb{R}$ and $\mathbb{N}$ denote the set of real numbers and natural numbers respectively.

Definition 2.1. *[12] Let $F : X^2 \to X$ be a mapping. A point $(x,y) \in X^2$ is said to be a coupled fixed point if $F(x,y) = x$ and $F(y,x) = y$.*

Definition 2.2. *[21] Let $F : X^N \to X$ be an N-variable mapping. A point $x \in X$ is said to be a multivariate fixed point of order N if $F(x,x,\ldots,x) = x$.*

Note that, if $(x,y) \in X^2$ is a coupled fixed point of F, then $(y,x) \in X^2$ is also a coupled fixed point of F ; Similarly, if $(x,y) \in X^2$ is the unique coupled fixed point of F, then $x \in X$ is a unique multivariate fixed point of order 2.

Definition 2.3. *[22] Let (X,d) be a metric space and T be a self map on X. Define $D_T(x)$ and $D_T(x,y)$ by*

$$D_T(x) = \sup\{d(u,v) : u,v \in \{x,Tx,T^2x,\ldots\}\}$$
$$D_T(x,y) = \sup\{d(u,v) : u,v \in \{x,Tx,T^2(x),\ldots,$$
$$y,Ty,T^2y,\ldots\}\}$$

for any $x,y \in X$, that is $D_T(x)$ is the diameter of the orbit $\{x,Tx,T^2x,\ldots\}$ of x.

Definition 2.4. *[9] Let X be a nonempty set and $b \geq 1$ be a given real number. A function $d : X^2 \to [0,\infty)$ is said to be b-metric if for all $x,y,z \in X$,*

(B1) $d(x,y) = 0 \Leftrightarrow x = y$

(B2) $d(x,y) = d(y,x)$

(B3) $d(x,z) \leq b[d(x,y) + d(y,z)]$.

The pair (X,d) is called a b-metric space.

3. A coupled fixed point theorem via diameter of an orbit

We start this section, with the definition of diameter of an orbit.

Definition 3.1. *Let (X,d) be a b-metric space with $b \geq 1$ and $F : X^2 \to X$ be a mapping that satisfies $F^0(x,y) = x$ and $F^n(x,y) = F^{n-1}(F(x,y),F(y,x))$ for all $n \geq 1$. Let $\mathscr{O}_F : X^{2N} \to \mathscr{P}(X)$ and $\mathscr{D}_F : X^{2N} \to \mathbb{R}$ be the mappings defined by*

$$\mathscr{O}_F(\mathbf{x}) = \bigcup_{i=1}^{N} A_i,$$

where $A_i = \{x_{2i-1}, F(x_{2i-1},x_{2i}), F^2(x_{2i-1},x_{2i}),\ldots\}$.

$$\mathscr{D}_F(\mathbf{x}) = b\sup\{d(u,v) : u,v \in \mathscr{O}_F(x)\},$$

for all $\mathbf{x} = (x_1,x_2,\ldots,x_{2N}) \in X^{2N}$. Then $\mathscr{O}_F(\mathbf{x})$ is called an orbit at $\mathbf{x}$ and $\mathscr{D}_F(\mathbf{x})$ is called diameter of an orbit at $\mathbf{x}$.

Example 3.2. *Let $X = [0,1]$. Clearly (X,d) is a b-metric space with $d(x,y) = |x-y|^2$. Here $b = 2$. Let $F : X^2 \to X$ be a mapping defined by $F(x,y) = \frac{x}{3} + \frac{y}{2}$. Then we have*

(a.) $\mathscr{O}_F(0,1) = \left\{0, \frac{1}{2}, \frac{1}{3},\ldots\right\}$ and $\mathscr{O}_F(1,0) = \left\{1, \frac{1}{3}, \frac{13}{36},\ldots\right\}$

(b.) $\mathscr{O}_F(1,0,0,1) = \left\{0, \frac{1}{2},\ldots 1, \frac{1}{3},\ldots\right\}$ and
   $\mathscr{O}_F\left(0, \frac{1}{2}, \frac{1}{2}, 0\right) = \left\{0, \frac{1}{4}, \frac{1}{6},\ldots, \frac{1}{2}, \frac{1}{6},\ldots\right\}$

