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Common fixed point theorem satisfying rational contraction in complex valued dislocated metric space

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Abstract. In this article we introduce a notion of fixed point theorem satisfying rational contraction in complex valued dislocated metric space and also support our main theorem to provide an example.

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Contents

1. Introduction

In Mathematical analysis, general topology and functional analysis the fixed point theory play a very important role. Many applications of fixed point theory in computer science, engineering field, image processing and mathematics etc. Banach contraction mapping principle play a crucial role in the fixed poin theory. The concept of dislocated metric space was first introduced by Hitzler in 2001. He generalized the Banach contraction mapping principle in the dislocated metric space. The beauty of dislocated metric space that the self distance between two points need not be necessarily zero. The logical programming, topology, electronic engineering and computer science etc. these are the fields which the dislocated metric space play a very vital role. Azam et al. introduced the complex valued metric spaces and proved Banach contraction mapping principle. So many researchers proved many contraction principle by this complex valued metric spaces. Ozgur edge and Ismet karaca introduced the complex valued disclocated metric spaces. Now we are going to prove the complex valued dislocated metric spaces in the fixed point theorem satisfying rational contraction mapping. Before entering into our main results we shall recall some basic definition and results which are needful.

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2. Preliminaries

We recollect some basic definitions and notions which is useful for proving our main results.
Let C be the set of complex numbers and $v_1, v_2 \in C$. Define a partial order $\leq$ on C as follows:
$v_1 \leq v_2$ if and only if $Re(v_1) \leq Re(v_2)$, $Im(v_1) \leq Im(v_2)$.
Consequently, one can infer that $v_1 \leq v_2$ if one of the following conditions is satisfied:
(i) $Re(v_1) = Re(v_2)$, $Im(v_1) < Im(v_2)$,
(ii)$Re(v_1) < Re(v_2)$, $Im(v_1) = Im(v_2)$,
(iii)$Re(v_1) < Re(v_2)$, $Im(v_1) < Im(v_2)$,
(iv)$Re(v_1) = Re(v_2)$, $Im(v_1) = Im(v_2)$.
In particular,we write $v_1 \lneq v_2$ if $v_1 \neq v_2$ and one of $(i), (ii)$ and (iii) is satisfied and we write $v_1 < v_2$ if only (iii) is satisfied. Notice that
(a) If $0 \leq v_1 \lneq v_2$, then $|v_1| < |v_2|$,
(b) If $v_1 \leq v_2$ and $v_2 < v_3$ then $v_1 < v_3$,
(c) If $p, q \in R$ and $p \leq q$ then $pv \leq qv$ for all $v \in C$.
Now we define a complex valued dislocated metric space

Definition 2.1. *Consider γ_d be a non void set and define a function $\gamma_d : H \times H \to C$ satisfies the following conditions such that for all $u, r, w \in H$*
(1)$\gamma_d(u,r) = \gamma_d(r,u)$
(2)$\gamma_d(u,r) = \gamma_d(r,u) = 0$ if and only if $u = r$
(3)$\gamma_d(u,r) \leq \gamma_d(u,w) + \gamma_d(w,r)$
Then γ_d is said to be complex valued dislocated metric space and call (H, γ_d) is a complex valued dislocated metric space.

Example 2.2. *Consider the function that $\gamma_d : H \times H \to C$ be defined by $\gamma_d(u,r) = max\{u, r\}$ where $H = C$ then it is called as complex valued dislocated metric space.*

Remark 2.3. *Every complex valued metric space is a complex valued dislocated metric space but converse need not be true.*

Definition 2.4. *Consider (H, γ_d) be a complex valued dislocated metric space and define a sequence $\{u_n\}$ in H for each $u \in H$*
(i)let the sequence $\{u_n\}$ be convergent to u in (H, γ_d) is said to be complex valued dislocated metric space then for each $\epsilon > 0$ we can find $n_0 \in N$ such that $\gamma_d(u_n, u) < \epsilon$ for each $n > n_0$ which is denoted by $u_n \to u$
(ii)Consider the sequence $\{u_n\}$ be cauchy sequence in (H, γ_d) is called complex valued dislocated metric space if $lim_{n\to\infty}\gamma_d(u_n, u_{n+b}) = 0$ for each $b > 0$
(iii)Let (H,γ_d) be a complex valued complete dislocated metric space if every complex valued cauchy sequence in H converges to some $u \in H$.

