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Integer partitions revisited using multisets and binary matrices

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Abstract
In this paper, partition of a positive integer is represented as a multiset and binary matrix. New formulae and theorems of integer partition are developed by using the results of multiset theory and the binary matrix concept. Few properties of this binary matrix are found. Some well known theorems of integer partitions are proved using this binary matrix and multiset representations.

Keywords
Multisets, Integer partitions,Conjugate of a partition, Self Conjugate, Binary matrix, 1-matrix.

AMS Subject Classification
26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05.

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Contents

1. Introduction

Multiset is a collection of elements in which elements are allowed to repeat.[9] In some situations,the classical definition of set proves inadequate and multisets become a very useful tool in such situations. Multisets are of interest in many areas of mathematics and Computer Science.

N.J. Wildberger[7] gave many contributions to multiset theory. In his paper, he distinguishes between set, ordered set,multiset and list. Relation and operation with multisets are also established.

Girish K.P[9] developed relations and functions in multiset context. According to D.Singh[3], multiset representations are found in prime factorization, Zeros and poles of meromorphic functions and in many other situations. Multisets having negative integers as multipliers are known as Hybrid sets[11].

Partition of multisets are discussed by E.A.Bender[6]. In this paper, we are considering partition of an integer as a multiset.

A partition of a positive integer n is a non increasing sequence of positive integers that add upto n[12]. The partition set M of a positive integer n is a multiset whose elements are from the set $\{1,2,\cdots,n\}$ and the sum of the element is n. For example, $15 = 4+3+3+2+2+1$. The corresponding multiset is $\{4,3,3,2,2,1\}$.

Integer partitions were first studied by Euler and then by Hardy and Ramanuja[1]. H. L. Alder's [2] studies lead to develop generating functions to find the number of partitions in different cases. A.D.Healy [5] defines partition as a non decreasing sequence. But we are considering the partition of an integer as a non increasing sequence.

This paper is an attempt to connect integer partition with multisets and binary matrix. In section 4, some results of integer partition,especially in the case of conjugates are found and proved by using the concepts of multisets and binary matrix. In section 5, alternate proofs are giving for some theorems of integer partitions. In section 6, the binary matrix of a partition is divided into blocks and new results are obtained.

2. Multisets

Definition 2.1. *[9] A collection of elements containing duplicate is called multiset. If X is a set of elements, a multiset M drawn from X is represented by a function C_M defined as $C_M : X \to N$, where N is the set of non negative integers. For each $x \in X$, $C_M(x)$ is the characteristic value of x in M and indicates the number of occurrence of x in M.*

The word multiset is often shortened to **Mset**.

Notation 2.2. *Let M be a mset from $X = \{x_1, x_2, \cdots, x_n\}$ with x_1 appearing k_1 times, x_2 appearing k_2 times and so on x_n appearing k_n times. Then M is written as*
$M = \{k_1|x_1, k_2|x_2, \cdots, k_n|x_n\}$.

Definition 2.3. *[9] Let M_1 and M_2 be two msets drawn from a set X. M_1 is a* submultiset *of M_2 ($M_1 \subseteq M_2$) if $C_{M_1}(x) \leq C_{M_2}(x)$ for all x in X.*

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Definition 2.4. *[9] Two msets M_1 and M_2 are* equal *if $M_1 \subseteq M_2$ and $M_2 \subseteq M_1$.*

Definition 2.5. *[9] Let M_1 and M_2 be two multisets defined on a set X. Use $M_1 \oplus M_2$ to denote a multiset M such that $C_M(x_i) = C_{M_1}(x_i) + C_{M_2}(x_i)$ for each $x_i \in X$.*

Definition 2.6. *[9] Let M_1 and M_2 be two multisets defined on a set X. Use $M_1 \ominus M_2$ to denote a multiset M such that $C_M(x_i) = \max\{C_{M_1}(x_i) - C_{M_2}(x_i), 0\}$ for each $x_i \in X$.*

3. Integer Partition

Definition 3.1. *[10] A Partition of a positive integer n is a non increasing sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of positive integers λ_i such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.*

Example 3.2. *For $n = 5$, there exists precisely 7 partitions. They are*
5,
4+1
3+2,
3+1+1,
2+2+1,
2+1+1+1 and
1+1+1+1+1.