(c.) $\mathscr{D}_F(0,1) = 2\sup\{d(u,v) : u,v \in \mathscr{O}_F(0,1)\} = \frac{1}{2}$

(d.) $\mathscr{D}_F(1,0) = 2\sup\{d(u,v) : u,v \in \mathscr{O}_F(1,0)\} = 2$

(e.) $\mathscr{D}_F(1,0,0,1) = 2\sup\{d(u,v) : u,v \in \mathscr{O}_F(1,0,0,1)\} = 2$

(f.) $\mathscr{D}_F\left(0, \frac{1}{2}, \frac{1}{2}, 0\right) = 2\sup\left\{d(u,v) : u,v \in \mathscr{O}_F\left(0, \frac{1}{2}, \frac{1}{2}, 0\right)\right\}$
   $= 1$

Remarks

(i.) From the above example, it is easy to see that $\mathscr{D}_F(x,y)$ and $\mathscr{D}_F(y,x)$ need not be equal for all $x,y \in X$.

(ii.) $\mathscr{D}_F(x,y) \leq \mathscr{D}_F(x,y,u,v)$ and $\mathscr{D}_F(u,v) \leq \mathscr{D}_F(x,y,u,v)$ for all $x,y,u,v \in X$.

(iii.) If $\mathscr{D}_F(x,y,u,v) = 0$, then $\mathscr{D}_F(x,y) = \mathscr{D}_F(u,v) = 0$.

(iv.) If $\mathscr{D}_F(x,y) = \mathscr{D}_F(y,x) = 0$, then (x,y) is a coupled fixed point of F.

(v.) If $\mathscr{D}_F(x,y,y,x) = 0$, then x is a multivariate fixed point of order 2.

(vi.) If (x,y) is a coupled fixed point, $\mathscr{D}_F(x,y,y,x) = bd(x,y)$.

(vii.) If $\mathscr{D}_F(x,y,y,x) = bd(x,y)$, then (x,y) need not be a coupled fixed point. For, in the above example we have $\mathscr{D}_F\left(0, \frac{1}{2}, \frac{1}{2}, 0\right) = 1 = 2d(0, \frac{1}{2})$. But $\left(0, \frac{1}{2}\right)$ is not a coupled fixed point.

Theorem 3.3. *Let (X,d) be a complete b-metric space and let $F : X^2 \to X$ be a mapping. Suppose $\mathscr{D}_F(x,y) < \infty$ for all $x,y \in X$ and if there exists a function $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:*

(i.) $\phi(t) < t$ holds for all $t \in (0,\infty)$.

(ii.) For any $\epsilon > 0$, there exists $\delta > 0$ such that $\epsilon < t < \epsilon + \delta \Rightarrow \phi(t) \leq \epsilon$.

(iii.) For any $x, y, u, v \in X$,

$$bd\left(F(x,y), F(u,v)\right) \leq \phi\left(\mathcal{D}_F(x,y,u,v)\right)$$

holds.

Then F *has a unique coupled fixed point.*

Proof. Step 1. If $(x,y) = (u,v)$ or if

$$\lim_{n\to\infty} \mathcal{D}_F(F^n(x,y), F^n(y,x)) = \lim_{n\to\infty} \mathcal{D}_F(F^n(u,v), F^n(v,u)) = 0,$$

for some $(x,y), (u,v) \in X^2$. We prove the following:

$$\lim_{n\to\infty} \mathcal{D}_F(F^n(x,y), F^n(y,x), F^n(u,v), F^n(v,u)) = 0 \quad (3.1)$$

Let $n \in N$, then we have
$$\{F^{n+1}(x,y), F^{n+2}(x,y), \ldots, F^{n+1}(u,v), F^{n+2}(u,v), \ldots\}$$
$$\subseteq \{F^n(x,y), F^{n+1}(x,y), \ldots, F^n(u,v), F^{n+1}(u,v), \ldots\}$$

and
$$\mathcal{D}_F(F^{n+1}(x,y), F^{n+1}(y,x), F^{n+1}(u,v), F^{n+1}(v,u))$$
$$\leq \mathcal{D}_F(F^n(x,y), F^n(y,x), F^n(u,v), F^n(v,u)).$$

