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Mixed problem with an pure integral two-space-variables condition for a third order fractional parabolic equation

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Abstract

In this paper, we establish sufficient conditions for the existence and uniqueness of a solution, in a functional weighted Sobolev space, for Caputo fractional differential equations with integral conditions. The proof uses a functional analysis method presented, which it based on energy inequality and the density of the range of operator generated by the problem.

Keywords

Fractional differential equations, fractional Caputo derivative, Energy inequality, density of operator, the rang of operator.

AMS Subject Classification

26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05.

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Contents

1. Introduction

Fractional differential equations (FDEs) are generalizations of differential equations of integer order to an arbitrary order. These generalization play a crucial role in engineering, physics and applied mathematics. Therefore, they have generated a lot of interest from engineers and scientist in recent years. Since FDEs have memory, nonlocal relations in space and time , and complex phenomena can be modeled by using these equations. Indeed, we can find numerous applications in viscoelasticity, electro-chemistry, control theory, porous media, fluid flow, rheology, diffusive transport, electrocal network, electromagnetic theory, probability, signal processing, and many other physical processes.

Problem which combine local and integral conditions for a second order parabolic equations is investigated by the potential method by Cannon [12] and kamynin [26], by Fourier's method by Ionkin [24] and by energy inequality method in [30] and [5].

Existence and uniqueness of solution to parabolic fractional differential equations with integral conditions have been studied by Ossaif Taki-Eddine and Bouziani Abdelfatah [29].

Mixed problem with an integral two space- variables condition for a third order parabolic equation has been studied by Ossaif Taki-Eddine and Bouziani Abdelfatah [28].

Our work is a generalization on a third order parabolic Fractional Differential Equations with the Caputo derivative.

2. Preliminaries and formulation of the problem

Let $\Gamma(.)$ denote the gamma function. For any positive integer $0<\alpha<1$, the Caputo derivative is defined as follow

$$\mathrm{D}_t^{\alpha} v(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial v(x,\tau)}{\partial \tau} \times \frac{1}{(t-\tau)^{\alpha}} d\tau. \quad (2.1)$$

In the rectangle $\Omega = (0,1)\times(0,T)$, with $T<+\infty$, we consider the third order linear fractional parabolic equation

$$\mathtt{D}_t^{\alpha} u - \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial u}{\partial x}\right) = F(x,t), \tag{2.2}$$

with the initial condition

$$lu = u(x,0) = \phi(x), \quad x \in (0,1), \tag{2.3}$$

local boundary conditions

$$\frac{\partial u}{\partial x}_{|x=i} = 0, \quad i \in \{0,\alpha,\beta,1\}, \quad t \in (0,T), \tag{2.4}$$

$$\frac{\partial^2 u}{\partial x^2}_{|x=0} = 0, \quad t \in (0,T), \tag{2.5}$$

$$\frac{\partial^2 u}{\partial x^2}_{|x=1} = 0, \quad t \in (0,T), \tag{2.6}$$

and the weighted integral conditions :

$$\int_0^{\alpha} u(x,t)dx + \int_{\beta}^1 u(x,t)dx = E(t), \quad t \in (0,T), \tag{2.7}$$

$$\int_0^{\alpha} xu(x,t)dx + \int_{\beta}^1 xu(x,t)dx = G(t), \quad t \in (0,T). \tag{2.8}$$

$F(x,t)$, $\phi(x)$ are the known functions and $a(x,t)$, $E(t)$ and $G(t)$ satisfy the following conditions :

Condition 1
The coefficient $a(x,t)$ is a real-value belonging to $\mathbf{C}^2(\overline{\Omega})$ such that

1. $c_0 \leq a(x,t) \leq c_1$;
2. $\frac{1}{2}a(x,t) - \frac{\partial a(x,t)}{\partial x} \geq 0$;
3. $a(x,t) - 3\frac{\partial a(x,t)}{\partial x} \geq 0.$

In condition 1 and the rest of the paper, $c_i, i = 1,...,6$, denote strictly positive constants

Condition 2

1. $0 < \alpha < \beta < 1, \quad \alpha + \beta = 1$;
2. $G(t) = \frac{1}{2}\left(\alpha^2 + 1 - \beta^2\right)E(t) = \alpha E(t)$;
3. $\alpha^2 + 1 - \beta^2 = \alpha + 1 - \beta = 2\alpha.$

In this paper, sufficient conditions for existence and uniqueness of solution in a functional weighted Sobolev space for Caputo fractional equations are established.

Since the boundary conditions are inhomogeneous, we construct a function

$$\begin{aligned} w(x,t) = {} & \frac{-6(\alpha^2+1-\beta^2)x + 4(\alpha^3+1-\beta^3)}{4(\alpha^3-\beta^3) - 3(\alpha^2-\beta^2)+1}E(t) \\ & + \frac{12x-6}{4(\alpha^3-\beta^3)-3(\alpha^2-\beta^2)+1}G(t), \end{aligned}$$

and we introduce a new function $\tilde{u}(x,t) = u(x,t) - w(x,t)$.

Then problem (2.2) - (2.8) can be formulated as

$$\mathtt{D}_t^{\alpha} \tilde{u} - \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial \tilde{u}}{\partial x}\right) = g(x,t), \tag{2.9}$$

$$l\tilde{u} = \tilde{u}(x,0) = \psi(x), \quad x \in (0,1), \tag{2.10}$$

$$\frac{\partial \tilde{u}}{\partial x}_{|x=i} = 0, \quad i \in \{0,\alpha,\beta,1\}, \quad t \in (0,T), \tag{2.11}$$

$$\frac{\partial^2 \tilde{u}}{\partial x^2}_{|x=0} = 0, \quad t \in (0,T), \tag{2.12}$$

$$\frac{\partial^2 \tilde{u}}{\partial x^2}_{|x=1} = 0, \quad t \in (0,T), \tag{2.13}$$

$$\int_0^{\alpha} \tilde{u}(x,t)dx + \int_{\beta}^1 \tilde{u}(x,t)dx = 0, \quad t \in (0,T), \tag{2.14}$$

$$\int_0^{\alpha} x\tilde{u}(x,t)dx + \int_{\beta}^1 x\tilde{u}(x,t)dx = 0, \quad t \in (0,T), \tag{2.15}$$

where

$$\begin{aligned} g(x,t) &= F(x,t) - \mathtt{D}_t^{\alpha} w(x,t) + \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial w(x,t)}{\partial x}\right), \\ \psi(x) &= \phi(x) - w(x,0). \end{aligned}$$

Again, introducing a new function $v = \tilde{u} - \tilde{u}(x,0) = \tilde{u} - \psi(x)$, problem (2.9) - (2.15) can be formulated as

$$\mathtt{D}_t^{\alpha} v - \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial v}{\partial x}\right) = f(x,t), \tag{2.16}$$

$$lv = v(x,0) = 0, \quad x \in (0,1), \tag{2.17}$$

$$\frac{\partial v}{\partial x}_{|x=i}=0,\quad i\in\{0,\alpha,\beta,1\},\quad t\in(0,T), \tag{2.18}$$

$$\frac{\partial^2 v}{\partial x^2}_{|x=0}=0,\quad t\in(0,T), \tag{2.19}$$

$$\frac{\partial^2 v}{\partial x^2}_{|x=1}=0,\quad t\in(0,T), \tag{2.20}$$

$$\int_0^\alpha v(x,t)dx+\int_\beta^1 v(x,t)dx=0,\quad t\in(0,T), \tag{2.21}$$

$$\int_0^\alpha xv(x,t)dx+\int_\beta^1 xv(x,t)dx=0,\quad t\in(0,T), \tag{2.22}$$

where

$$\begin{aligned} f(x,t) &= g(x,t)+\frac{\partial^2 a(x,t)}{\partial x^2}\frac{\partial\psi}{\partial x}\\ &= F(x,t)+\frac{\partial^2 a(x,t)}{\partial x^2}\frac{\partial\psi}{\partial x}+\frac{\partial^2 a(x,t)}{\partial x^2}\frac{\partial w(x,t)}{\partial x},\\ \frac{\partial\psi}{\partial x} &= \frac{\partial\phi}{\partial x}-\frac{\partial w(x,0)}{\partial x}. \end{aligned}$$