Definition 3.3. *[2] Partition of a positive number n into m parts is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ such that λ_i are all positive integers with $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$ and $\lambda_i \geq \lambda_j$ for $i < j$.*

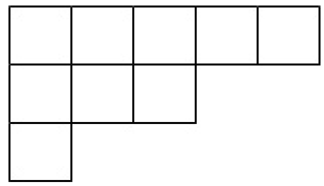
Notation 3.4. *$P(n,m)$ denotes the set of all partitions of n into m parts.*

Example 3.5. $P(7,4)$
$= \{4+1+1+1, 3+2+1+1, 2+2+2+1\}$

.

3.1 Partition Diagram (Ferrers Diagram or Young Diagram)[13]

The Ferrers diagram of an integer partition gives us a very useful tool for visualizing partitions, and sometimes for proving identities. It is constructed by stacking left-justified rows of cells, where the number of cells in each row corresponds to the size of a part. The first row corresponds to the largest part, the second row corresponds to the second largest part, and so on.As an illustration partition diagram for the partition $9 = 5 + 3 + 1$ is



Definition 3.6. *[13]* Conjugate of a partition λ *of a positive integer n is that whose diagram is the transpose (in the sense of matrices) of that of λ.*

Example 3.7. *Conjugate of $\lambda = 5+3+1$ is*
$\lambda^* = 3+2+2+1+1$.

Note 3.8. *Conjugate of a partition λ of a positive integer n is also a partition of n.*

Definition 3.9. *[13] If the conjugate of a partition λ of a positive integer n is same as that of λ, then the partition is known as* self conjugate. *In other words, if its Ferrers diagram is symmetric about the diagonal.*

Example 3.10. *The partition 4+2+1+1 is self conjugate.*

4. Integer Partition as a Multiset and Binary matrix

Lemma 4.1. *Partition of a positive integer n is a* Multiset *and by definition* 3.1*, partition of a positive integer n is a non increasing sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of positive integers λ_i such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. So this sequence λ is also a multiset.*

Lemma 4.2. *If λ is a partition of a positive integer n partitioned into m parts, and $M = \{a_1|x_1, a_2|x_2, \cdots, a_k|x_k\}$ such that $x_1 > x_2 > \cdots > x_k$ is its corresponding mset, then M can be represented as a matrix A of order $m \times x_1$ whose elements are zeros and ones. That is, A is a* Binary Matrix.

Example 4.3. . *The partition 4+4+3+2+2+2+1 of 18 is the mset $\{2|4, 1|3, 3|2, 1|1\}$ and the corresponding binary matrix* *is the* 7×4 *matrix*

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Lemma 4.4. *If $M = \{a_1|x_1, a_2|x_2, \cdots, a_k|x_k\}$ is the mset corresponding to a partition λ of a positive integer n that is partitioned into m parts, then $(i) a_1 + a_2 + \cdots + a_k = m$.*
$(ii) a_1x_1 + a_2x_2 + \cdots + a_kx_k = n$.

Note 4.5. *In standard representation, the mset corresponding to the partition of a positive integer n is taken as $M = \{a_1|x_1, a_2|x_2, \cdots, a_k|x_k\}$ with $x_1 > x_2 > \cdots > x_k$.*

Lemma 4.6. *If A is the binary matrix corresponding to a partition of a positive integer n, partitioned into m parts having mset $M = \{a_1|x_1, a_2|x_2, \cdots, a_k|x_k\}$, then $\sum_{j=1}^{x_1} \sum_{i=1}^{m} a_{ij} = n$, where a_{ij} is the element in the i^{th} row and j^{th} column of matrix A.*

Lemma 4.7. *If A is the matrix corresponding to an mset M of a partition λ, and if B is the matrix corresponding to the mset M^c of the conjugate partition of λ, then $B = A^T$.*

