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Magnetic trajectories on oriented surfaces

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Abstract
We study magnetic trajectories of a magnetic field on an oriented surface S in three-dimensional Euclidean space. Defining the Lorentz force of a magnetic field on S, we give the Lorentz force equation for the associated magnetic trajectories. We have derived the Killing magnetic flow equations with regard to the geodesic curvature, geodesic torsion and normal curvature of the curve γ on S. Finally we examine magnetic trajectories on some familiar surfaces in three-dimensional Euclidean space.

Keywords
Magnetic curve, Lorentz force equation, geodesic curvature.

AMS Subject Classification
82D40, 53A05, 35A15.

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Contents

1. Introduction

Shape of the magnetic flow in a magnetic field is known a magnetic trajectory. A magnetic trajectory of a magnetic flow is significant in the study of the Hall effect. The Landau-Hall problem is the study of the motion of a charged particle in the existence of a constant and static magnetic field on a Riemannian surface. In this case, a particle of mass and charge evolve with velocity vector satisfying the Lorentz force law [5]. So, the problem includes the study of solutions of the Lorentz force equation associated with a Killing magnetic field. The Lorentz force equation and Hall effect allow to some implementations in many research areas in physics, chemistry, atmospheric and computer sciences, etc. [1]. Recent studies of characterization of magnetic flow in a magnetic field have brought variational perspective. Therefore the techniques of the differential geometry could be used in the construction of the equations. So the solutions of the equations could be interpreted with a more geometric point of view [1, 2, 4, 5, 7, 8, 9, 10]. For example, in [1], authors have completely solved the Landau-Hall problem for Killing magnetic fields in Riemannian manifolds with constant sectional curvature by using the techniques of the differential geometry, variational methods and Jacobi elliptic functions. Also in [2], authors show that magnetic trajectories of a Killing magnetic field correspond to the centerlines of the Kirchhoff elastic rods. Any divergence-free vector field in three-dimensional Euclidean space defines a magnetic field. The magnetic trajectory of a magnetic flow is a curve such that it stands for the trajectory of a charged particle moving on the surface S under the action of a magnetic field F defined as a closed $2-$form on (see, [7]). The Lorentz force of F denoted by a skew symmetric operator ϕ is given by

$$F(X,Y) = <\phi(X), Y> \tag{1.1}$$

for $X, Y \in \chi(S)$. On a oriented surface S, the magnetic trajectories of F satisfies the Lorentz equation (or known Newton equation)

$$\phi(\gamma') = \nabla_{\gamma'}\gamma'$$

(see, [7]). On the other hand, a vector field V on a surface S is a Killing vector field if and only if it satisfies the Killing equation,

$$\mathscr{L}_V < X,Y > = < \nabla_X V, Y > + < \nabla_Y V, Z > = 0$$

for every vector fields X, Y, Z on S, where ∇ is the covariant derivative (see [3]). Since a Killing field on S is determined by a vector and its gradient, its divergence is free and it automatically defines a magnetic flow which is known Killing magnetic flow [1].

In this paper we examine the magnetic curves representing the magnetic trajectories of charged particles on oriented surfaces in three-dimensional Euclidean space $\mathbb{R}^3$. By using the Darboux frame equations along a curve γ on S, we characterize the magnetic field V. Then we derive the Killing magnetic flow equations. Finally we apply to this formulation curves on some surfaces in $\mathbb{R}^3$.

2. Preliminaries

Consider a connected oriented surface S in three-dimensional Euclidean space $\mathbb{R}^3$. F denotes a complete differential $2-$form in a open subset U of S with $F = d\omega$, where ω is a $1-$form. If we define Γ as a set of C^∞ curves that combine two fixed point of U, the Lorentz force equation is a minimizer of the functional $\mathscr{F} : \Gamma \to \mathbb{R}$ defined by

$$\mathscr{F}(\gamma) = \frac{1}{2}\int_\gamma <\gamma',\gamma'> dt + \int_\gamma \omega(\gamma')\, dt.$$

Then the Euler-Lagrange equation of the functional $\mathscr{F}$ is derived as

$$\phi(\gamma') = \nabla_{\gamma'}\gamma'. \tag{2.1}$$

An extremal of $\mathscr{F}$ corresponds to a solution of the Lorentz force equation [1].