Thus, $\{\mathcal{D}_F(F^n(x,y), F^n(y,x), F^n(u,v), F^n(v,u))\}$ is monotonic decreasing sequence that is bounded below and hence converges to greatest lower bound. Now we claim that, if $\lim_{n\to\infty} \mathcal{D}_F(F^n(x,y), F^n(y,x), F^n(u,v), F^n(v,u)) = \epsilon$, then $\epsilon = 0$. Suppose $\epsilon > 0$, then by the construction of $\mathcal{D}_F$, we have the following two cases.

i. $\epsilon < \mathcal{D}_F(F^n(x,y), F^n(y,x), F^n(u,v), F^n(v,u))$ for all n.

ii. $\mathcal{D}_F(F^n(x,y), F^n(y,x), F^n(u,v), F^n(v,u)) = \epsilon$
for all $n \geq N$.

First let us assume that

$$\epsilon < \mathcal{D}_F(F^n(x,y), F^n(y,x), F^n(u,v), F^n(v,u))$$

for all n. Let $\delta \in (0, \infty)$ be such that $\epsilon < t < \epsilon + \delta$, then $\phi(t) \leq \epsilon$. Since ϵ is the greatest lower bound of the sequence $\{\mathcal{D}_F(F^n(x,y), F^n(y,x), F^n(u,v), F^n(v,u))\}$, there exists an $r \in \mathbb{N}$ such that

$$\mathcal{D}_F(F^r(x,y), F^r(y,x), F^r(u,v), F^r(v,u)) < \epsilon + \delta.$$

Let $m, n \geq r$, then

$$\begin{aligned}
\epsilon &< \mathcal{D}_F(F^{\max\{m,n\}}(x,y), F^{\max\{m,n\}}(y,x), \\
&\qquad F^{\max\{m,n\}}(u,v), F^{\max\{m,n\}}(v,u)) \\
&\leq \mathcal{D}_F(F^m(x,y), F^m(y,x), F^n(u,v), F^n(v,u)) \\
&\leq \mathcal{D}_F(F^{\min\{m,n\}}(x,y), F^{\min\{m,n\}}(y,x), \\
&\qquad F^{\min\{m,n\}}(u,v), F^{\min\{m,n\}}(v,u)) \\
&\leq \mathcal{D}_F(F^r(x,y), F^r(y,x), F^r(u,v), F^r(v,u)) \\
&< \epsilon + \delta.
\end{aligned}$$

Thus we have,

$$\begin{aligned}
bd&\left(F^{(m+1)}(x,y), F^{(n+1)}(u,v)\right) \\
&= bd\left(F(F^m(x,y), F^m(y,x)), F(F^n(u,v), F^n(v,u))\right) \\
&\leq \phi\left(\mathcal{D}_F(F^m(x,y), F^m(y,x), F^n(u,v), F^n(v,u))\right) \\
&\leq \epsilon.
\end{aligned}$$

Subsequently we attain the inequality,

$$\epsilon < \mathcal{D}_F(F^{r+1}(x,y), F^{r+1}(y,x), F^{r+1}(u,v), F^{r+1}(v,u)) \leq \epsilon$$

which is a contradiction, in both the cases of our assumption. For if $(x,y) = (u,v)$, then the inequality follows obviously. Suppose

$$\lim_{n\to\infty} \mathcal{D}_F(F^n(x,y), F^n(y,x)) = \lim_{n\to\infty} \mathcal{D}_F(F^n(u,v), F^n(v,u)) = 0,$$

there exists an $N \in \mathbb{N}$ such that

$$\mathcal{D}_F(F^n(x,y), F^n(y,x)) = \mathcal{D}_F(F^n(u,v), F^n(v,u)) < \frac{\epsilon}{2}$$

for all $n \geq N$ and it is easy to see that $r \geq N$. Thus

$$\mathcal{D}_F(F^r(x,y), F^r(y,x)) = \mathcal{D}_F(F^r(u,v), F^r(v,u)) < \frac{\epsilon}{2}.$$

But since m, n are arbitrary, we have

$$\epsilon < \mathcal{D}_F(F^{r+1}(x,y), F^{r+1}(y,x), F^{r+1}(u,v), F^{r+1}(v,u)) \leq \epsilon.$$