Lemma 4.8. *If A is the matrix corresponding to a partition λ, then A is symmetric if and only if λ is self conjugate.*

Theorem 4.9. *Let $M = \{a_1|x_1, a_2|x_2, \cdots, a_k|x_k\}$ be the mset corresponding to a partition λ of a positive integer n. If $M^c = \{b_1|y_1, b_2|y_2, \cdots, b_l|y_l\}$ is the mset corresponding to λ^*, the conjugate of λ, then*
***(i)** $k = l$.*
(ii)

$$y_j = \sum_{i=1}^{k-(j-1)} a_i,$$

for $j = 1, 2, \cdots, k$.
***(iii)** $b_j = x_{k-(j-1)} - x_{k-(j-2)}$, for $j = 1, 2, \cdots, k$.*

Proof:- **(i)** Suppose λ is a partition of n into m parts. Then by lemma 4.4,

$$a_1 + a_2 + \cdots + a_k = m$$

But by the definition of conjugate,

$$y_1 = m$$

So

$$a_1 + a_2 + \cdots + a_k = y_1$$

From the Partition diagram and definition of conjugate, we have,

$$y_2 = m - a_k$$

Hence,

$$a_1 + a_2 + \cdots + a_{k-1} = y_2$$

Similarly

$$a_1 + a_2 + \cdots + a_{k-2} = y_3$$

and so on. Proceeding like this,

$$a_1 = y_k.$$

Since $a_1, a_2, \cdots, a_k$ are all positive and there is exactly l , $y_j s$,

$$k \leq l \tag{4.1}$$

Conjugate of a conjugate is the original partition. So starting from M^c we get

$$b_1 + b_2 + \cdots + b_l = x_1$$

$$b_1 + b_2 + \cdots + b_{l-1} = x_2$$

$\cdots\cdots\cdots$
$\cdots\cdots\cdots$

$$b_1 = x_l$$

Since there exists exactly k, $x_i s$ and $b_1, b_2, \cdots, b_l$ are all positive,

$$l \leq k \tag{4.2}$$

From (4.1) and (4.2)

$$k = l$$

(ii) We have already obtained that

$$y_1 = a_1 + a_2 + \cdots + a_k$$

$$y_2 = a_1 + a_2 + \cdots a_{k-1}$$

$$\cdots\cdots\cdots$$

$$y_k = a_1$$

In general,

$$y_j = \sum_{i=1}^{k-(j-1)} a_1$$

(iii) $b_1 = x_l = x_k$, since $l = k$

$$b_1 + b_2 = x_{l-1} = x_{k-1}$$

So,

$$b_2 = x_{k-1} - x_k$$

$$b_1 + b_2 + b_3 = x_{l-2} = x_{k-2}$$

Therefore

$$b_3 = x_{k-2} - x_{k-1}$$
$$\cdots\cdots\cdots$$

In general,

$$b_j = x_{k-(j-1)} - x_{k-(j-2)}.$$

Corollary 4.10. *If $M = \{a_1|x_1, a_2|x_2, \cdots, a_k|x_k\}$ is the mset corresponding to a self conjugate partition λ , then*

$$x_j = \sum_{i=1}^{k-(j-1)} a_i.$$

Proof:- Let

$$M^c = \{b_1|y_1, b_2|y_2, \cdots, b_k|y_k\}$$

be the mset corresponding to the conjugate λ^* of λ. Since λ is self conjugate, $M = M^c$
By the equality of msets, $a_i = b_i\ \forall i$ and $x_i = y_i\ \forall i$.
So by theorem 4.9,

$$y_j = \sum_{i=1}^{k-(j-1)} a_i.$$

Therefore,

$$x_j = \sum_{i=1}^{k-(j-1)} a_i.$$

Corollary 4.11. *Let* **M** *be the set of all msets corresponding to the partitions of an integer n. Define a function f on* **M** *as $f(M) = M^c$, where M^c is the mset corresponding to the conjugate of the partition that represents M. Then f is a bijection.*

Proof:- f is well defined because, for every mset M of a partition λ, there is a unique mset M^c corresponding to the conjugate λ^*.
If M_1 and M_2 are two msets from **M** having $f(M_1) = f(M_2)$, then by the definition of conjugate and by theorem 4.9, $M_1 = M_2$. Which implies f is an injection. Now, for every M in **M** there is an M^c in **M** such that $f(M^c) = M$. So f is a surjection. Hence f is a bijection.