We state the two lemmas which are useful to prove our main theorem

Lemma 2.5. *Consider (H,γ_d) be a complex valued dislocated metric space.Let $\{u_n\}$ be sequence in H. Then $\{u_n\}$ converges to u if and only if $|\gamma_d(u_n, u)| \to 0$ as $n \to \infty$*

Lemma 2.6. *Consider (H,γ_d) be a complex valued dislocated metric space.Let $\{u_n\}$ be sequence in H. Then $\{u_n\}$ is a complex valued dislocated metric cauchy sequence if and only if $|\gamma_d(u_n, u_{n+b})| \to 0$ as $n \to \infty$*

3. Main Results

In this section, we prove the theorem by using new rational contraction mapping in complex valued dislocated metric space.
Now we first define the rational contraction mapping in complex valued dislocated metric space

Definition 3.1. *Let* (H, γ_d) *be a complete complex valued dislocated metric space. Consider the function* $G, T : H \to H$ *which satisfies the rational contraction conditions*
$\gamma_d(Gu, Tr) \leq a[\gamma_d(u,r)] + \frac{3b[\gamma_d(u,Tr)]^2}{1+\gamma_d(u,r)+\gamma_d(r,Tr)} + c[\gamma_d(u,Tr) + \gamma_d(u,Gu)]$ *for each* $u, r \in H$ *and the non negativity constants are* a, b, c

Theorem 3.2. *Let* (H, γ_d) *be a complete complex valued dislocated metric space. Consider the function* $G, T : H \to H$ *which satisfies the rational contraction conditions of (3.1) with* $2a + 6b + 3c < 1$ *.Then G has unique common fixed point.*

Proof. Let u_0 be the arbitrary point in H. Now define $u_{k+1} = Gu_k$,$u_{k+2} = Tu_{k+1}$, for each $k \in Z^+$ Therefore,

$$\begin{aligned}
\gamma_d(u_{k+1}, u_{k+2}) &= \gamma_d(Gu_k, Tu_{k+1}) \\
&\leq a[\gamma_d(u_k, u_{k+1})] + \frac{3b[\gamma_d(u_k, Tu_{k+1})]^2}{1+\gamma_d(u_k,u_{k+1})+\gamma_d(u_{k+1},Tu_{k+1})} + \\
&\quad c[\gamma_d(u_k, Tu_{k+1}) + \gamma_d(u_k, Gu_k)] \\
&\leq a[\gamma_d(u_k, u_{k+1})] + \frac{3b[\gamma_d(u_k,u_{k+2})]^2}{1+\gamma_d(u_k,u_{k+1})+\gamma_d(u_{k+1},u_{k+2})} + \\
&\quad c[\gamma_d(u_k, u_{k+2}) + \gamma_d(u_k, u_{k+1})] \\
&\leq a[\gamma_d(u_k, u_{k+1})] + \frac{3b[\gamma_d(u_k,u_{k+1})+\gamma_d(u_{k+1},u_{k+2})]^2}{1+\gamma_d(u_k,u_{k+1})+\gamma_d(u_{k+1},u_{k+2})} + \\
&\quad c[\gamma_d(u_k, u_{k+1}) + \gamma_d(u_{k+1}, u_{k+2}) + \gamma_d(u_k, u_{k+1})]
\end{aligned}$$

$$\begin{aligned}
|\gamma_d(u_{k+1}, u_{k+2})| &\leq a|\gamma_d(u_k, u_{k+1})| + 3b|\gamma_d(u_k, u_{k+1}) + \gamma_d(u_{k+1}, u_{k+2})| + \\
&\quad c|2\gamma_d(u_k, u_{k+1}) + \gamma_d(u_{k+1}, u_{k+2})|
\end{aligned}$$

