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# Some product type hyperbolic Young functions

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**Abstract**

In this paper we have defined some product type hyperbolic convex function and product type hyperbolic Young function and prove some results based on this. We also characterize an integral representation of the product type hyperbolic convex functions in this paper.

**Keywords**

Hyperbolic numbers, Null cone, Product type function, Product type $\mathscr{D}$-convex function, Product type $\mathscr{D}$-Young function.

**AMS Subject Classification**

30G35, 47A10.

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## 1. Introduction

Convexity plays a very important role in the study of analysis. One such concept is the study of convex function which is specially important in the study of optimization problems. In Fourier analysis W.H. Young found certain convex functions $\varphi : \mathbb{R} \to \bar{\mathbb{R}}^+$ which satisfies $\varphi(0) = 0$, $\varphi(-x) = \varphi(x)$ and $\lim_{x\to\infty} \varphi(x) = \infty$ The importance of studying convex functions were only recognized from the work of W.H. Young in 1912. But their role in the study of abstract analysis evolves only with the fundamental ideas of Z.W. Birnbaum and W.Orlicz in the year 1931. However study of convex function and Young functions were done on the set of real numbers system which is very rich in structure. We have made an attempt to study the concept of convex functions and Young function on the set of hyperbolic number system which is considered to be an affordable replacement for the real number system.The study of hyperbolic numbers which we denote by $\mathscr{D}$ was initiated by James Cockle way back in 1848 [7] and then in 1893 Lie and Scheffers [16] continued this study. Hyperbolic number system has widely been studied due to its commutative Clifford algebraic properties.The importance of hyperbolic numbers can be seen from the fact that the Minkowski geometry were developed solely using this system of numbers ( see, [3],[9],[27],[28], [4]). Hyperbolic numbers were used in studying various other areas of mathematics and physics such as in function theory, Fourier transformation, relativistic quantum physics and many more. Many papers has appeared studying hyperbolic numbers from various points of view (see, [5], [22], [6], [11] ,[13]) and references therein. During the past several years focus has been on developing hyperbolic numbers as an affordable replacement for the real number system. However a recent paper [10] has appeared which studied this system of numbers as the only(natural) generalization of real numbers, into Archimedean f-algebra of dimension two. They generalized the fundamental properties of real numbers to this number system. In this paper we shall define some special types of $\mathscr{D}$-convex function and $\mathscr{D}$-Young's functions on the set of hyperbolic number and derive a $\mathscr{D}$-integral representation for them. We shall also prove $\mathscr{D}$-Young's inequality.

## 2. Preliminaries

Now we shall go through a brief review of hyperbolic numbers system. The hyperbolic numbers denoted by $\mathscr{D}$ is a commutative ring of all numbers of the form $Z = a + kb$, a,b $\in \mathbb{R}$, with $k$ satisfying $k^2 = 1$.

$$i.e \qquad \mathscr{D} = \left\{ a + kb : a, b \in \mathbb{R}, k^2 = 1, k \notin \mathbb{R} \right\}.$$

It is very well known that we can also decompose every element $Z \in \mathscr{D}$ as

$$Z = x_1 e + x_2 e^\dagger \tag{2.1}$$

where $e = \frac{1}{2}(1+k)$ and $e^\dagger = \frac{1}{2}(1-k)$ are two zero divisors in the set $\mathscr{D}$ and $x_1 = a+b$ and $x_2 = a-b$. The two zero divisors satisfies the following properties

$$ee^\dagger = 0,\ e+e^\dagger = 1,\ e-e^\dagger = k,\ e^2 = e \text{ and } \left(e^\dagger\right)^2 = e^\dagger$$

We call equation (2.1), the idempotent decomposition of $\mathscr{D}$. All the zero divisors are either of the form $x_1 e$ or $x_2 e^\dagger$ with $x_1 \neq 0$ and $x_2 \neq 0$ and we denote the set of all zero divisors of $\mathscr{D}$ by $\mathscr{NC}$ and call it the null cone of $\mathscr{D}$. It is also important to mention here that $\mathscr{D}$ is a module over itself. Also the set of positive hyperbolic number denoted by $\mathscr{D}^+$ are the set of all those hyperbolic numbers whose idempotent components are non negative.

$$\mathscr{D}^+ = \left\{\alpha e + \beta e^\dagger : \alpha, \beta \geq 0\right\} \tag{2.2}$$

We shall define a partial order relation on $\mathscr{D}$ as follows. Given $x, y \in \mathscr{D}$, we write $x \leq' y$ if $y - x \in \mathscr{D}^+$. It is easy to see that this relation is reflexive, symmetric and antisymmetric and so it defines a partial order relation on $\mathscr{D}$. Also for $x, y \in \mathscr{D}$, if $x \leq' y$, then we say that $y$ is $\mathscr{D}$-larger than $x$ and $x$ is $\mathscr{D}$-smaller than $y$. The notion of upper and lower bounds also exists in the context of hyperbolic plane. Given a subset $S$ of $\mathscr{D}$ we can define $\mathscr{D}$-upper bounds and $\mathscr{D}$-lower bounds of this set $S$. Using this bounds, this set can be made $\mathscr{D}$-bounded from above and $\mathscr{D}$-bounded from below if it exists. Now if the set is $\mathscr{D}$-bounded from above as well as from below then we say that the set is $\mathscr{D}$-bounded.