Corollary 4.12. *If a partition λ with mset M of a positive integer n have all its parts different, then the parts of the conjugate will be $k, k-1, \cdots, 1$.*

Proof:- Let $M = \{a_1|x_1, a_2|x_2, \cdots, a_k|x_k\}$ be the mset of λ. Since all the parts are different, $a_1 = a_2 = \cdots = a_k = 1$. Let $M^c = \{b_1|y_1, b_2|y_2, \cdots, b_k|y_k\}$ be the mset corresponding to the conjugate. Then by theorem 4.9,

$$y_j = \sum_{i=1}^{k-(j-1)} a_i = \sum_{i=1}^{k-(j-1)} 1 = k-(j-1)$$

. So $y_1 = k$, $y_2 = k-1$, $\cdots$, $y_k = 1$. That is, the parts of the conjugate are $k, k-1, \cdots, 1$.

Theorem 4.13. *There exist at least one self conjugate partition for every positive integer $n > 2$.*

Proof:-If n is even, say $n = 2m$, then, $\{1|m, 1|2, (m-2)|1\}$ is the required partition.
If n is odd, say $n = 2m+1$, then, $\{1|(m+1), m|1\}$ is the required partition.

Theorem 4.14. *For every even number n with $n = p+q$, p and q odd and distinct, there exists a self conjugate partition depending on this p and q.*

Proof:- Since p is odd $\lambda_1 = \{1|((p+1)/2), ((p-1)/2)|1\}$ is a self conjugate partition for p. Similarly, $\lambda_2 = \{1|((q+1)/2), ((q-1)/2)|1\}$ is a self conjugate partition for q.
Now, to find a self conjugate partition of n by combining these λ_1 and λ_2 $1|((p+1)/2)$ is labeled as first term. The $(p-1)/2$, 1's are labeled from 2 to $(p+1)/2$. $1|((q-1)/2)$ is labeled as 2.
The $(q-1)/2$, 1's are labeled from 3 to $(q+3)/2$.
Thus the partition of n becomes
$\lambda = \{1|((p+1)/2), 1|((q+3)/2), ((q-1)/2)|2, (((p-q)/2)-1)|1\}$, which is self conjugate.

Corollary 4.15. *Corresponding to a Goldbach partition (A partition of an even number into a pair of primes), we have a unique self conjugate partition and hence a symmetric binary matrix.*

In general $M_1 \ominus M_2$ is not equal to $M_2 \ominus M_1$. The following theorem explains a situation in which these two msets are equivalent by viewing through integer partition.

Theorem 4.16. *If M_1 and M_2 are msets of two partitions of a positive integer n, then the msets $M_1 \ominus M_2$ and $M_2 \ominus M_1$ represent the partitions of a common integer.*

Proof:- Consider $M_1 = \{a_1|x_1, a_2|x_2, \cdots, a_k|x_k\}$ and $M_2 = \{b_1|y_1, b_2|y_2, \cdots, b_l|y_l\}$. Suppose M_1 and M_2 contains no term in common. Then clearly, both $M_1 \ominus M_2$ and $M_2 \ominus M_1$ represents the integer n. Next, if M_1 and M_2 contains exactly one term in common with different multiplicities. for precise, let $x_p = y_q$ for some p, $1 \le p \le k$, and some q, $1 \le q \le l$.
If $a_p = b_q$, then both $M_1 \ominus M_2$ and $M_2 \ominus M_1$ represents the integer $n - a_p x_p$.
If $a_p > b_q$, then both of the msets represents $n - b_q y_q$.
For $a_p < b_q$, both of the above msets represents the partition of the integer $n - a_p x_p$.