Hence, instead of looking for the function u, we seek the function v. The solution of problem (2.2), (2.3), (2.4), (2.5), (2.6), (2.7) and (2.8) will be simply given by the formula

$$\begin{aligned} u(x,\ t) &= \tilde{u}(x,\ t)+w(x,\ t)\\ &= v(x,\ t)+w(x,\ t)+\tilde{u}(x,0)\\ &= v(x,\ t)+w(x,\ t)+\psi(x) \end{aligned}$$

The solution of problem (2.16) - (2.22) can be considered as a solution of the operator equation

$$\mathbf{L}v=f. \tag{2.23}$$

The operator L maps from $\mathbb{E}$ to $\mathbb{F}$, where $\mathbb{E}$ is the Banach space consisting of functions $v\in\mathbf{L}^2(\Omega)$ such that

$$\mathtt{D}_t^\alpha v,\quad \frac{\partial v}{\partial x},\ \frac{\partial^2 v}{\partial^2 x},\quad \frac{\partial^3 v}{\partial^3 x},\quad \mathtt{D}_t^\alpha\frac{\partial^2 v}{\partial x^2}\in\mathbf{L}^2(\Omega).$$

The norm in $\mathbb{E}$ is defined by

$$\begin{aligned} \|v\|_{\mathbb{E}}^2 = &\int_0^T\int_0^\alpha\left(\int_x^\alpha \mathtt{D}_t^\alpha v d\xi\right)^2 dxdt\\ &+\int_0^T\int_\beta^1\left(\int_\beta^x \mathtt{D}_t^\alpha v d\xi\right)^2 dxdt\\ &+\sup_{0\le t\le T}\left(\int_0^\alpha(5-x)\left(\mathtt{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)\right)^2 dx\right.\\ &+\int_\beta^1(\frac{5}{4}-x)\left(\mathtt{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)\right)^2 dx\\ &\left.+\int_\alpha^\beta(\beta-\alpha)\left(\mathtt{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)\right)^2 dx\right), \end{aligned} \tag{2.24}$$

and $\mathbb{F}$ is the Hilbert space with the finite norm

$$\|Lv\|_{\mathbb{F}}^2=\int_\Omega f^2dxdt. \tag{2.25}$$

Theorem 2.1. *Let conditions* 1 *and* 2 *be fulfilled . Then for any fuction* $v\in\mathbf{D}(\mathbf{L})$, *we have the inequality*

$$\|v\|_{\mathbb{E}}\le c\|\mathbf{L}v\|_{\mathbb{F}}, \tag{2.26}$$

where c is a positive constant indépendent of v.

Proof. Multiplying the equation (2.16) by

$$Mv=\begin{cases} M_1v, & 0\le x\le\alpha,\\ M_2v, & \alpha\le x\le\beta,\\ M_3v, & \beta\le x\le 1,\end{cases}$$

where

$$\begin{aligned} M_1v = &4\int_x^\alpha \mathtt{D}_t^\alpha v d\xi\\ &-\int_x^\alpha\left(\int_\xi^\alpha \mathtt{D}_t^\alpha v d\eta-(1-\xi)\mathtt{D}_t^\alpha v\right)d\xi, \end{aligned} \tag{2.27}$$

$$M_2v=(x-\alpha)\int_x^\beta \mathtt{D}_t^\alpha v d\xi+(x-\beta)\int_\alpha^x \mathtt{D}_t^\alpha v d\xi, \tag{2.28}$$

$$\begin{aligned} M_3v = &-\frac{1}{4}\int_\beta^x \mathtt{D}_t^\alpha v d\xi\\ &-\int_\beta^x\left(\int_\beta^\xi \mathtt{D}_t^\alpha v d\eta+(1-\xi)\mathtt{D}_t^\alpha v\right)d\xi, \end{aligned} \tag{2.29}$$

and integrating over $\Omega^T=(0,1)\times(0,T)$.

1. On the interval $(0,\alpha)$, we denote $\Omega_\alpha^T=\Omega_\alpha=(0,\alpha)\times(0,T)$, we get

$$\begin{aligned} \int_{\Omega_\alpha} f(x,t)M_1vdxdt = &\int_{\Omega_\alpha}\mathtt{D}_t^\alpha v\left(4\int_x^\alpha \mathtt{D}_t^\alpha v d\xi\right.\\ &\left.-\int_x^\alpha\left(\int_\xi^\alpha \mathtt{D}_t^\alpha v d\eta-(1-\xi)\mathtt{D}_t^\alpha v\right)d\xi\right)dxdt\\ &-\int_{\Omega_\alpha}\frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial v}{\partial x}\right)\left(4\int_x^\alpha \mathtt{D}_t^\alpha v d\xi\right)dxdt\\ &+\int_{\Omega_\alpha}\frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial v}{\partial x}\right)\\ &\times\left(\int_x^\alpha\left(\int_\xi^\alpha \mathtt{D}_t^\alpha v d\eta-(1-\xi)\mathtt{D}_t^\alpha v\right)d\xi\right)dxdt. \end{aligned} \tag{2.30}$$

Integrating by parts each term of the right hand-side of

(2.30) and using the conditions (2.17)-(2.22), we get

$$\int_{\Omega_\alpha} \mathtt{D}_t^\alpha v \left(4\int_x^\alpha \mathtt{D}_t^\alpha v d\xi - \int_x^\alpha \left(\int_\xi^\alpha \mathtt{D}_t^\alpha v d\eta - (1-\xi)\mathtt{D}_t^\alpha v\right) d\xi\right) dxdt$$
$$= \frac{5}{2}\int_0^T \left(\int_0^\alpha \mathtt{D}_t^\alpha v dx\right)^2 dt - 2\int_0^T \left(\int_0^\alpha \mathtt{D}_t^\alpha v dx\right)\left(\int_0^\alpha x\mathtt{D}_t^\alpha v dx\right) dt + \frac{3}{2}\int_{\Omega_\alpha} \left(\int_x^\alpha \mathtt{D}_t^\alpha v d\xi\right)^2 dxdt, \qquad (2.31)$$

$$-\int_{\Omega_\alpha} \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial v}{\partial x}\right)\left(4\int_x^\alpha \mathtt{D}_t^\alpha v d\xi - \int_x^\alpha \left(\int_\xi^\alpha \mathtt{D}_t^\alpha v d\eta - (1-\xi)\mathtt{D}_t^\alpha v\right) d\xi\right) dxdt$$
$$= \int_{\Omega_\alpha} (5-x)a(x,t)\mathtt{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)^2 dxdt. \qquad (2.32)$$

Replacing $M_1 v$ in (2.30) by its representation (2.27), we get

$$\int_{\Omega_\alpha} f.M_1 v dxdt = \int_{\Omega_\alpha} f\left(4\int_x^\alpha \mathtt{D}_t^\alpha v d\xi - \int_x^\alpha \left(\int_\xi^\alpha \mathtt{D}_t^\alpha v d\eta - (1-\xi)\mathtt{D}_t^\alpha v\right) d\xi\right) dxdt$$
$$= \int_{\Omega_\alpha} f.\left(4\int_x^\alpha \mathtt{D}_t^\alpha v d\xi\right) dxdt - \int_{\Omega_\alpha}\left(f.\int_x^\alpha \left(\int_\xi^\alpha \mathtt{D}_t^\alpha v d\eta\right) d\xi\right) dxdt + \int_{\Omega_\alpha}\left(f.\int_x^\alpha (1-\xi)\mathtt{D}_t^\alpha v d\xi\right) dxdt. \qquad (2.33)$$

Integrating by parts the last term of the right-hand side of (2.33), we obtain

$$\int_{\Omega_\alpha}\left(f.\int_x^\alpha (1-\xi)\mathtt{D}_t^\alpha v d\xi\right) dxdt = \int_{\Omega_\alpha}\left(f.(1-x)\int_x^\alpha \mathtt{D}_t^\alpha v d\xi\right) dxdt - \int_{\Omega_\alpha} f.\int_x^\alpha \left(\int_\xi^\alpha \mathtt{D}_t^\alpha v d\eta\right) d\xi dxdt. \qquad (2.34)$$

Substituting (2.34) into (2.33), we obtain

$$\int_{\Omega_\alpha} fM_1 v dxdt = \int_{\Omega_\alpha} f.\left(4\int_x^\alpha \mathtt{D}_t^\alpha v d\xi\right) dxdt + \int_{\Omega_\alpha}\left(f.(1-x)\int_x^\alpha \mathtt{D}_t^\alpha v d\xi\right) dxdt - 2\int_{\Omega_\alpha} f.\int_x^\alpha \left(\int_\xi^\alpha \mathtt{D}_t^\alpha v d\eta\right) d\xi dxdt. \qquad (2.35)$$