Since
$|1 + d(u_k, u_{k+1}) + d(u_{k+1}, u_{k+2})| > |d(u_k, u_{k+1}) + d(u_{k+1}, u_{k+2})|$
Now

$$\begin{aligned}
|\gamma_d(u_{k+1}, u_{k+2})| &\leq a|\gamma_d(u_k, u_{k+1})| + 3b|\gamma_d(u_k, u_{k+1})| + 3b|\gamma_d(u_{k+1}, u_{k+2})| + \\
&\quad 2c|\gamma_d(u_k, u_{k+1})| + c|\gamma_d(u_{k+1}, u_{k+2})|
\end{aligned}$$

Therefore $|\gamma_d(u_{k+1}, u_{k+2})| \leq \frac{a+3b+2c}{1-(3b+c)}|\gamma_d(u_k, u_{k+1})|$
Similarly,

$$\begin{aligned}
\gamma_d(u_{k+2}, u_{k+3}) &= \gamma_d(Gu_{k+1}, Tu_{k+2}) \\
&\leq a[\gamma_d(u_{k+1}, u_{k+2})] + \frac{3b[\gamma_d(u_{k+1}, Tu_{k+2})]^2}{1+\gamma_d(u_{k+1},u_{k+2})+\gamma_d(u_{k+2},Tu_{k+2})} + \\
&\quad c[\gamma_d(u_{k+1}, Tu_{k+2}) + \gamma_d(u_{k+1}, Gu_{k+1})] \\
&\leq a[\gamma_d(u_{k+1}, u_{k+2})] + \frac{3b[\gamma_d(u_{k+1},u_{k+3})]^2}{1+\gamma_d(u_{k+1},u_{k+2})+\gamma_d(u_{k+2},u_{k+3})} + \\
&\quad c[\gamma_d(u_{k+1}, u_{k+3}) + \gamma_d(u_{k+1}, u_{k+2})] \\
&\leq a[\gamma_d(u_{k+1}, u_{k+2})] + \frac{3b[\gamma_d(u_{k+1},u_{k+2})+\gamma_d(u_{k+2},u_{k+3})]^2}{1+\gamma_d(u_{k+1},u_{k+2})+\gamma_d(u_{k+2},u_{k+3})} + \\
&\quad c[\gamma_d(u_{k+1}, u_{k+2}) + \gamma_d(u_{k+2}, u_{k+3}) + \gamma_d(u_{k+1}, u_{k+2})]
\end{aligned}$$

$$\begin{aligned}
|\gamma_d(u_{k+2}, u_{k+3})| &\leq a|\gamma_d(u_{k+1}, u_{k+2})| + 3b|\gamma_d(u_{k+1}, u_{k+2}) + \gamma_d(u_{k+2}, u_{k+3})| + \\
&\quad c|2\gamma_d(u_{k+1}, u_{k+2}) + \gamma_d(u_{k+2}, u_{k+3})|
\end{aligned}$$

Since
$|1 + d(u_{k+1}, u_{k+2}) + d(u_{k+2}, u_{k+3})| > |d(u_{k+1}, u_{k+2}) + d(u_{k+2}, u_{k+3})|$
Now

$$\begin{aligned}
|\gamma_d(u_{k+2}, u_{k+3})| &\leq a|\gamma_d(u_{k+1}, u_{k+2})| + 3b|\gamma_d(u_{k+1}, u_{k+2})| + 3b|\gamma_d(u_{k+2}, u_{k+3})| + \\
&\quad 2c|\gamma_d(u_{k+1}, u_{k+2})| + c|\gamma_d(u_{k+2}, u_{k+3})|
\end{aligned}$$