Since magnetic fields are seen to be divergence-free vector fields in three-dimension, if $V \in \chi(S)$ has a zero divergence, it defines a magnetic vector field on S. Let the differential $2-$form F be a magnetic field with the skew-symmetric Lorentz force operator ϕ on S. A curve γ which is the associated magnetic trajectories on S satisfies the Lorentz force equation. The cross product $X \times Y$ on S can be defined by

$$<X \times Y, Z> = \Omega_3(X,Y,Z),$$

where Ω_3 is area element of S. Then ϕ associated with a magnetic field $F = i_V\Omega_3$ can be obtained as follows

$$<\phi(X),Y> = F(X,Y) = i_V\Omega_3(V,X,Y) = <V \times X, Y>.$$

So one can see that

$$\phi(X) = V \times X, \tag{2.2}$$

i.e., ϕ is defined via cross product on S. Combining (2.1) and (2.2), the Lorentz force equation can be written by

$$\phi(\gamma') = \nabla_{\gamma'}\gamma' = V \times \gamma'$$

for a curve γ on S.

Assume that $\gamma : I \to S$ is a parametrized curve with arc length s, $0 \le s \le \ell$. The curvature and the torsion functions of γ are resp. denoted by $\kappa(s)$ and $\tau(s)$. Let $T(s) = \gamma'(s)$ and n denote the unit tangent vector to γ and the unit normal vector field of S, resp. Then the trihedron $\{T,Q,n\}$ along γ determines an orthonormal frame field, known as Darboux frame field. The Darboux frame field is defined as follows

$$T(s) = \gamma'(s),\ Q(s) = n(s) \times T(s),\ n(s) = n(\gamma(s)) \tag{2.3}$$

and the frame has the following derivative equations

$$\begin{pmatrix} T' \\ Q' \\ n' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ Q \\ n \end{pmatrix}, \tag{2.4}$$

where κ_g is the geodesic curvature, κ_n is the normal curvature and τ_g is the geodesic torsion of γ.

The square curvature κ^2 and the torsion τ of γ on S have the following relations

$$\kappa^2 = \kappa_g^2 + \kappa_n^2 \tag{2.5}$$

and

$$\tau = \tau_g + \frac{\kappa_n'\kappa_g - \kappa_g'\kappa_n}{\kappa_g^2 + \kappa_n^2}, \tag{2.6}$$

(see [6], [11]).

3. Magnetic Fields According to Darboux Frame

Let S denote a connected oriented surface in three-dimensional Euclidean space and F denote a magnetic field on S. We first give a proposition which expresses the Lorentz force in terms of the Darboux frame field.

Proposition 1. Suppose that $\gamma : I \subset \mathbb{R} \to S \subset \mathbb{R}^3$ is a parametrized magnetic curve and $\{T,Q,n\}$ is the Darboux frame field along γ. Then the Lorentz force ϕ in the Darboux frame $\{T,Q,n\}$ is written as follows

$$\phi(T) = \kappa_g Q + \kappa_n n, \tag{3.1}$$

$$\phi(Q) = -\kappa_g T + \omega n \tag{3.2}$$

and

$$\phi(n) = -\kappa_n T - \omega Q, \tag{3.3}$$

where the function ω associated with each magnetic curve is the quasislope measured according to the magnetic field V.

Proof. The unit tangent vector at a point $\gamma(s)$ of γ is $T(s) = \gamma'(s)$. Then from Eq. (2.1) and Darboux formulas (2.4), we have

$$\phi(T) = \nabla_T T = \kappa_g Q + \kappa_n n.$$

We can write the linear expansion of $\phi(Q)$, $\phi(n) \in S$ as follows

$$\phi(Q) =<\phi(Q), T>T+<\phi(Q), Q>Q+<\phi(Q), n>n$$

and

$$\phi(n) =<\phi(n), T>T+<\phi(n), Q>Q+<\phi(n), n>n,$$

resp. Taking into consideration Eq. (2.2) and Eq. (3.1), we get

$$\begin{aligned} <\phi(Q), T> &=< V\times Q, T>=-<V\times T, Q> \\ &=-<\phi(T), Q>=-\kappa_g \end{aligned}$$

and

$$\begin{aligned} <\phi(n), T> &=< V\times n, T>=-<V\times T, n> \\ &=-<\phi(T), n>=-\kappa_n. \end{aligned}$$