Integrating by parts the last term of the right-hand side of (2.35), we

$$-2\int_{\Omega_\alpha} f.\int_x^\alpha \left(\int_\xi^\alpha \mathtt{D}_t^\alpha v d\eta\right) d\xi dxdt = 2\int_0^T \left(\int_0^\alpha x\mathtt{D}_t^\alpha v d\eta d\xi\right)\left(\int_0^\alpha f(x)dx\right) dt + 2\int_{\Omega_\alpha}\left(\int_x^\alpha \mathtt{D}_t^\alpha v d\xi\right)\left(\int_x^\alpha f(\xi)\xi\right) dxdt. \qquad (2.36)$$

Putting, (2.36) into (2.35) and using the Cauchy inequality, we can estimate

$$\int_{\Omega_\alpha} fM_1 v dxdt \le \frac{4}{2\varepsilon_1}\int_{\Omega_\alpha}\left(\int_x^\alpha \mathtt{D}_t^\alpha v d\xi\right)^2 dxdt + \frac{4\varepsilon_1}{2}\int_{\Omega_\alpha} f^2 dxdt + \frac{\varepsilon_2}{2}\int_{\Omega_\alpha}(1-x)^2 f^2 dxdt + \frac{1}{2\varepsilon_2}\int_{\Omega_\alpha}\left(\int_x^\alpha \mathtt{D}_t^\alpha v d\xi\right)^2 dxdt$$
$$+ \frac{1}{\varepsilon_3}\int_0^T \left(\int_0^\alpha x\mathtt{D}_t^\alpha v dx\right)^2 dt + \varepsilon_3\int_0^T \left(\int_0^\alpha f dx\right)^2 dt + \varepsilon_4\int_{\Omega_\alpha}\left(\int_x^\alpha f d\xi\right)^2 dxdt + \frac{1}{\varepsilon_4}\int_{\Omega_\alpha}\left(\int_x^\alpha \mathtt{D}_t^\alpha v d\xi\right)^2 dxdt, \qquad (2.37)$$

where

$$\int_{\Omega_\alpha}\left(\int_x^\alpha f d\xi\right)^2 dxdt \le 4\int_{\Omega_\alpha}(1-x)^2 f^2 dxdt + 2\int_0^T \left(\int_0^\alpha f dx\right)^2 dt \le 4\int_{\Omega_\alpha} f^2 dxdt + 2\int_0^T \left(\int_0^\alpha f dx\right)^2 dt.$$

Consequently, (2.37) becomes

$$\int_{\Omega_\alpha} f.M_1 v dxdt \le \frac{2}{\varepsilon_1}\int_{\Omega_\alpha}\left(\int_x^\alpha \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt$$
$$+2\varepsilon_1\int_{\Omega_\alpha} f^2 dxdt + \frac{1}{2\varepsilon_2}\int_{\Omega_\alpha}\left(\int_x^\alpha \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt$$
$$+\frac{\varepsilon_2}{2}\int_{\Omega_\alpha} f^2 dxdt + \varepsilon_3\int_0^\tau\left(\int_0^\alpha f dx\right)^2 dt$$
$$+\frac{1}{\varepsilon_3}\int_0^T\left(\int_0^\alpha \mathrm{D}_t^\alpha v dx\right)^2 dt + 4\varepsilon_4\int_{\Omega_\alpha} f^2 dxdt$$
$$+2\varepsilon_4\int_0^T\left(\int_0^\alpha f dx\right)^2 dt$$
$$+\frac{1}{\varepsilon_4}\int_{\Omega_\alpha}\left(\int_x^\alpha \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt. \qquad (2.38)$$

2. On the interval (α,β), we denote $\Omega_{\alpha,\beta}^T = \Omega_{\alpha,\beta} = (\alpha,\beta)\times(0,T)$, we get

$$\int_{\Omega_{\alpha,\beta}} f.M_2 v dxdt$$
$$= \int_{\Omega_{\alpha,\beta}} \mathrm{D}_t^\alpha v\left((x-\alpha)\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right.$$
$$\left.+(x-\beta)\int_\alpha^x \mathrm{D}_t^\alpha v d\xi\right) dxdt$$
$$-\int_{\Omega_{\alpha,\beta}} \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial v}{\partial x}\right)$$
$$\times\left((x-\alpha)\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right.$$
$$\left.+(x-\beta)\int_\alpha^x \mathrm{D}_t^\alpha v d\xi\right) dxdt. \qquad (2.39)$$

Integrating by parts each term of the right hand-side of (2.39) and using the conditions (2.17)-(2.22), we get

$$\int_{\Omega_{\alpha,\beta}} \mathrm{D}_t^\alpha v\left((x-\alpha)\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right.$$
$$\left.+(\beta-x)\int_\alpha^x \mathrm{D}_t^\alpha v d\xi\right) dxdt$$
$$= \int_{\Omega_{\alpha,\beta}}\left(\int_\alpha^\beta \mathrm{D}_t^\alpha v d\xi\right)\left(\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right) dxdt$$
$$+\frac{1}{2}(\beta-\alpha)\int_{\Omega_{\alpha,\beta}}\left(\int_\alpha^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt, \qquad (2.40)$$

$$-\int_{\Omega_{\alpha,\beta}} \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial v}{\partial x}\right)\left((x-\alpha)\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right.$$
$$\left.+(\beta-x)\int_\alpha^x \mathrm{D}_t^\alpha v d\xi\right) dxdt$$
$$= \int_{\Omega_{\alpha,\beta}} (\beta-\alpha) a(x,t)\mathrm{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)^2 dxdt. \qquad (2.41)$$

Replacing $M_2 v$ in (2.39) by its representation (2.28), we have

$$\int_{\Omega_{\alpha,\beta}} f M_2 v dxdt$$
$$= \int_{\Omega_{\alpha,\beta}} f\left((x-\alpha)\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right.$$
$$\left.+(\beta-x)\int_\alpha^x \mathrm{D}_t^\alpha v d\xi\right) dxdt. \qquad (2.42)$$

By virtue of Cauchy inequality, from (2.42), we obtain

$$\int_{\Omega_{\alpha,\beta}} f.M_2 v dxdt$$
$$\le \frac{1}{2}\int_{\Omega_{\alpha,\beta}}\left(\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt$$
$$+\frac{1}{2}\int_{\Omega_{\alpha,\beta}} (x-\alpha)^2 f^2 dxdt$$
$$+\frac{1}{2}\int_{\Omega_{\alpha,\beta}} (\beta-x)^2 f^2 dxdt$$
$$+\frac{1}{2}\int_{\Omega_{\alpha,\beta}}\left(\int_\alpha^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt. \qquad (2.43)$$

3. On the interval $(\beta,1)$, we denote $\Omega_\beta^T = \Omega_\beta = (\beta,1)\times(0,T)$, we get

$$\int_{\Omega_\alpha} f.M_3 v dxdt = \int_{\Omega_\beta} \mathrm{D}_t^\alpha v.\left(-\frac{1}{4}\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right.$$
$$\left.-\int_\beta^x\left(\int_\beta^\xi \mathrm{D}_t^\alpha v d\eta + (1-\xi)\mathrm{D}_t^\alpha v\right) d\xi\right) dxdt$$
$$-\int_{\Omega_\beta} \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial v}{\partial x}\right)\left(-\frac{1}{4}\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right) dxdt$$
$$+\int_{\Omega_\beta} \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial v}{\partial x}\right)$$
$$\times\left(\int_\beta^x\left(\int_\beta^\xi \mathrm{D}_t^\alpha v d\eta + (1-\xi)\mathrm{D}_t^\alpha v\right) d\xi\right) dxdt. \qquad (2.44)$$