5. Alternate proofs for the theorems of Integer partitions

In this section, our aim is to prove some important theorems of integer partitions(which includes euler's theorem) with the use of multiset theory and binary matrix concept. By this attempt, we are trying to emphasis the wide application of multiset theory and integer partitions.

Theorem 5.1. *(Euler) The number of partitions of a positive integer n into distinct parts is equal to the number of partitions of n into odd parts.*

Proof:- Let $M_1 = \{a_1|x_1, a_2|x_2, \cdots, a_k|x_k\}$ be the mset corresponding to the partition λ in which all parts are distinct. Then $a_1 = a_2 = \cdots = a_k = 1$. So M_1 becomes $\{1|x_1, 1|x_2, \cdots, 1|x_k\}$. Construct a mset M_2 from M_1 in such a way that the element $1|x_i$ is taken as it is if x_i is odd and if x_i is even, then take $\{2^r|y\}$ where $x_i = 2^r y$ with y odd and r a nonnegative integer. Then this M_2 is also a partition of n. Moreover, M_2 has all its parts odd. If we are considering a mapping from such M_1 to M_2 , then it constitute a bijection from the set of all partitions with distinct parts to the set of all partitions with odd parts. The injection of the mapping follows from the construction of M_2 itself. For surjection, consider a partition of n with all parts odd. Let $M = \{a_1|x_1, a_2|x_2, \cdots, a_k|x_k\}$ be the mset of that partition. Then each a_i can be uniquely expressed as $a_i = b_1 + b_2 + \cdots + b_s$, where every $b_j = 2^\alpha$ for some nonnegative α. Construct a mset by replacing the element $a_i|x_i$ by the elements $1|b_1x_i, 1|b_2x_i, \cdots, 1|b_sx_i$. This new mset is the image of M under the above mentioned mapping.

Theorem 5.2. *The number of partitions of a positive integer n whose largest part is t is equal to the number of partitions of n with t parts.*

Proof:- We are proving this theorem with the help of the binary matrix that we had discussed. Consider a partition λ of n whose largest part is t. let $M = \{a_1|x_1, a_2|x_2, \cdots, a_k|x_k\}$ be the corresponding mset and let A be the equivalent binary matrix associated with this M having order $m \times p$. Then

$$m = \sum_{i=1}^{k} a_i, p = x_1 \tag{5.1}$$

Since $x_1 > x_2 > \cdots > x_k$, x_1 is the largest part and hence

$$t = x_1. \tag{5.2}$$

If M^c is the mset corresponding to the conjugate of λ, and B is the equivalent binary matrix, then $B = A^T$.So order of B is $p \times m$. Since B corresponds to M^c,

$$p = \sum_{i=1}^{k} b_i, m = y. \tag{5.3}$$

Comparing equations (3),(4) and (5), $\sum_{i=1}^{k} b_i = t$. This implies that M^c has t parts. Since M^c is also a partition of n and the mapping from M to M^c is a bijection(corollary 4.12), we have the theorem.

Corollary 5.3. *The number of partitions of n with at most t parts is equal to the number of partitions of n with parts smaller than or equal to t.*

Proof:- By theorem 5.2, the number of partitions of n with t parts is equal to the number of partitions of n with largest part t. So the number of partitions of n with $t-1$ parts is equal to the number of partitions of n with largest part $t-1$. Proceeding like this, the number of partitions of n with only 1 part is equal to the number of partitions of n with largest part 1. summing up these results, we have the corollary.

Theorem 5.4. *The number of partitions of a positive integer n in which all parts are odd and distinct is equal to the number of partitions of n which are self conjugates.*