Therefore
$|\gamma_d(u_{k+2}, u_{k+3})| \leq \frac{a+3b+2c}{1-(3b+c)}|\gamma_d(u_{k+1}, u_{k+2})|$

Since $a+3b+2c<1$ therefore $\alpha=\frac{a+3b+2c}{1-(3b+c)}<1$
Then, we have
$|\gamma_d(u_{n+1},u_{n+2})|\le\alpha|\gamma_d(u_n,u_{n+1})|\le ....\le\alpha^{n+1}|\gamma_d(u_0,u_1)|$
Therefore for every $m>n$ we have
$|\gamma_d(u_n,u_m)|\le|\gamma_d(u_n,u_{n+1})|+|\gamma_d(u_{n+1},u_{n+2})|+........+|\gamma_d(u_{m-1},u_m)|$
$|\gamma_d(u_n,u_m)|\le[\alpha^n+\alpha^{n+1}+........+\alpha^{m-1}]|\gamma_d(u_0,u_1)|$
$\le\frac{\alpha^n}{1-\alpha}|\gamma_d(u_0,u_1)|$
$\gamma_d(u_n,u_m)|\le\frac{\alpha^n}{1-\alpha}|\gamma_d(u_0,u_1)|\to 0$ as $n,m\to\infty$
Hence $\{u_n\}$ is a cauchy sequence. Since H is complete there must exist $x\in H$ such that $\{u_n\}\to x$ as $n\to\infty$
Suppose on contrary that $x\neq Gx$ so $\gamma_d(x,Gx)=y$
Now
$y\le\gamma_d(x,x_{k+2})+\gamma_d(u_{k+2},Gx)$
$\le\gamma_d(x,u_{k+2})+\gamma_d(Tu_{k+1},Gx)$
$\le a[\gamma_d(x,u_{k+1})]+\frac{3b[\gamma_d(x,Tu_{k+1})]^2}{1+\gamma_d(x,u_{k+1})+\gamma_d(u_{k+1},Tu_{k+1})}+c[\gamma_d(x,Tu_{k+1})+\gamma_d(x,Gx)]$
$\le a[\gamma_d(x,u_{k+1})]+\frac{3b[\gamma_d(x,u_{k+2})]^2}{1+\gamma_d(x,u_{k+1})+\gamma_d(u_{k+1},u_{k+2})}+c[\gamma_d(x,u_{k+2})+\gamma_d(x,Gx)]$
$|y|\le a|\gamma_d(x,u_{k+1})|+3b|\gamma_d(x,u_{k+1})+\gamma_d(u_{k+1},u_{k+2})|+c|\gamma_d(x,u_{k+1})+\gamma_d(u_{k+1},u_{k+2})+\gamma_d(x,Gx)|$
Since
$|1+\gamma_d(x,u_{k+1})+\gamma_d(u_{k+1},u_{k+2})|>|\gamma_d(x,u_{k+1})+\gamma_d(u_{k+1},u_{k+2})|$
Therefore
$|y|\le a|\gamma_d(x,u_{k+1})|+3b|\gamma_d(x,u_{k+1})|+3b|\gamma_d(u_{k+1},u_{k+2})|+c|\gamma_d(x,u_{k+1})|+$
$c|\gamma_d(u_{k+1},u_{k+2})|+c|\gamma_d(x,Gx)|$
Letting $n\to\infty$ we have
$|\gamma_d(x,Gx)|\le\frac{a+6b+2c}{1-c}|\gamma_d(x,x)|$ Since $a+6b+2c<1$
Therefore, we have $|\gamma_d(x,Gx)|\to 0$ which is the contradiction.
Hence $Gx=x$ similarly we prove that $Tx=x$
To prove the uniqueness of common fixed point of G and T, let $d\in H$ be the another common fixed point of G and T, we have
$\gamma_d(x,d)=\gamma_d(Gx,Td)\le a[\gamma_d(x,d)]+\frac{3b[\gamma_d(x,Td)]^2}{1+\gamma_d(x,d)+\gamma_d(d,Td)}+c[\gamma_d(x,Td)+\gamma_d(x,Gx)]$