Integrating by parts each integral of the right hand-side of (2.44) and using the conditions (2.17)-(2.22), we obtain

$$\int_{\Omega_\beta} \mathrm{D}_t^\alpha u \left(-\frac{1}{4}\int_\beta^x \mathrm{D}_t^\alpha u d\xi - \int_\beta^x \left(\int_\beta^\xi \mathrm{D}_t^\alpha u d\eta + (1-\xi)\mathrm{D}_t^\alpha u\right) d\xi\right) dxdt$$
$$= \frac{3}{2}\int_{\Omega_\beta} \left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + 2\int_0^T \left(\int_\beta^1 \mathrm{D}_t^\alpha v dx\right)\left(\int_\beta^1 x\mathrm{D}_t^\alpha v dx\right) dt - \frac{17}{8}\int_0^T \left(\int_\beta^1 \mathrm{D}_t^\alpha v dx\right)^2 dt, \qquad (2.45)$$

$$-\int_{\Omega_\beta} \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial v}{\partial x}\right)\left(-\frac{1}{4}\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right) dxdt + \int_{\Omega_\beta} \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial v}{\partial x}\right) \times \left(\int_\beta^x \left(\int_\beta^\xi \mathrm{D}_t^\alpha v d\eta + (1-\xi)\mathrm{D}_t^\alpha v\right) d\xi\right) dxdt$$
$$= \int_{\Omega_\beta} (\frac{5}{4} - x) a(x,t)\mathrm{D}_t^{\frac{\alpha}{2}} \left(\frac{\partial v}{\partial x}\right)^2 dxdt. \qquad (2.46)$$

Replacing $M_3 v$ in (2.44) by its representation (2.29), and integrating by parts the terms of the right-hand, we obtain

$$\int_{\Omega_\beta} f.M_3 v dxdt = \int_{\Omega_\beta} f.\left(-\frac{1}{4}\int_\beta^x \mathrm{D}_t^\alpha v d\xi - \int_\beta^x \left(\int_\beta^\xi \mathrm{D}_t^\alpha v d\eta + (1-\xi)\mathrm{D}_t^\alpha v\right) d\xi\right) dxdt$$
$$= \int_{\Omega_\beta} f.\left(-\frac{1}{4}\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right) dxdt - \int_{\Omega_\beta} f\left((1-x)\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right) dxdt - 2\int_{\Omega_\beta} f\left(\int_\beta^x \int_\beta^\xi \mathrm{D}_t^\alpha v d\eta d\xi\right) dxdt. \qquad (2.47)$$

Integrating by parts the last integral of the right hand-side of (2.47), we have

$$-2\int_{\Omega_\beta} f\left(\int_\beta^x \int_\beta^\xi \mathrm{D}_t^\alpha v d\eta d\xi\right) dxdt = -2\int_0^T \left(\int_\beta^1 f dx\right)\left(\int_\beta^1 \mathrm{D}_t^\alpha v dx - \int_\beta^1 x\mathrm{D}_t^\alpha v dx\right) dt + 2\int_{\Omega_\beta} \left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)\left(\int_\beta^x f d\xi\right) dxdt. \qquad (2.48)$$

Substituting (2.48) into (2.47), and using Cauchy's $\varepsilon-$inequality. Observe that

$$\int_{\Omega_\beta} f.M_3 v dxdt \le \frac{1}{8\varepsilon_5}\int_{\Omega_\beta} \left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + \frac{\varepsilon_5}{8}\int_{\Omega_\beta} f^2 dxdt + \frac{\varepsilon_6}{2}\int_{\Omega_\beta} (1-x)^2 f^2 dxdt + \frac{1}{2\varepsilon_6}\int_{\Omega_\beta} \left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + \varepsilon_7 \int_0^T \left(\int_\beta^1 f dx\right)^2 dt + \frac{1}{\varepsilon_7}\int_0^T \left(\int_\beta^1 \mathrm{D}_t^\alpha v dx\right)^2 dt + \varepsilon_8 \int_0^T \left(\int_\beta^1 f dx\right)^2 dt + \frac{1}{\varepsilon_8}\int_0^T \left(\int_\beta^1 x\mathrm{D}_t^\alpha v dx\right)^2 dt + \frac{1}{\varepsilon_9}\int_{\Omega_\beta} \left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + \varepsilon_9 \int_{\Omega_\beta} \left(\int_\beta^x f d\xi\right)^2 dxdt. \qquad (2.49)$$

Estimated the last integral of the right hand-side of (2.49)

$$\int_{\Omega_\beta} \left(\int_\beta^x f d\xi\right)^2 dxdt \le 4\int_{\Omega_\beta} (x-\beta) f^2 dxdt \le 4\int_{\Omega_\beta} f^2 dxdt. \qquad (2.50)$$

Therefore, by formulas (2.49) and (2.50), we have

$$\int_{\Omega_\beta} f M_3 v dxdt \le \frac{\varepsilon_5}{8}\int_{\Omega_\beta} f^2 dxdt + \frac{1}{8\varepsilon_5}\int_{\Omega_\beta} \left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + \frac{\varepsilon_6}{2}\int_{\Omega_\beta} f^2 dxdt + \frac{1}{2\varepsilon_6}\int_{\Omega_\beta} \left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + 2\varepsilon_{10}\int_0^T \left(\int_\beta^1 f dx\right)^2 dt + \frac{2}{\varepsilon_{10}}\int_0^T \left(\int_\beta^1 \mathrm{D}_t^\alpha v dx\right)^2 dt + \frac{1}{\varepsilon_9}\int_{\Omega_\beta} \left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + 4\varepsilon_9 \int_{\Omega_\beta} f^2 dxdt, \qquad (2.51)$$

where $\varepsilon_{10} = \varepsilon_7 + \varepsilon_8$.

Substituting (2.31), (2.32) and (2.38) into (2.30), on Ω_α, we obtain

$$\begin{aligned}
&\frac{5}{2}\int_0^T \left(\int_0^\alpha \mathrm{D}_t^\alpha v dx\right)^2 dt - 2\int_0^T \left(\int_0^\alpha \mathrm{D}_t^\alpha v dx\right) \\
&\times \left(\int_0^\alpha x \mathrm{D}_t^\alpha v dx\right) dt \\
&+ \frac{3}{2}\int_{\Omega_\alpha} \left(\int_x^\alpha \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt \\
&+ \int_{\Omega_\alpha} (5-x) a(x,t) \frac{\partial v}{\partial x}\frac{\partial}{\partial x}\mathrm{D}_t^\alpha v dxdt \\
&\leq 2\varepsilon_1 \int_{\Omega_\alpha} f^2 dxdt \\
&+ \frac{2}{\varepsilon_1}\int_{\Omega_\alpha} \left(\int_x^\alpha \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt \\
&+ \frac{1}{2\varepsilon_2}\int_{\Omega_\alpha} \left(\int_x^\alpha \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt \\
&+ \frac{\varepsilon_2}{2}\int_{\Omega_\alpha} f^2 dxdt \\
&+ \varepsilon_3 \int_0^T \left(\int_0^\alpha f dx\right)^2 dt \\
&+ \frac{1}{\varepsilon_3}\int_0^T \left(\int_0^\alpha \mathrm{D}_t^\alpha v dx\right)^2 dt \\
&+ 4\varepsilon_4 \int_{\Omega_\alpha} f^2 dxdt + 2\varepsilon_4 \int_0^T \left(\int_0^\alpha f dx\right)^2 dt \\
&+ \frac{1}{\varepsilon_4}\int_{\Omega_\alpha} \left(\int_x^\alpha \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt. \qquad (2.52)
\end{aligned}$$

So, we get

$$\begin{aligned}
&\frac{5}{2}\int_0^T \left(\int_0^\alpha \mathrm{D}_t^\alpha v dx\right)^2 dt - 2\int_0^T \left(\int_0^\alpha \mathrm{D}_t^\alpha v dx\right) \\
&\times \left(\int_0^\alpha x \mathrm{D}_t^\alpha v dx\right) dt + \frac{3}{2}\int_{\Omega_\alpha} \left(\int_x^\alpha \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt \\
&+ \int_{\Omega_\alpha} (5-x) a(x,t) \frac{\partial v}{\partial x}.\frac{\partial}{\partial x}\mathrm{D}_t^\alpha v dxdt \\
&\leq \left(2\varepsilon_1 + \frac{\varepsilon_2}{2} + 4\varepsilon_4\right)\int_{\Omega_\alpha} f^2 dxdt \\
&+ \left(\frac{2}{\varepsilon_1} + \frac{1}{2\varepsilon_2} + \frac{1}{\varepsilon_4}\right)\int_{\Omega_\alpha} \left(\int_x^\alpha \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt \\
&+ (\varepsilon_3 + 2\varepsilon_4)\int_0^T \left(\int_0^\alpha f dx\right)^2 dt \\
&+ \frac{1}{\varepsilon_3}\int_0^T \left(\int_0^\alpha \mathrm{D}_t^\alpha v dx\right)^2 dt. \qquad (2.53)
\end{aligned}$$