Proof:- Let $M = \{a_1|x_1, a_2|x_2, \cdots, a_k|x_k\}$ be the mset of a partition with all parts odd and distinct. Then $a_1 = a_2 = \cdots = a_k = 1$, $x_1 > x_2 > \cdots > x_k$. Also, $x_1 + x_2 + \cdots + x_k = n$ and $x_1, x_2, \cdots, x_k$ are all odd. Let $y_i = \frac{x_i+1}{2}$ and $z_i = \frac{x_i-1}{2}$. Then $y_i + z_i = x_i$ and $y_i = z_i + 1$. Consider the mset $\{1|y_i, z_i|1\}$. This represents a self conjugate partition of x_i. If A_i is the binary matrix of this partition, then it will be a square, symmetric matrix of order y_i with its first row and column have all elements 1 and other elements 0. From the choice of y_i, we have $y_1 > y_2 > \cdots > y_k$. Construct k square matrices B_i of same order y_i as follows: Take $B_1 = A_1$. In B_2, all elements of first row and column are taken as 0. In second row and column, starting from the second element y_2 elements are 1 and the rest are 0. Similarly in B_3, first two rows have all elements 0. In the third row and column from third element onwards y_3 elements are taken as 1 and the others as 0. Continuing like this, in B_k, first $k-1$ rows and columns have elements as zero only and in k^{th} row and column, from k^{th} element to $y_k + k^(th)$ all elements are 1 and the remaining are zeros. Let $C = B_1 + B_2 + \cdots + B_k$. Then this C is a symmetric matrix and hence the corresponding partition is self conjugate.
If A is the binary matrix corresponding to M, then the mapping which maps this A to C is a bijection. the mapping is well defined since this C is uniquely determined from M. That is, for an A, there is one and only one C. on the other hand, if $M_1 = \{b_1|y_1, b_2|y_2, \cdots b_k|y_k\}$ is the mset of a self conjugate partition with binary matrix C. Let $x_1 = 2\sum_j c_{1j} - 1$, $x_2 = 2\sum_j c_{2j} - 3$, $x_3 = 2\sum_j c_{3j} - 5, \cdots$ $x_k = 2\sum_j c_{kj} - (2k-1)$. Then $M_2 = \{1|x_1, 1|x_2, \cdots, 1|x_k\}$is a partition of n in which the parts are odd and distinct. Moreover the matrix of this M_2 is the pre image of C.
So we establish a one one correspondence between the set of partitions which are self conjugate to the set of partitions having all its parts odd and distinct.
This proves the theorem.

6. Dividing the Binary Matrix into Blocks

Result 6.1. *$M = \{a_1|x_1, a_2|x_2, \cdots, a_k|x_k\}$ and $M^c = \{b_1|y_1, b_2|y_2, \cdots, b_k|y_k\}$ be the msets corresponding to the partitions λ and its conjugate λ^*. Let A be the binary matrix of M. Divide A into k^2 submatrices $A_{11}, A_{12}, \cdots, A_{kk}$ of orders $a_1 \times b_1, a_1 \times b_2 \cdots, a_k \times b_k$. Then these submatrices are either* Zero Matrices *or* 1- matrices *(matrices having all*

its elements 1).
More precisely, A_{ij} is a 1- matrix for $j = 1, 2, \cdots, k-(i-1)$ and 0 matrix for all other j.

Result 6.2. *The number of 1- matrices for the mset $M = \{a_1|x_1, a_2|x_2, \cdots, a_k|x_k\}$ is $\frac{k(k+1)}{2}$.*
***Proof:**- By result 6.1, A_{ij} is a 1- matrix for $j = 1, 2, \cdots, k-(i-1)$. So for $i = 1$, A_{1j} is a 1- matrix for $j = 1, 2, \cdots, k$. That is, there are k, 1- matrices for $i = 1$. For $i = 2$, there exists $k-1$ such matrices. Proceeding like this, for $i = k$, there is only one 1- matrix. So the total number of 1- matrices is equal to $k+(k-1)+\cdots+1 = \frac{k(k+1)}{2}$.*

7. Conclusion and Future Work

In this paper we introduce a binary matrix and submatrices for a partition of a positive integer. The submatrices were created by using the concept of conjugate of a partition and a formula is developed to find the conjugate by applying the properties of multisets.

With the rapid development of network and multimedia technologies, digital communication has become an important part of our day-to-day life. So security plays a crucial role in transferring data. One such way to secure transformation is Cryptography. Matrices are used in Cryptography. Further studies may lead to find the possibilities of using the above discussed binary matrix in cryptography.

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