Suppose $\mathcal{D}_F(F^n(x,y), F^n(y,x), F^n(u,v), F^n(v,u)) = \epsilon$ for all $n \geq N$ and

$$\lim_{n\to\infty} \mathcal{D}_F(F^n(x,y), F^n(y,x)) = \lim_{n\to\infty} \mathcal{D}_F(F^n(u,v), F^n(v,u)) = 0.$$

Then there exists an $N_1 \in \mathbb{N}$ such that

$$\mathcal{D}_F(F^n(x,y), F^n(y,x)) = \mathcal{D}_F(F^n(u,v), F^n(v,u)) < \phi(\epsilon)$$

for all $n \geq N_1$. Let $r = \max\{N, N_1\}$, then it follows that

$$\mathcal{D}_F(F^r(x,y), F^r(y,x), F^r(u,v), F^r(v,u)) = \epsilon$$

and

$$\mathcal{D}_F(F^r(x,y), F^r(y,x)) = \mathcal{D}_F(F^r(u,v), F^r(v,u)) < \phi(\epsilon).$$

Let $r \leq m$ and $r \leq n$, then

$$\begin{aligned}
\epsilon &< \mathcal{D}_F(F^{\max\{m,n\}}(x,y), F^{\max\{m,n\}}(y,x), \\
&\qquad F^{\max\{m,n\}}(u,v), F^{\max\{m,n\}}(v,u)) \\
&\leq \mathcal{D}_F(F^m(x,y), F^m(y,x), \\
&\qquad F^n(u,v), F^n(v,u)) \\
&\leq \mathcal{D}_F(F^{\min\{m,n\}}(x,y), F^{\min\{m,n\}}(y,x), \\
&\qquad F^{\min\{m,n\}}(u,v), F^{\min\{m,n\}}(v,u)) \\
&\leq \mathcal{D}_F(F^r(x,y), F^r(y,x), F^r(u,v), F^r(v,u)) \\
&= \epsilon.
\end{aligned}$$

Hence,

$$bd\left(\mathbf{F}^{m+1}(x,y),\mathbf{F}^{n+1}(u,v)\right)$$
$$= bd\left(\mathbf{F}(\mathbf{F}^m(x,y),\mathbf{F}^m(y,x)),\mathbf{F}(\mathbf{F}^n(u,v),\mathbf{F}^n(v,u))\right)$$
$$\leq \phi\left(\mathcal{D}_{\mathbf{F}}(\mathbf{F}^m(x,y),\mathbf{F}^m(y,x),\mathbf{F}^n(u,v),\mathbf{F}^n(v,u))\right)$$
$$= \phi(\epsilon).$$

But since m,n are arbitrary, we have

$$\epsilon = \mathcal{D}_{\mathbf{F}}(\mathbf{F}^{r+1}(x,y),\mathbf{F}^{r+1}(y,x),\mathbf{F}^{r+1}(u,v),\mathbf{F}^{r+1}(v,u)) \leq \phi(\epsilon) < \epsilon$$

which leads us to a contradiction. Suppose if we take $(x,y) = (u,v)$ instead of

$$\lim_{n\to\infty}\mathcal{D}_{\mathbf{F}}(\mathbf{F}^n(x,y),\mathbf{F}^n(y,x)) = \lim_{n\to\infty}\mathcal{D}_{\mathbf{F}}(\mathbf{F}^n(u,v),\mathbf{F}^n(v,u)) = 0,$$

then by letting $r = N$, in the above discussion we obtain a similar form of contradiction. Thus we have

$$\lim_{n\to\infty}\mathcal{D}_{\mathbf{F}}(\mathbf{F}^n(x,y),\mathbf{F}^n(y,x),\mathbf{F}^n(u,v),\mathbf{F}^n(v,u)) = 0.$$

Step 2. Now we prove the theorem. Let $x,y \in X$, then by (3.1), it follows that $\lim_{n\to\infty}\mathcal{D}_{\mathbf{F}}(\mathbf{F}^n(x,y),\mathbf{F}^n(y,x)) = 0$. Thus for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\mathcal{D}_{\mathbf{F}}(\mathbf{F}^n(x,y),\mathbf{F}^n(y,x)) < \epsilon$ for all $n \geq N$ and hence

$$bd(\mathbf{F}^r(x,y),\mathbf{F}^m(x,y)) \leq \mathcal{D}_{\mathbf{F}}(\mathbf{F}^n(x,y),\mathbf{F}^n(y,x)) < \epsilon$$