Substituting (2.40), (2.41) and (2.43) into (2.39), on $\Omega_{\alpha,\beta}$, we obtain

$$\begin{aligned}
&\int_{\Omega_{\alpha,\beta}} \left(\int_\alpha^\beta \mathrm{D}_t^\alpha v d\xi\right)\left(\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right) dxdt \\
&+ \frac{1}{2}(\beta-\alpha)\int_{\Omega_{\alpha,\beta}} \left(\int_\alpha^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt \\
&+ (\beta-\alpha)\int_{\Omega_{\alpha,\beta}} a(x,t)\mathrm{D}_t^{\frac{\alpha}{2}} \left(\frac{\partial v}{\partial x}\right)^2 dxdt \\
&\leq \frac{1}{2\varepsilon}\int_{\Omega_{\alpha,\beta}} \left(\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt \\
&+ \frac{\varepsilon}{2}\int_{\Omega_{\alpha,\beta}} (x-\alpha)^2 f^2 dxdt \\
&+ \frac{1}{2\varepsilon'}\int_{\Omega_{\alpha,\beta}} \left(\int_\alpha^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt \\
&+ \frac{\varepsilon'}{2}\int_{\Omega_{\alpha,\beta}} (\beta-x)^2 f^2 dxdt. \qquad (2.54)
\end{aligned}$$

With

$$\begin{aligned}
&\int_{\Omega_{\alpha,\beta}} \left(\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt \leq \\
&\int_{\Omega_{\alpha,\beta}} \left(\int_\alpha^\beta \mathrm{D}_t^\alpha v d\xi\right)\left(\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right) dxdt. \qquad (2.55)
\end{aligned}$$

That implies

$$\begin{aligned}
&\int_{\Omega_{\alpha,\beta}} \left(\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt \\
&+ \frac{1}{2}(\beta-\alpha)\int_{\Omega_{\alpha,\beta}} \left(\int_\alpha^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt \\
&+ c_0(\beta-\alpha)\int_{\Omega_{\alpha,\beta}} a(x,t)\mathrm{D}_t^{\frac{\alpha}{2}} \left(\frac{\partial v}{\partial x}\right)^2 dxdt \\
&\leq \frac{1}{2\varepsilon}\int_{\Omega_{\alpha,\beta}} \left(\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt \\
&+ \frac{\varepsilon}{2}\int_{\Omega_{\alpha,\beta}} (x-\alpha)^2 f^2 dxdt \\
&+ \frac{1}{2\varepsilon'}\int_{\Omega_{\alpha,\beta}} \left(\int_\alpha^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt \\
&+ \frac{\varepsilon'}{2}\int_{\Omega_{\alpha,\beta}} (\beta-x)^2 f^2 dxdt. \qquad (2.56)
\end{aligned}$$

Combining the same terms of (2.56), we have

$$\begin{aligned}
&\left(1-\frac{1}{2\varepsilon}\right)\int_{\Omega_{\alpha,\beta}} \left(\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt \\
&+ \left(\frac{1}{2}(\beta-\alpha) - \frac{1}{2\varepsilon'}\right)\int_{\Omega_{\alpha,\beta}} \left(\int_\alpha^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt \\
&+ c_0\int_{\Omega_{\alpha,\beta}} (\beta-\alpha)\mathrm{D}_t^{\frac{\alpha}{2}} \left(\frac{\partial v}{\partial x}\right)^2 dxdt
\end{aligned}$$

$$\leq \left(\frac{\varepsilon}{2}+\frac{\varepsilon'}{2}\right)\int_{\Omega_{\alpha,\beta}} f^2 dxdt. \tag{2.57}$$

If we put $\varepsilon=1,\quad \varepsilon'=\dfrac{1}{\beta-\alpha}+1$, $c_1=\left(\dfrac{1}{2}(\beta-\alpha)-\dfrac{1}{2\varepsilon'}\right)=\dfrac{(\beta-\alpha)^2}{2(1+\beta-\alpha)}$ and $c_2=\dfrac{\varepsilon}{2}+\dfrac{\varepsilon'}{2}$, the inequality (2.57) implies

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_{\alpha,\beta}}\left(\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt\\&+c_0\int_{\Omega_{\alpha,\beta}}(\beta-\alpha)\mathrm{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)^2 dxdt\\&+c_1\int_{\Omega_{\alpha,\beta}}\left(\int_\alpha^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt\\&\leq c_2\int_{\Omega_{\alpha,\beta}} f^2 dxdt.\end{aligned} \tag{2.58}$$

Substituting (2.45), (2.46) and (2.51) into (2.44), on Ω_β, we have

$$\begin{aligned}&\frac{3}{2}\int_{\Omega_\beta}\left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt-\frac{17}{8}\int_0^T\left(\int_\beta^1 \mathrm{D}_t^\alpha v dx\right)^2 dt\\&+2\int_0^T\left(\int_\beta^1 \mathrm{D}_t^\alpha v dx\right)\left(\int_\beta^1 x\mathrm{D}_t^\alpha v dx\right)dt\\&+\int_\beta^1\left(\frac{5}{4}-x\right)\int_0^T a(x,t)\frac{\partial v}{\partial x}.\frac{\partial}{\partial x}\mathrm{D}_t^\alpha v dtdx\\&\leq\frac{\varepsilon_5}{8}\int_{\Omega_\beta} f^2 dxdt+\frac{1}{8\varepsilon_5}\int_{\Omega_\beta}\left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt\\&+\frac{\varepsilon_6}{2}\int_{\Omega_\beta} f^2 dxdt+\frac{1}{2\varepsilon_6}\int_{\Omega_\beta}\left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt\\&+2\varepsilon_{10}\int_0^T\left(\int_\beta^1 f dx\right)^2 dt+\frac{2}{\varepsilon_{10}}\int_0^T\left(\int_\beta^1 \mathrm{D}_t^\alpha v dx\right)^2 dt\\&+\frac{1}{\varepsilon_9}\int_{\Omega_\beta}\left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt+4\varepsilon_9\int_{\Omega_\beta} f^2 dxdt\\&\leq\frac{\varepsilon_5}{8}\int_{\Omega_\beta} f^2 dxdt+\frac{1}{8\varepsilon_5}\int_{\Omega_\beta}\left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt\\&+\frac{\varepsilon_6}{2}\int_{\Omega_\beta} f^2 dxdt+\frac{1}{2\varepsilon_6}\int_{\Omega_\beta}\left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt\\&+2\varepsilon_{10}\int_0^T\left(\int_\beta^1 f dx\right)^2 dt+\frac{2}{\varepsilon_{10}}\int_0^T\left(\int_\beta^1 \mathrm{D}_t^\alpha v dx\right)^2 dt\\&+\frac{1}{\varepsilon_9}\int_{\Omega_\beta}\left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt\\&+4\varepsilon_9\int_{\Omega_\beta} f^2 dxdt.\end{aligned} \tag{2.59}$$

Then,

$$\begin{aligned}&\frac{3}{2}\int_{\Omega_\beta}\left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt-\frac{17}{8}\int_0^T\left(\int_\beta^1 \mathrm{D}_t^\alpha v dx\right)^2 dt\\&+2\int_0^T\left(\int_\beta^1 \mathrm{D}_t^\alpha v dx\right)\left(\int_\beta^1 x\mathrm{D}_t^\alpha v dx\right)dt\\&+\int_\beta^1\left(\frac{5}{4}-x\right)\int_0^T a(x,t)\frac{\partial v}{\partial x}.\frac{\partial}{\partial x}\mathrm{D}_t^\alpha v dtdx\\&\leq\left(\frac{1}{8\varepsilon_5}+\frac{1}{2\varepsilon_6}+\frac{1}{\varepsilon_9}\right)\int_{\Omega_\beta}\left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt\\&+2\varepsilon_{10}\int_0^T\left(\int_\beta^1 f dx\right)^2 dt\\&+\frac{2}{\varepsilon_{10}}\int_0^T\left(\int_\beta^1 \mathrm{D}_t^\alpha v dx\right)^2 dt\\&+\left(\frac{\varepsilon_5}{8}+\frac{\varepsilon_6}{2}+4\varepsilon_9\right)\int_{\Omega_\beta} f^2 dxdt.\end{aligned} \tag{2.60}$$