Since ϕ is a skew-symmetric operator, we get

$$<\phi(Q), Q>=<\phi(n), n>=0.$$

On the other hand, we have from (2.3)

$$\omega =<\phi(Q), n>=-<V\times n, Q>=-<\phi(n), Q>.$$

Proposition 2. A parametrized curve $\gamma : I\to S$ is a magnetic trajectory of a magnetic field V iff it is written along γ by

$$V=\omega T-\kappa_n Q+\kappa_g n. \tag{3.4}$$

Proof. Suppose that γ is a magnetic curve along a magnetic field V and the Darboux frame along γ is given by $\{T, Q, n\}$. Then, V can written as

$$V =<V, T>T+<V, Q>Q+<V, n>n.$$

To find coefficient of V, we use the Lorentz force in Darboux frame equations $(3.1-3.3)$:

$$\begin{aligned} \omega &= <\phi(Q), n>=<V\times Q, n>=<V, T>, \\ \kappa_n &= <\phi(T), n>=<V\times T, n>=-<V, Q> \end{aligned}$$

and

$$-\kappa_g =<\phi(Q), T>=<V\times Q, T>=-<V, n>.$$

4. Killing Magnetic Flow Equations

Let $\gamma : I\to S$ be a curve on oriented surface in three-dimensional Euclidean space. Suppose the V is a vector field along γ. A variation of γ in the direction of V can be defined by

$$\begin{aligned} \Upsilon : \quad [0,1]\times(-\varepsilon, \varepsilon) &\to S \\ (s,t) &\to \Upsilon(s,t) \end{aligned}$$

satisfying

$$\Upsilon(s,0)=\gamma(s), \ \left(\frac{\partial \Upsilon(s,t)}{\partial t}\right)_{t=0}=V(s), \left(\frac{\partial \Upsilon(s,t)}{\partial s}\right)_{t=0}=\gamma'(s).$$

One can write $\upsilon(s,t)=\left\|\frac{\partial \Upsilon(s,t)}{\partial s}\right\|$, $\kappa(s,t)$ and $\tau(s,t)$ (see [1]).

Lemma 3. We consider that $\gamma : I\subset \mathbb{R}\to S$ is a curve in $\mathbb{R}^3$ and V is a vector field along the curve γ. Then we have the following equalities

$$V(\upsilon)=\left(\frac{\partial \upsilon(s,t)}{\partial t}\right)_{t=0} =<\nabla_T V, T>\upsilon, \tag{4.1}$$

$$V(\kappa)=\left(\frac{\partial \kappa(s,t)}{\partial t}\right)_{t=0} =\frac{1}{\kappa}<\nabla_T^2 V, \nabla_T T>-2\kappa<\nabla_T V, T> \tag{4.2}$$

and

$$\begin{aligned} V(\tau)=\left(\frac{\partial \tau(s,t)}{\partial t}\right)_{t=0} &= \left(\frac{1}{\kappa^2}<\nabla^2_{\frac{d}{ds}} V, T\times\nabla_T T>\right)' \\ &+\tau<\nabla_T V, T>+<\nabla_T V, T\times\nabla_T T>. \end{aligned} \tag{4.3}$$

Proposition 4. Let γ be a regular curve on S in three-dimensional Euclidean space $\mathbb{R}^3$. If $V(s)$ is the restriction to $\gamma(s)$ of a Killing vector field, then we have for V in $\mathbb{R}^3$

$$V(\upsilon)=V(\kappa)=V(\tau)=0. \tag{4.4}$$

Theorem 5. Let γ be a parametrized curve on an oriented surface S in $\mathbb{R}^3$. Suppose that $V=\omega T-\kappa_n Q+\kappa_g n$, where ω is the quasislope, a Killing vector field along γ. Then magnetic trajectories are curves on S satisfying following differential equations