for all $r,m,n \geq N$. Therefore $\{\mathbf{F}^n(x,y)\}$ is a Cauchy sequence in X and hence converges to some point $p \in X$. Similarly, we can prove that $\{\mathbf{F}^n(y,x)\}$ converges to some point $q \in X$. By (3.1), it follows that,

$$\lim_{n\to\infty}\mathcal{D}_{\mathbf{F}}(\mathbf{F}^n(p,q),\mathbf{F}^n(q,p))$$
$$= 0 = \lim_{n\to\infty}\mathcal{D}_{\mathbf{F}}(\mathbf{F}^n(x,y),\mathbf{F}^n(y,x),\mathbf{F}^n(p,q),\mathbf{F}^n(q,p)).$$

Thus for every $\epsilon > 0$, there exists an $N_1 \in \mathbb{N}$ such that

$$\mathcal{D}_{\mathbf{F}}(\mathbf{F}^n(x,y),\mathbf{F}^n(y,x),\mathbf{F}^n(p,q),\mathbf{F}^n(q,p)) < \frac{\epsilon}{2}\ \text{for all}\ n \geq N_1$$

and hence

$$bd(\mathbf{F}^n(x,y),\mathbf{F}^n(p,q))$$
$$< \mathcal{D}_{\mathbf{F}}(\mathbf{F}^n(x,y),\mathbf{F}^n(y,x),\mathbf{F}^n(p,q),\mathbf{F}^n(q,p))$$
$$< \frac{\epsilon}{2}$$

for all $n \geq N_1$. Since $\lim_{n\to\infty}\mathbf{F}^n(x,y) = p$, there exists an $N_2 \in \mathbb{N}$ such that

$$d(\mathbf{F}^n(x,y),p) < \frac{\epsilon}{2b}\ \text{for all}\ n \geq N_2.$$

Thus for any $n \geq \max\{N_1,N_2\}$, we have

$$d(\mathbf{F}^n(p,q),p) \leq b\left(d(\mathbf{F}^n(p,q),\mathbf{F}^n(x,y)) + d(\mathbf{F}^n(x,y),p)\right)$$
$$< b\left(\frac{\epsilon}{2b} + \frac{\epsilon}{2b}\right) = \epsilon$$

and hence $\{\mathbf{F}^n(p,q)\}$ converges to $p \in X$.

$$i.e, \lim_{n\to\infty}d(p,\mathbf{F}^n(p,q)) = 0.$$

Now since

$$\mathcal{D}_{\mathbf{F}}(p,q) = \sup\{\mathcal{D}_{\mathbf{F}}(\mathbf{F}(p,q),\mathbf{F}(q,p)),d(p,\mathbf{F}^n(p,q)) : n \in \mathbb{N}\},$$

it follows that $\mathcal{D}_{\mathbf{F}}(p,q) = \mathcal{D}_{\mathbf{F}}(\mathbf{F}(p,q),\mathbf{F}(q,p))$. We claim that $\mathcal{D}_{\mathbf{F}}(p,q) = 0$. Suppose that $\mathcal{D}_{\mathbf{F}}(p,q) = \epsilon > 0$, then (since $\lim_{n\to\infty}\mathcal{D}_{\mathbf{F}}(\mathbf{F}^n(p,q),\mathbf{F}^n(q,p)) = 0$) there exists an $r \in \mathbb{N}$ such that

$$\epsilon = \mathcal{D}_{\mathbf{F}}(p,q) = \cdots = \mathcal{D}_{\mathbf{F}}(\mathbf{F}^{r-1}(p,q),\mathbf{F}^{r-1}(q,p))$$
$$= \mathcal{D}_{\mathbf{F}}(\mathbf{F}^r(p,q),\mathbf{F}^r(q,p))$$
$$> \mathcal{D}_{\mathbf{F}}(\mathbf{F}^{r+1}(p,q),\mathbf{F}^{r+1}(q,p)).$$

This implies that,

$$\epsilon = \mathcal{D}_{\mathbf{F}}(\mathbf{F}^r(p,q),\mathbf{F}^r(q,p)) = b\sup\{d(\mathbf{F}^r(p,q),\mathbf{F}^n(p,q)) : n > r\}.$$