We are adding between (2.53) and (2.60), we obtain

$$\begin{aligned}&\frac{3}{2}\int_{\Omega_\alpha}\left(\int_x^\alpha \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt\\&+\frac{3}{2}\int_{\Omega_\beta}\left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt\\&+\frac{3}{8}\int_0^T\left(\int_\beta^1 \mathrm{D}_t^\alpha v dx\right)^2 dt\\&+\int_{\Omega_\alpha}(5-x)a(x,t)\mathrm{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)^2 dxdt\\&+\int_{\Omega_\beta}\left(\frac{5}{4}-x\right)a(x,t)\mathrm{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)^2 dxdt\\&\leq\left(2\varepsilon_1+\frac{\varepsilon_2}{2}+4\varepsilon_4\right)\int_{\Omega_\alpha} f^2 dxdt\\&+\left(\frac{2}{\varepsilon_1}+\frac{1}{2\varepsilon_2}+\frac{1}{\varepsilon_4}\right)\int_{\Omega_\alpha}\left(\int_x^\alpha \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt\\&+(\varepsilon_3+2\varepsilon_4)\int_0^T\left(\int_0^\alpha f dx\right)^2 dt\\&+\left(\frac{1}{\varepsilon_3}+\frac{2}{\varepsilon_{10}}\right)\int_0^T\left(\int_0^\alpha \mathrm{D}_t^\alpha v dx\right)^2 dt\\&+\left(\frac{\varepsilon_5}{8}+\frac{\varepsilon_6}{2}+4\varepsilon_9\right)\int_{\Omega_\beta} f^2 dxdt\\&+\left(\frac{1}{8\varepsilon_5}+\frac{1}{2\varepsilon_6}+\frac{1}{\varepsilon_9}\right)\int_{\Omega_\beta}\left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt\\&+2\varepsilon_{10}\int_0^T\left(\int_\beta^1 f dx\right)^2 dt.\end{aligned} \tag{2.61}$$

That implies

$$\frac{3}{2}\int_{\Omega_\alpha}\left(\int_x^\alpha \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + \frac{3}{2}\int_{\Omega_\beta}\left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + \frac{3}{8}\int_0^T\left(\int_\beta^1 \mathrm{D}_t^\alpha v dx\right)^2 dt + \int_{\Omega_\alpha}(5-x)a(x,t)\mathrm{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)^2 dxdt + \int_{\Omega_\beta}(\frac{5}{4}-x)a(x,t)\mathrm{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)^2 dxdt \leq \left(2\varepsilon_1+\frac{\varepsilon_2}{2}+4\varepsilon_4\right)\int_{\Omega_\alpha} f^2 dxdt + \left(\frac{2}{\varepsilon_1}+\frac{1}{2\varepsilon_2}+\frac{1}{\varepsilon_4}\right)\int_{\Omega_\alpha}\left(\int_x^\alpha \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + (\varepsilon_3+2\varepsilon_4)\int_0^T\left(\int_0^\alpha f dx\right)^2 dt + \left(\frac{1}{\varepsilon_3}+\frac{2}{\varepsilon_{10}}\right)\int_0^T\left(\int_0^\alpha \mathrm{D}_t^\alpha v dx\right)^2 dt + \left(\frac{\varepsilon_5}{8}+\frac{\varepsilon_6}{2}+4\varepsilon_9\right)\int_{\Omega_\beta} f^2 dxdt + \left(\frac{1}{8\varepsilon_5}+\frac{1}{2\varepsilon_6}+\frac{1}{\varepsilon_9}\right)\int_{\Omega_\beta}\left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + 2\varepsilon_{10}\int_0^\tau\left(\int_\beta^1 f dx\right)^2 dt. \tag{2.62}$$

If we put $\varepsilon_1=4, \varepsilon_2=2, \varepsilon_3=8, \varepsilon_4=2, \varepsilon_5=\varepsilon_6=1, \varepsilon_9=2$ and $\varepsilon_{10}=4$, we get

$$\frac{1}{4}\int_{\Omega_\alpha}\left(\int_x^\alpha \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + \frac{3}{8}\int_{\Omega_\beta}\left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + c_0\int_{\Omega_\alpha}(5-x)\mathrm{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)^2 dxdt + c_0\int_{\Omega_\beta}(\frac{5}{4}-x)\mathrm{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)^2 dxdt \leq 17\int_{\Omega_\alpha} f^2 dxdt + \frac{69}{8}\int_{\Omega_\beta} f^2 dxdt \; 12\int_0^T\left(\int_0^\alpha f dx\right)^2 dt + 8\int_0^\tau\left(\int_\beta^1 f dx\right)^2 dt. \tag{2.63}$$

We are adding between (2.58) and (2.63), we get

$$\int_{\Omega_\alpha}\left(\int_x^\alpha \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + \int_{\Omega_\beta}\left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + \int_{\Omega_\alpha}(5-x)\mathrm{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)^2 dxdt + \int_{\Omega_\beta}(\frac{5}{4}-x)\mathrm{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)^2 dxdt + \int_{\Omega_{\alpha,\beta}}\left(\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + \int_{\Omega_{\alpha,\beta}}\left(\int_\alpha^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + \int_{\Omega_{\alpha,\beta}}(\beta-\alpha)\mathrm{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)^2 dxdt$$

$$\leq \frac{c_4}{c_3}\left(\int_{\Omega_\alpha} f^2 dxdt + \int_{\Omega_\beta} f^2 dxdt + \int_0^T\left(\int_0^\alpha f dx\right)^2 dt + \int_0^T\left(\int_\beta^1 f dx\right)^2 dt + \int_{\Omega_{\alpha,\beta}} f^2 dxdt\right) \leq c_5\left(\int_{\Omega_\alpha} f^2 dxdt + \int_{\Omega_\beta} f^2 dxdt + \int_{\Omega_{\alpha,\beta}} f^2 dxdt + \int_0^T\left(\left(\int_0^\alpha f dx\right)^2 + \left(\int_\beta^1 f dx\right)^2\right) dt. \tag{2.64}$$

With $c_3=\min\left(\frac{1}{4},\ c_0,\ c_1\right), c_4=\max(17,c_2)$ et $c_5=\frac{c_4}{c_3}$.

Therefore, we get

$$\int_{\Omega_\alpha}\left(\int_x^\alpha \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + \int_{\Omega_\beta}\left(\int_\beta^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + \int_{\Omega_\alpha}(5-x)\mathrm{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)^2 dxdt + \int_{\Omega_\beta}(\frac{5}{4}-x)\mathrm{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)^2 dxdt + \int_{\Omega_{\alpha,\beta}}\left(\int_x^\beta \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + \int_{\Omega_{\alpha,\beta}}\left(\int_\alpha^x \mathrm{D}_t^\alpha v d\xi\right)^2 dxdt + \int_{\Omega_{\alpha,\beta}}(\beta-\alpha)\mathrm{D}_t^{\frac{\alpha}{2}}\left(\frac{\partial v}{\partial x}\right)^2 dxdt$$

$$\leq c_5 \left(\int_\Omega f^2 dxdt + \int_0^T \left(\left(\int_0^\alpha f dx \right)^2 dt + \left(\int_\beta^1 f dx \right)^2 \right) dt \right) \leq c_6 \int_\Omega f^2 dxdt. \tag{2.65}$$

where $c_6 = 1 + c_5$.

The right-hand side of (2.65) is independent of τ, hence replacing the left-hand side by its upper bound with respect to τ from 0 to T. Thus inequality (2.26) holds, where $c = (c_6)^{\frac{1}{2}}$. □

Proposition 2.2. *The operator L from $\mathbb{E}$ to $\mathbb{F}$ is closable .*

Proof. Suppose that $v_n \in \mathbf{D}(L)$ is a sequence such that

$$v_n \overset{n\to+\infty}{\longrightarrow} 0 \quad \text{in} \quad \mathbb{E}, \tag{2.66}$$

$$Lv_n \overset{n\to+\infty}{\longrightarrow} f \quad \text{in} \quad \mathbb{F}, \tag{2.67}$$

We must show $f \equiv 0$. Equation (2.66) implies that

$$v_n \overset{n\to+\infty}{\longrightarrow} 0 \quad \text{in} \quad \mathbf{D}'(\Omega). \tag{2.68}$$

By vertue of the continuity of derivation of $\mathbf{D}'(\Omega)$ in $\mathbf{D}'(\Omega)$, we have

$$\mathscr{L} v_n \overset{n\to+\infty}{\longrightarrow} 0 \quad \text{in} \quad \mathbf{D}'(\Omega). \tag{2.69}$$

We see via (2.67) that

$$\mathscr{L} v_n \overset{n\to+\infty}{\longrightarrow} f \quad \text{in} \quad L_2(\Omega), \tag{2.70}$$

then

$$\mathscr{L} v_n \overset{n\to+\infty}{\longrightarrow} f \quad \text{in} \quad \mathbf{D}'(\Omega). \tag{2.71}$$

By virtue of the uniqueness of the limit in $\mathbf{D}'(\Omega)$, (2.69) and (2.71) imply that $f \equiv 0$. □

Definition 2.3. *A solution of the equation*

$$\overline{L}v = f, \tag{2.72}$$

is called a strong solution of problem (2.16), (2.17), (2.18), (2.19), (2.20), (2.21) and (2.22).