$$b\kappa_g+c\kappa_n=0 \tag{4.5}$$

and

$$\begin{aligned} &\left(\frac{1}{\kappa_g^2+\kappa_n^2}\right)'(c\kappa_g-b\kappa_n)+\frac{1}{\kappa_g^2+\kappa_n^2}\left(\kappa_g\left(a\kappa_n+b\tau_g+c'\right)\right. \\ &\left.-\kappa_n\left(a\kappa_g+b'-c\tau_g\right)-b\left(\kappa_g\tau_g+\kappa_n'\right)+c\left(\kappa_g'-\kappa_n\tau_g\right)\right) \\ &+\kappa_g\left(\omega\kappa_n-\kappa_n\tau_g+\kappa_g'\right)-\kappa_n\left(\omega\kappa_g-\kappa_n'-\kappa_g\tau_g\right)=0, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} a&=-\omega\left(\kappa_g^2+\kappa_n^2\right)+\tau_g\left(\kappa_g^2+\kappa_n^2\right)+\kappa_g\kappa_n'-\kappa_n\kappa_g', \\ b&=-\kappa_n''+\omega\kappa_g'-2\kappa_g'\tau_g-\kappa_g\tau_g'-\omega\kappa_n\tau_g+\kappa_n\tau_g^2, \\ c&=\kappa_g''-2\kappa_n'\tau_g-\kappa_n\tau_g'-\kappa_g\tau_g^2+\omega\kappa_g\tau_g+\omega\kappa_n' \end{aligned}$$

Proof. Assume that V is a Killing vector field along γ on S. Along any magnetic trajectory γ, we have $V=\omega T-\kappa_n Q+\kappa_g n$. If V is Killing vector field, we calculate

$$\begin{aligned} V(\upsilon) =&< \omega' T+\left(\omega\kappa_g-\kappa_n'-\kappa_g\tau_g\right)Q \\ &+\left(\omega\kappa_n-\kappa_n\tau_g+\kappa_g'\right)n, T>. \end{aligned}$$

From Eq.(4.1) and Eq. (4.4), we have

$$\omega' = 0,$$

that is ω is a constant, and

$$\nabla_T V = \left(\omega\kappa_g - \kappa_n' - \kappa_g\tau_g\right)Q + \left(\omega\kappa_n - \kappa_n\tau_g + \kappa_g'\right)n. \quad (4.7)$$

We calculate first derivative of (4.7) as follows

$$\begin{aligned}\nabla_T^2 V = &\left(-\omega\left(\kappa_g^2+\kappa_n^2\right) + \tau_g\left(\kappa_g^2+\kappa_n^2\right) + \kappa_g\kappa_n' - \kappa_n\kappa_g'\right)T \\ &\left(-\kappa_n'' + \omega\kappa_g' - 2\kappa_g'\tau_g - \kappa_g\tau_g' - \omega\kappa_n\tau_g + \kappa_n\tau_g^2\right)Q \\ &\left(\kappa_g'' - 2\kappa_n'\tau_g - \kappa_n\tau_g' - \kappa_g\tau_g^2 + \omega\kappa_g\tau_g + \omega\kappa_n'\right)n \\ &= aT + bQ + cn.\end{aligned} \quad (4.8)$$

Substituting Eq. (4.8) into Eq. (4.2) and using Eq. (4.4), we get

$$\begin{aligned}V(\kappa) = &\left(-\kappa_n'' + \omega\kappa_g' - 2\kappa_g'\tau_g - \kappa_g\tau_g' - \omega\kappa_n\tau_g + \kappa_n\tau_g^2\right)\kappa_g \\ &\left(\kappa_g'' - 2\kappa_n'\tau_g - \kappa_n\tau_g' - \kappa_g\tau_g^2 + \omega\kappa_g\tau_g + \omega\kappa_n'\right)\kappa_n \\ &= b\kappa_g + c\kappa_n = 0.\end{aligned}$$

Finally, we find to $V(\tau)$ as follows

$$\begin{aligned}V(\tau) = &\left(\tfrac{1}{\kappa^2}\right)' < \nabla_T^2 V, T\times\nabla_T T > + \left(\tfrac{1}{\kappa^2}\right) < \nabla_T^3 V, T\times\nabla_T T > \\ &+ \left(\tfrac{1}{\kappa^2}\right) < \nabla_T^2 V, T\times\nabla_T^2 T > + < \nabla_T V, T\times\nabla_T T > .\end{aligned} \quad (4.9)$$

By using the Darboux trihedron and formulas (2.4), we can obtain the following cross products

$$T\times\nabla_T T = \kappa_g n + -\kappa_n Q \quad (4.10)$$

and

$$T\times\nabla_T^2 T = -\left(\kappa_g\tau_g - \kappa_n'\right)Q + \left(\kappa_g' - \kappa_n\tau_g\right)n. \quad (4.11)$$

Taking first derivative of (4.8), we obtain

$$\begin{aligned}\nabla_T^3 V = &\left(a' - b\kappa_g - c\kappa_n\right)T + \left(a\kappa_g + b' - c\tau_g\right)Q \\ &+ \left(a\kappa_n + b\tau_g c'\right)n.\end{aligned} \quad (4.12)$$

Substituting Eqs. (2.5), (4.5), (4.10), (4.12) and (4.8) into (4.9), we derive Eq. (4.6).