Thus for any $n > r$, we have

$$bd(\mathbf{F}^r(p,q),\mathbf{F}^n(p,q))$$
$$= bd\left(\mathbf{F}(\mathbf{F}^{r-1}(p,q),\mathbf{F}^{r-1}(q,p)),\mathbf{F}(\mathbf{F}^{n-1}(p,q),\mathbf{F}^{n-1}(q,p))\right)$$
$$\leq \phi\left(\mathcal{D}_{\mathbf{F}}(\mathbf{F}^{r-1}(p,q),\mathbf{F}^{r-1}(q,p),\mathbf{F}^{n-1}(p,q),\mathbf{F}^{n-1}(q,p))\right)$$
$$= \phi\left(\mathcal{D}_{\mathbf{F}}(\mathbf{F}^{r-1}(p,q),\mathbf{F}^{r-1}(q,p))\right)$$
$$= \phi(\epsilon).$$

Since n is arbitrary, we have

$$\epsilon = b\sup\{d(\mathbf{F}^r(p,q),\mathbf{F}^n(p,q)) : n > r\} \leq \phi(\epsilon) < \epsilon,$$

which leads to a contradiction. Hence $\mathcal{D}_{\mathbf{F}}(p,q) = 0$. Similarly, we can prove that $\mathcal{D}_{\mathbf{F}}(q,p) = 0$. Thus it follows that, $\mathbf{F}(p,q) = p$ and $\mathbf{F}(q,p) = q$; i.e, (p,q) is a coupled fixed point of $\mathbf{F}$. To prove the uniqueness, if (u,v) is another coupled fixed point of $\mathbf{F}$, then

$$bd(p,u) = bd\left(\mathbf{F}(p,q),\mathbf{F}(u,v)\right) \leq \phi\left(\mathcal{D}_{\mathbf{F}}(p,q,u,v)\right) < bd(p,u),$$

and

$$bd(q,v) = bd\left(\mathbf{F}(q,p),\mathbf{F}(v,u)\right) \leq \phi\left(\mathcal{D}_{\mathbf{F}}(q,p,v,u)\right) < bd(y,v)$$

which in turn implies that $p = u$ and $q = v$ as desired. $\quad\square$

Example 3.4. *Let $X = [0,1]$. Clearly (X,d) is a complete b-metric space with $d(x,y) = |x - y|^2$. Here $b = 2$. Let $\mathbf{F} : X^2 \to X$ be defined by $\mathbf{F}(x,y) = \frac{x+y}{100}$. Then, clearly $\mathcal{D}_{\mathbf{F}}(x,y) = \left|\frac{99x-y}{100}\right|^2 < \infty$. Let $\phi : [0,\infty) \to [0,\infty)$ be defined by $\phi(t) = \frac{t}{2}$. Let $\epsilon > 0$, then by letting $\delta = \frac{\epsilon}{2}$, we have $\phi(t) = \frac{t}{2} < \epsilon$, for any $t \in (\epsilon,\frac{3\epsilon}{2})$. Thus it follows that,*

$$2d\left(\mathbf{F}(x,y),\mathbf{F}(u,v)\right) = 2\left|\frac{x+y}{100} - \frac{u+v}{100}\right|^2$$
$$= 2\left|\frac{(x-u)+(y-v)}{100}\right|^2$$
$$< |x-u|^2$$
$$= \phi\left(\mathcal{D}_{\mathbf{F}}(x,y,u,v)\right)$$

for all $x, y, u, v \in X$. *Thus by Theorem 3.3,* $0 \in X$ *is a unique coupled fixed point of* F.

4. Application to nonlinear integro-differential equations

Let X be set of all continuous functions from $[\frac{-\pi}{2}, \frac{\pi}{2}]$ to $[\frac{-\pi}{2}, \frac{\pi}{2}]$ and let $d : X^2 \to \mathbb{R}$ be a function defined by $d(x(t), y(t)) = \sup\limits_{t \in [-\pi/2, \pi/2]} |x(t) - y(t)|^2$. Then (X, d) is a complete b-metric space. Here $b = 2$. Let $K : X^2 \to X$ be a mapping that satisfies

$$\left| \int_{-\pi/2}^{\pi/2} (k(x(s), y(s)) - k(u(s), v(s))) \, ds \right| \leq 1$$