Since points of the graph of $\overline{L}$ are limits of sequences of points of the graph of L, we extend (2.26) to apply to strong solutions by taking the limits.

Corollary 2.4. *Under the conditions of Theorem 2.1, there is a constant $C > 0$ independent of v such that*

$$\|v\|_{\mathbb{E}} \leq \|\overline{L}v\|_{\mathbb{F}}, \qquad v \in \mathbf{D}'(\Omega). \tag{2.73}$$

Corollary 2.5. *Assert that, if a strong solution exists, it is unique and depends continuously on f, if v is considered in the topology of $\mathbb{E}$ and f is considered in the topology of $\mathbb{F}$.*

Corollary 2.6. *The rang $R(\overline{L})$ of the operator $\overline{L}$ is closed in $\mathbb{F}$ and $R(\overline{L}) = \overline{R(L)}$, where $R(L)$ is the range of L.*

3. Solvability of the problem

To show the existence of solutions, we prove that $R(L)$ is dense in $\mathbb{F}$ for all $v \in \mathbf{D}(L)$ and for all arbitrary $f \in \mathbb{F}$.

Theorem 3.1. *Suppose the conditions of Theorem 2.1 are satisfied. Then the problem (2.16)-(2.22) admits a unique strong solution $v = \overline{L}^{-1} f = \overline{L^{-1}} f$.*

Proof. First we prove that $R(L)$ is dense in $\mathbb{F}$ for all $v \in \mathbf{D}(L)$. □

Proposition 3.2. *Let the conditions of Theorem (3.1) be satisfied, if, for $\omega \in L^2(\Omega)$ and for all $v \in \mathbf{D}(L)$, we have*

$$\int_\Omega Lv\omega dxdt = 0, \tag{3.1}$$

then ω vanishes almost everywhere in Ω

Proof. The scalar product of $\mathbb{F}$ is defined by

$$(Lv,\ \omega)_{\mathbb{F}} = \int_\Omega Lv\omega dxdt, \tag{3.2}$$

then, equality (3.1) can be written as

$$\int_\Omega \mathtt{D}_t^\alpha v\omega dxdt = \int_\Omega \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial v}{\partial x}\right) \omega dxdt. \tag{3.3}$$

If we put

$$v = \mathfrak{F}_t(z(x,\tau)) = \int_0^t z(x,\tau)d\tau,$$

where

$$z, \quad \frac{\partial z}{\partial x}, \quad \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial \mathfrak{F}_t(z(x,\tau))}{\partial x}\right), \quad \mathtt{D}_t^\alpha z, \quad \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial \mathfrak{F}_t(z(x,\tau))}{\partial x}\right) \in L^2(\Omega).$$

As a result of (3.3), we obtain

$$\int_\Omega \mathtt{D}_t^\alpha \mathfrak{F}_t(z(x,\tau))\omega dxdt = \int_\Omega \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial \mathfrak{F}_t(z(x,\tau))}{\partial x}\right) \omega dxdt. \tag{3.4}$$

In terms of the given function ω, and from the equality (3.4) we give the function ω in terms of z as

$$\omega = \begin{cases} \omega_1, & 0 \leq x \leq \alpha, \\ \omega_2, & \alpha \leq x \leq \beta, \\ \omega_3, & \beta \leq x \leq 1, \end{cases} \tag{3.5}$$

where

$$\omega_1 = \int_x^\alpha \int_0^\xi \mathfrak{F}_t(z(\eta,\tau))d\eta d\xi, \tag{3.6}$$

$$\omega_2 = \int_x^\beta \int_\alpha^\xi \mathfrak{F}_t(z(\eta,\tau))d\eta d\xi, \tag{3.7}$$

$$\omega_3 = \int_x^1 \int_\beta^\xi \mathfrak{F}_t(z(\eta,\tau))d\eta d\xi. \tag{3.8}$$

So, $\omega \in L^2(\Omega)$, and z satisfy the same conditions of the function v and $\frac{\partial^2 z}{\partial x^2}|_{x=\alpha} = 0, \qquad \frac{\partial^2 z}{\partial x^2}|_{x=\beta} = 0.$

Replacing ω in (3.4) by its representation (3.5) and integrating by parts each term of (3.4) with the use of conditions of z, we obtain

- On the interval $\Omega_\alpha = (0,\alpha)\times(0,T)$, we have

$$\int_{\Omega_\alpha} \mathtt{D}_t^\alpha \mathfrak{F}_t(z(x,\tau))\omega_1 dxdt = \int_{\Omega_\alpha} \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial \mathfrak{F}_t(z(x,\tau))}{\partial x}\right)\omega_1 dxdt. \tag{3.9}$$

Integrating by parts each integral of (3.9) and by using the conditions of the function z, we get

$$\int_{\Omega_\alpha} \mathtt{D}_t^\alpha \mathfrak{F}_t(z(x,\tau))\omega_1 dxdt = \int_{\Omega_\alpha}\left(\mathtt{D}_t^{\frac{\alpha}{2}}\left(\int_\alpha^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt, \tag{3.10}$$

$$\begin{aligned}&\int_{\Omega_\alpha} \frac{\partial^2}{\partial x^2}\left(a\frac{\partial \mathfrak{F}_t(z(x,\tau))}{\partial x}\right) \times\left(\int_x^\alpha\int_0^\xi \mathfrak{F}_t(z(\eta,\tau))d\eta d\xi\right)dxdt\\ &= -\frac{1}{2}\int_0^\tau a\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2|_{x=0}^{x=\alpha}dt + \frac{1}{2}\int_{\Omega_\alpha}\frac{\partial a}{\partial x}\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2 dxdt\\ &\le \int_0^\tau\left(-\frac{1}{2}a + \frac{\partial a}{\partial x}\right)\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2 dt. \end{aligned} \tag{3.11}$$

Substituting (3.10) and (3.11), we have

$$\int_{\Omega_\alpha}\left(\mathtt{D}_t^{\frac{\alpha}{2}}\left(\int_\alpha^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt \le \int_0^\tau\left(-\frac{1}{2}a + \frac{\partial a}{\partial x}\right)\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2 dt. \tag{3.12}$$

Since $-\frac{1}{2}a(x,\ t) + \frac{\partial a(x,\ t)}{\partial x} \le 0$, we have

$$\int_{\Omega_\alpha}\left(\mathtt{D}_t^{\frac{\alpha}{2}}\left(\int_\alpha^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt \le 0. \tag{3.13}$$

- On the interval $\Omega_{\alpha,\beta} = (\alpha,\beta)\times(0,T)$, we obtain

$$\int_{\Omega_{\alpha,\beta}} \mathtt{D}_t^\alpha \mathfrak{F}_t(z(x,\tau))\omega_2 dxdt = \int_{\Omega_{\alpha,\beta}} \frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial \mathfrak{F}_t(z(x,\tau))}{\partial x}\right)\omega_2 dxdt. \tag{3.14}$$

Integrating by parts each term of (3.14) and taking account conditions of the function z

$$\begin{aligned}&\int_{\Omega_{\alpha,\beta}} \mathtt{D}_t^\alpha \mathfrak{F}_t(z(x,\tau))\omega_2 dxdt\\ &= \int_{\Omega_{\alpha\beta}} \mathtt{D}_t^\alpha\left(\mathfrak{F}_t(z(\eta,\tau))\right) \times\left(\int_x^\beta\int_\alpha^\xi \mathfrak{F}_t(z(\eta,\tau))d\eta d\xi\right)dxdt\\ &= \int_{\Omega_{\alpha,\beta}}\left(\mathtt{D}_t^{\frac{\alpha}{2}}\left(\int_\alpha^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt.\end{aligned} \tag{3.15}$$