Definition 6. Any regular curve on oriented surface S is called the magnetic trajectory of a magnetic field V if it satisfies the differential equation system (4.5) and (4.6).

5. Applications

Magnetic trajectories on a plane: The plane curves have identically zero torsion and normal curvature [11]. A regular curve γ on a plane is a magnetic trajectory of a magnetic field V iff it can be written along γ as follows

$$V = \omega T + \kappa_g n.$$

Thus we can give the following corollary.

Corollary 7. Any regular curve γ with constant geodesic curvature on a plane is a magnetic trajectory of the magnetic field V. So, the parts of a circle and the geodesic on a plane are magnetic trajectories of the magnetic field V.

Magnetic trajectories on a sphere: We consider the sphere with radius r,

$$\mathbb{S}^2(r) = \left\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1^2+x_2^2+x_3^2 = r^2\right\}.$$

The geodesic torsion τ_g vanishes for all curves on $\mathbb{S}^2(r)$ and normal curvature $\kappa_n^2 = \frac{1}{r^2}$ [11]. Then a regular curve γ on $\mathbb{S}^2(r)$ is a magnetic trajectory of a magnetic field V iff it can be written along γ as

$$V = \omega T - \frac{1}{r}Q + \kappa_g n.$$

For given magnetic trajectory of the magnetic field V, the Killing magnetic flow equations (4.5) and (4.6) reduce to

$$\omega\frac{r}{2}\left(\kappa_g^2\right)' + \kappa_g'' = 0 \quad (5.1)$$

and

$$\begin{aligned}\left(\tfrac{r^2}{r^2\kappa_g^2+1}\right)'\left(\kappa_g''\kappa_g - \tfrac{1}{r}\omega\kappa_g'\right) + \tfrac{r^2}{r^2\kappa_g^2+1}\left(\kappa_g'''\kappa_g - \tfrac{1}{r}\omega\kappa_g'' + \kappa_g''\kappa_g'\right) \\ +\kappa_g\kappa_g' = 0.\end{aligned} \quad (5.2)$$

Theorem 8. The Killing magnetic flow equations which gives the magnetic trajectory of the magnetic field V on the sphere is given by differential equations (5.1) and (5.2).

Corollary 9. Any regular curve γ with constant geodesic curvature (such as small circles) on the sphere $\mathbb{S}^2(r)$ is a magnetic trajectory of the magnetic field V. So, the geodesic on a sphere $\mathbb{S}^2(r)$ is a magnetic trajectory of the magnetic field V.

Magnetic trajectories on a cylinder: Let the cylinder be parametrized by

$$x(u,v) = \left(r\cos\frac{u}{r}, r\sin\frac{u}{r}, v\right),$$

where r is radius of the circle. Then for an arbitrary arc γ on the cylinder

$$\kappa_g = \frac{d\theta}{ds},\ \kappa_n = -\frac{1}{r}\cos^2\theta \text{ and } \tau_g = \frac{1}{r}\cos\theta\sin\theta,$$

where $\theta = \theta(s)$ is the angle between the $u-$coordinate curve through $\gamma(s)$ and the arc γ. The geodesics on the cylinder are characterized by $\theta =$constant (see **[**11**]**). Then, we clearly see the following corollary;

Corollary 10. All geodesics (straight lines, circles and helices) on a cylinder are magnetic trajectories of the magnetic field V.

Conclusion. In this work we consider magnetic curves on oriented surface S in Euclidean $3-$space $\mathbb{R}^3$. To deriving the

Killing magnetic flow equations which determine magnetic curves on S, we use the Darboux frame field which made it possible to know more about the curvatures of a curve on the surface. We apply this formulation to give results about magnetic curves on a plane, a sphere and a cylinder surfaces. We show that geodesics of these surfaces are magnetic curves. Moreover we exhibit some examples for magnetic trajectories in the magnetic field on these surfaces with different approach in [1], [5] and [7].

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