and let $\psi \in X$ be such that $\sup\limits_{t \in [-\pi/2, \pi/2]} \left| \int_{-\pi/2}^{t} \psi(t) \, dt \right| \leq 1$. Let

$$x'(t) = \int_{-\pi/2}^{\pi/2} k(x(s), y(s)) \psi(s) \, ds,$$

$$y'(t) = \int_{-\pi/2}^{\pi/2} k(x(s), y(s)) \psi(s) \, ds, \qquad (4.1)$$

be the system of nonlinear integro-differential equations. Let $F : X^2 \to X$ be a mapping defined by

$$F(x(t), y(t)) = \int_{-\pi/2}^{t} \int_{-\pi/2}^{\pi/2} k(x(s), y(s)) \psi(s) \, ds \, dt.$$

Then $\mathcal{D}_F(x(t), y(t)) \leq \pi^2 < \infty$ for all $x(t), y(t) \in X$ and $\mathcal{D}_F(x(t), y(t), u(t), v(t)) \leq \pi^2$ for all $x(t), y(t), u(t), v(t) \in X$. Now consider

$$2d(F(x(t), y(t)), F(u(t), v(t)))$$

$$= 2 \sup_{t \in [-\pi/2, \pi/2]} \left| \int_{-\pi/2}^{t} \int_{-\pi/2}^{\pi/2} \{k(x(s), y(s)) - k(u(s), v(s))\} \, ds \, \psi(t) \, dt \right|^2$$

$$\leq 2 \sup_{t \in [-\pi/2, \pi/2]} \left| \int_{-\pi/2}^{t} \psi(t) \, dt \right|^2 \leq 2$$

Define $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{t}{2}$. Then $\frac{t}{2} < t$ for all $t \in (0, \infty)$. Let $\epsilon > 0$, then by letting $\delta = \frac{\epsilon}{2}$, we have $\phi(t) = \frac{t}{2} < \epsilon$, for any $t \in (\epsilon, \frac{3\epsilon}{2})$. Thus,

$$bd(F(x(t), y(t)), F(u(t), v(t))) \leq \phi(\mathcal{D}_F(x(t), y(t), u(t), v(t)))$$

for all $x(t), y(t), u(t), v(t) \in X$ and hence By Theorem 3.3, the pair of integro-differential equations (4.1) has a unique solution in X.

Example 4.1. *Let* $k(x(s), y(s)) = \frac{x(s)y(s)}{2}$ *and* $\psi(t) = \cos t$, *then clearly*

$$\left| \int_{-\pi/2}^{\pi/2} (k(x(s), y(s)) - k(u(s), v(s))) \, ds \right| \leq 1$$

and

$$\sup_{t \in [-\pi/2, \pi/2]} \left| \int_{-\pi/2}^{t} \psi(t) \, dt \right| \leq 1.$$

Let

$$x'(t) = \int_{-\pi/2}^{\pi/2} k(x(s), y(s)) \psi(s) \, ds$$

$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} x(s) y(s) \cos t \, ds,$$

$$y'(t) = \int_{-\pi/2}^{\pi/2} k(x(s), y(s)) \psi(s) \, ds$$

$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} x(s) y(s) \cos t \, ds \qquad (4.2)$$

Define $F : X^2 \to X$ *by* $F(x(t), y(t)) = \frac{1}{2} \int_{-\pi/2}^{t} \int_{-\pi/2}^{\pi/2} x(s) y(s) \psi(t) \, ds \, dt$.
Then clearly $\mathcal{D}_F(x(t), y(t)) \leq \pi^2 < \infty$ *for all* $x(t), y(t) \in X$ *and* $\mathcal{D}_F((x(t), y(t), u(t), v(t))) \leq \pi^2$ *for all* $x(t), y(t), u(t), v(t) \in X$. *Now consider,*

$$2d(F(x(t), y(t)), F(u(t), v(t)))$$

$$= 2 \sup_{t \in [-\pi/2, \pi/2]} \left| \frac{1}{2} \int_{-\pi/2}^{t} \int_{-\pi/2}^{\pi/2} \{x(s) y(s) - u(s) v(s)\} \, ds \cos t \, dt \right|^2$$

$$= \sup_{t \in [-\pi/2, \pi/2]} \left| \int_{-\pi/2}^{t} \cos t \, dt \right|^2 \leq 1$$

Let $\phi : [0, \infty) \to [0, \infty)$ *be defined by* $\phi(t) = \frac{t}{2}$, *then we have*

$$bd(F(x(t), y(t)), F(u(t), v(t))) \leq \phi(\mathcal{D}_F(x(t), y(t), u(t), v(t)))$$

for all $x(t), y(t), u(t), v(t) \in X$. *Thus by Theorem 3.3, the pair of integro-differential equation (4.2) has a unique solution in X which is equal to* $\frac{2}{\pi} \sin t$.