$\le a[\gamma_d(x,d)]+\frac{3b[\gamma_d(x,d)]^2}{1+\gamma_d(x,d)+\gamma_d(d,d)}+c[\gamma_d(x,d)+\gamma_d(x,x)]$

$\le a[\gamma_d(x,d)]+\frac{3b[\gamma_d(x,d)+\gamma_d(d,d)]^2}{1+\gamma_d(x,d)+\gamma_d(d,d)}+c[\gamma_d(x,d)+\gamma_d(x,x)]$
$|\gamma_d(x,d)|\le a|\gamma_d(x,d)|+3b|\gamma_d(x,d)+\gamma_d(d,d)|+c|\gamma_d(x,d)+\gamma_d(x,x)|$
Since $|1+\gamma_d(x,d)+\gamma_d(d,d)|>|\gamma_d(x,d)+\gamma_d(d,d)|$
Now,
$|\gamma_d(x,d)|\le a|\gamma_d(x,d)|+3b|\gamma_d(x,d)|+3b|\gamma_d(d,d)|+c|\gamma_d(x,d)|+c|\gamma_d(x,x)|$
$|\gamma_d(x,d)|\le\frac{3b}{1-(a+3b+c)}|\gamma_d(d,d)|+\frac{c}{1-(a+3b+c)}|\gamma_d(x,x)|$
Since $a+6b+3c<1$ therefore we have $x=d$ which shows the uniqueness of common fixed point. ■

Corollary 3.3. *Let (H,γ_d) be a complete complex valued dislocated metric space. Consider the function $G,T: H\to H$ which satisfies the rational contraction conditions $\gamma_d(Gu,Tr)\le a[\gamma_d(u,r)]+c[\gamma_d(u,Tr)+\gamma_d(u,Gu)]$ for each $u,r\in H$ and the non negativity constants are a,c with $2a+3c<1$ .Then G has unique common fixed point.*

Corollary 3.4. *Let (H,γ_d) be a complete complex valued dislocated metric space. Consider the function $G: H\to H$ which satisfies the contraction conditions $\gamma_d(Gu^n,Gr^n)\le a[\gamma_d(u,r)]$ for each $u,r\in H$ and the non negativity constant a with $a<1$ .Then G has unique fixed point.*

Example 3.5. *Let X= C be set of complex numbers.Define $f: C\times C\to C$as follows where $z_1=x_1+iy_1$*

$z_2 = x_2 + iy_2$ .Then (C, f) is a complete complex valued dislocated metric space.
Define $G : C \to C$ as

$$G(x) = \begin{cases} 0, & \text{if } x, y \in Q. \\ 1 + 2i, & \text{if } x, y \in Q^c \\ 2 & \text{if } x \in Q^c, y \in Q \\ 5i & \text{if } x \in Q, y \in Q^c \end{cases}$$

Let us consider $x = \sqrt{3}$ and $y = 0$ we obtain,
$f(G(\sqrt{3}), L(0)) = f(3, 0) = 3 \preccurlyeq \alpha f(\sqrt{3}, 0) = \alpha\sqrt{3}$
Therefore ,$\alpha \succcurlyeq \sqrt{3}$, which is a contradiction as $0 \preccurlyeq \alpha \prec 1$
We notice that $G^2 z = 0$ so that $0 = f(G^2 z_1, G^2 z_2) \preccurlyeq \alpha f(z_1, z_2)$ which shows that G^2 satisfies the requirement of Bryant theorem and z=0 is the unique fixed point of T.

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