Then

$$\int_{\Omega_{\alpha,\beta}} \mathtt{D}_t^\alpha \mathfrak{F}_t(z(x,\tau))\omega_2 dxdt = \int_{\Omega_{\alpha,\beta}}\left(\mathtt{D}_t^{\frac{\alpha}{2}}\left(\int_\alpha^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt. \tag{3.16}$$

$$\begin{aligned}&\int_{\Omega_{\alpha,\beta}} \frac{\partial^2}{\partial x^2}\left(a\frac{\partial \mathfrak{F}_t(z(x,\tau))}{\partial x}\right) \times\left(\int_x^\beta\int_\alpha^\xi \mathfrak{F}_t(z(\eta,\tau))d\eta d\xi\right)dxdt\\ &= -\frac{1}{2}\int_0^\tau a\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2|_{x=\alpha}^{x=\beta}dt + \frac{1}{2}\int_{\Omega_{\alpha,\beta}}\frac{\partial a}{\partial x}\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2 dxdt.\end{aligned}$$

Combining the above expression and (3.16), we arrive at

$$\begin{aligned}&\int_{\Omega_{\alpha,\beta}}\left(\mathtt{D}_t^{\frac{\alpha}{2}}\left(\int_\alpha^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt\\ &= -\frac{1}{2}\int_0^\tau a\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2|_{x=\alpha}^{x=\beta}dt + \frac{1}{2}\int_{\Omega_{\alpha,\beta}}\frac{\partial a}{\partial x}\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2 dxdt.\end{aligned} \tag{3.17}$$

Estimated the right-hand side of (3.17), we get

$$\begin{aligned}&\int_{\Omega_{\alpha,\beta}}\left(\mathtt{D}_t^{\frac{\alpha}{2}}\left(\int_\alpha^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt\\ &= -\frac{1}{2}\int_0^\tau a\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2|_{x=\alpha}^{x=\beta}dt + \frac{1}{2}\int_{\Omega_{\alpha,\beta}}\frac{\partial a}{\partial x}\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2 dxdt\\ &\le \frac{1}{2}\int_0^\tau\left(-a + 3\frac{\partial a}{\partial x}\right)\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2 dt.\end{aligned} \tag{3.18}$$

Hence, if $a(x,\ t)-3\dfrac{\partial a(x,\ t)}{\partial x}\geq 0$, we have

$$\int_{\Omega_{\alpha,\beta}}\left(\mathrm{D}_t^{\frac{\alpha}{2}}\left(\int_\alpha^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt\leq 0. \tag{3.19}$$

- On the interval $\Omega_\beta=(\beta,1)\times(0,\tau)$, we obtain

$$\int_{\Omega_\beta}\mathrm{D}_t^\alpha\mathfrak{F}_t(z(x,\tau))\omega_3 dxdt = \int_{\Omega_\beta}\frac{\partial^2}{\partial x^2}\left(a(x,t)\frac{\partial\mathfrak{F}_t(z(x,\tau))}{\partial x}\right)\omega_3 dxdt. \tag{3.20}$$

Integrating by parts each term of (3.20) and using the conditions of the function z, we have

$$\int_{\Omega_\beta}\mathrm{D}_t^\alpha\mathfrak{F}_t(z(x,\tau))\omega_3 dxdt=\int_{\Omega_\beta}\mathrm{D}_t^\alpha\left(\mathfrak{F}_t(z(\eta,\tau))\right)\times\left(\int_x^1\int_\beta^\xi\mathfrak{F}_t(z(\eta,\tau))d\eta d\xi\right)dxdt,$$

hence

$$\int_{\Omega_\beta}\mathrm{D}_t^\alpha\mathfrak{F}_t(z(x,\tau))\omega_3 dxdt = \int_{\Omega_\beta}\left(\mathrm{D}_t^{\frac{\alpha}{2}}\left(\int_\beta^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt. \tag{3.21}$$

$$\begin{aligned}&\int_{\Omega_\beta}\frac{\partial^2}{\partial x^2}\left(a\frac{\partial\mathfrak{F}_t(z(x,\tau))}{\partial x}\right)\\&\times\left(\int_x^1\int_\beta^\xi\mathfrak{F}_t(z(\eta,\tau))d\eta d\xi\right)dxdt\\&=-\frac{1}{2}\int_0^\tau a\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2\Big|_{x=\beta}^{x=1}dt\\&+\frac{1}{2}\int_{\Omega_\beta}\frac{\partial a}{\partial x}\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2 dxdt.\end{aligned} \tag{3.22}$$

We now estimate the right-hand side of (3.22) as follows

$$\begin{aligned}&-\frac{1}{2}\int_0^\tau a\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2\Big|_{x=\beta}^{x=1}dt\\&+\frac{1}{2}\int_{\Omega_\beta}\frac{\partial a}{\partial x}\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2 dxdt\\&\leq\int_0^\tau\left(-\frac{1}{2}a+\frac{\partial a}{\partial x}\right)\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2 dt.\end{aligned} \tag{3.23}$$

Thus we have, by virtue of (3.22)

$$\begin{aligned}&\int_{\Omega_\beta}\frac{\partial^2}{\partial x^2}\left(a\frac{\partial\mathfrak{F}_t(z(x,\tau))}{\partial x}\right)\\&\times\left(\int_x^1\int_\beta^\xi\mathfrak{F}_t(z(\eta,\tau))d\eta d\xi\right)dxdt\\&\leq\int_0^\tau\left(-\frac{1}{2}a+\frac{\partial a}{\partial x}\right)\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2 dt.\end{aligned} \tag{3.24}$$

By combining (3.21) and (3.24), we arrive at

$$\begin{aligned}&\int_{\Omega_\beta}\left(\mathrm{D}_t^{\frac{\alpha}{2}}\left(\int_\beta^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt\\&\leq\int_0^\tau\left(-\frac{1}{2}a+\frac{\partial a}{\partial x}\right)\left(\mathfrak{F}_t(z(\xi,\tau))\right)^2 dt.\end{aligned} \tag{3.25}$$

Using that $\frac{1}{2}a(x,\ t)-\dfrac{\partial a(x,\ t)}{\partial x}\geq 0$, we have following estimated

$$\int_{\Omega_\beta}\left(\mathrm{D}_t^{\frac{\alpha}{2}}\left(\int_\beta^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt\leq 0. \tag{3.26}$$

A summation of (3.13), (3.19) and (3.26) leads to

$$\begin{aligned}&\int_{\Omega_\alpha}\left(\mathrm{D}_t^{\frac{\alpha}{2}}\left(\int_\alpha^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt\\&+\int_{\Omega_{\alpha,\beta}}\left(\mathrm{D}_t^{\frac{\alpha}{2}}\left(\int_\alpha^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt\\&+\int_{\Omega_\beta}\left(\mathrm{D}_t^{\frac{\alpha}{2}}\left(\int_\beta^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt\leq 0.\end{aligned} \tag{3.27}$$

Since

$$\begin{aligned}&\int_{\Omega_\alpha}\left(\mathrm{D}_t^{\frac{\alpha}{2}}\left(\int_\alpha^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt\\&+\int_{\Omega_{\alpha,\beta}}\left(\mathrm{D}_t^{\frac{\alpha}{2}}\left(\int_\alpha^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt\\&+\int_{\Omega_\beta}\left(\mathrm{D}_t^{\frac{\alpha}{2}}\left(\int_\beta^x \mathfrak{F}_t(z(x,\tau)d\xi)\right)\right)^2 dxdt=0,\end{aligned} \tag{3.28}$$

we conclude that $z=0$; hence $\omega=0$, which ends the proof of the proposition 3.2. □

We return to the proof of Theorem 3.1. We have already noted that it is sufficient to prove that the set $R(L)$ is dense in $\mathbb{F}$.

Suppose that, for some $\omega\in R(L)^\perp$ and for all $v\in\mathbf{D}(L)$, we have

$$(Lv,\ \omega)_{L^2(\Omega)}=\int_\Omega Lv\omega dxdt=0.$$

Hence Proposition 3.2 implies that $\omega=0$.
We have just proved that $R(L)^\perp=\{0_{\mathbb{F}}\}$, then $R(L)$ is dense in $\mathbb{F}$.

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