References

[1] G. V. R. Babu, D. T. Mosissa, Fixed Point in b-metric space via Simulation Functions, *Novi Sad Journal of Mathematics*, (2017), 133-147.

[2] I. A. Bakhtin, The contraction mapping principle in quasimetric spaces, *Funct. Anal. Unianowsk Gos. Ped. Inst.*, 30, (1989), 26-37.

[3] F. F. Browder, On the convergence of successive approximations for nonlinear functional equations, *Nederl. Akad. Wetensch. Proc. Ser. A 71=Indag. Math.*, 30, (1968), 27-35.

[4] V. W. Bryant, A remark on a fixed point theorem for iterated mappings, *Am. Math. Mon.*, 75, (1968), 399-400.

[5] D. W. Boyd and J. S. W. Wong, On nonlinear contractions, *Proc. Amer. Math. Soc.*, 20, (1969), 458-464.

[6] R. Caccioppoli, Un teorema generale sullesistenza de elemente uniti in una transformazione funzionale, *Rend. Acad. Naz. Linzei.*, 11, (1930), 31-49.

[7] J. Caristi, Fixed point theorem for mapping satisfying inwardness conditions, *Trans. Am. Math. Soc.*, 215, (1976), 241-251.

[8] L. B. Ciric, A generalization of Banach's contraction principle, *Proc. Am. Math. Soc.* 45, (1974), 267-273.

[9] S. Czerwik, Contraction Mappings in b-metric Spaces, *Acta Mathematica et Informatica Ostraviensis*, 1, (1993), 5-11.

[10] M. Demmaa, R. Saadatib, P. Vetroa, Fixed Point Results on *b*-Metric Space via Picard Sequences and b-Simulation Functions, *Iranian Journal of Mathematical Sciences and Informatics*, 11, (2016), 123-136.

[11] M. Geraghty, On contractive mappings, *Proc. Amer. Math. Soc.*, 40, (1973), 604-608.

[12] T. Gnana Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Analysis*, 65, (2006), 1379-1393.

[13] M. Hegedus and T. Szilagyi, Equivalent conditions and a new fixed point theorem in the theory of contractive type mappings, *Math.Jpn.*, 25, (1980), 147-157.

[14] J. Jachymski and I. Jozwik, Nonlinear contractive conditions: a comparison and related problems Fixed Point Theory and Its Applications, *Polish Acad. Sci.*, 77, (2007), 123-146.

[15] R. Kannan, Some results on fixed points U II, *Am. Math. Mon.*, 76, (1969), 405-408.

[16] A. Meir and E. Keeler, A theorem on contraction mapping, *J.Math.Anal.Appl.*, 213, (1969), 348-354.

[17] J. Matkowski, Integrable solutions of functional equations, *Diss. Math.*, 127, (1975).

[18] E. Rakotch, A note on contractive mappings, *Proc. Am. Math. Soc.*, 13, (1962), 459-465.

[19] B. Samet, C. Vetro and P. Vetro, Fixed point theorem for $\alpha - \psi$-contractive type mappings, *Nonlinear Anal.*, 75, (2012), 2154-2165.

[20] W. Sintunavarat, P. Kumam and Y. J. Cho, Coupled fixed point theorems for nonlinear contractions without mixed monotone property, *Fixed Point Theory and Applications*, 170, (2012), 1-16.

[21] Y. Su, A. Petrusel and Y. Jen-chih, Multivariate fixed point theorems for contractions and nonexpansive mappings with applications, *Fixed Point Theory and Applications*, 9, (2016), 1-19.

[22] T. Suzuki, A generalization of Hegedus-Szilagyi's fixed point theorem in complete metric spaces, *Fixed Point Theory and Applications*, 1, (2018), 1-10.