We further define the notion of the supremum of a given subset of $\mathscr{D}$. Supremum of $S \subset \mathscr{D}$ denoted by $\sup_{\mathscr{D}} S$ is defined usually as the least of all $\mathscr{D}$-upper bounds of the given set. Similarly $\inf_{\mathscr{D}} S$ is the greatest of all $\mathscr{D}$-lower bounds of the set. However due to the idempotent decomposition of $\mathscr{D}$, we can find a convenient expression as follows:

For a subset $S$ of $\mathscr{D}$ which is $\mathscr{D}$- bounded from above, we consider the set $C_1 = \left\{\alpha : \alpha e + \beta e^\dagger \in S\right\}$ and $C_2 = \left\{\beta : \alpha e + \beta e^\dagger \in S\right\}$. Then the supremum of the set S denoted by $\sup_{\mathscr{D}} S$ is defined as $\sup_{\mathscr{D}} S = \sup C_1 \cdot e + \sup C_2 \cdot e^\dagger$ where $\sup C_1$ and $\sup C_2$ is the supremum taken over the subset $C_1$ and $C_2$ of real numbers. Finally the hyperbolic modulus of any hyperbolic number $Z = \alpha e + \beta e^\dagger$ denoted by $|Z|_k$ is given by the formula $|Z|_k = Z \cdot Z^* = |\alpha|^2 e + |\beta|^2 e^\dagger$ where "*" denotes the $*$-conjugation(see [2]). For further basic properties and results on hyperbolic numbers (see [2] [10] and [18]). The set $\mathscr{BC}$ of bicomplex numbers is defined as

$$\begin{aligned} \mathscr{BC} &= \{Z = x_o + x_1 i + x_2 j + x_3 ij : x_o, x_1, x_2, x_3 \in \mathbb{R}\} && (2.3)\\ &= \{Z = z_1 + jz_2 : z_1, z_2 \in \mathbb{C}(i)\}, && (2.4)\end{aligned}$$

where $i$ and $j$ are two imaginary units satisfying $ij = ji$ with $i^2 = j^2 = -1$ and $\mathbb{C}(i)$ is the set of complex numbers with imaginary units $i$ and $\mathbb{C}(j)$ is the set of complex numbers with imaginary units $j$.The set of hyperbolic number $\mathscr{D}$ lies inside the set $\mathscr{BC}$ with $k = ij$. For more details on this one can refer to [2].

## 3. Product Type Hyperbolic Convex Functions

We shall begin with some definition which will be used throughout this paper.

**Definition 3.1.** *[21] A subset $\mathscr{A} \subset \mathscr{BC}$ is called a product type set if $\mathscr{A} = \Pi_{1,i}(\mathscr{A})e + \Pi_{2,i}(\mathscr{A})e^\dagger$ where $\Pi_{k,i}$ are the idempotent projection to $\mathbb{C}(i)$ for $k = 1,2$ and denote $\mathscr{A}_1 = \Pi_{1,i}(\mathscr{A})$ and $\mathscr{A}_2 = \Pi_{2,i}(\mathscr{A})$ where $\mathscr{BC}$ denotes the set of bicomplex numbers*

We can also take $\mathscr{D}$ the set of hyperbolic number instead of $\mathscr{BC}$ to get a product type subset of the set of hyperbolic numbers using the projections $\pi_k : \mathscr{D} \to \mathbb{R}$.

**Definition 3.2.** *[21] A function $\varphi_{\mathscr{BC}} : \mathscr{A} = \mathscr{A}_1 e + \mathscr{A}_2 e^\dagger \subseteq \mathscr{BC} \longmapsto \mathscr{BC}$ is called a product type function if $\exists$ maps $\varphi_1 : \mathscr{A}_1 \longmapsto \mathbb{C}$ and $\varphi_2 : \mathscr{A}_2 \longmapsto \mathbb{C}$ such that*

$$\varphi_{\mathscr{BC}}(x_1 e + x_2 e^\dagger) = \varphi_1(x_1)e + \varphi_2(x_2)e^\dagger \ \ \forall \ x_1 e + x_2 e^\dagger \in \mathscr{A}. \tag{3.1}$$

Similarly we can take $\mathscr{D}$ in place of $\mathscr{BC}$ and $\mathbb{R}$ in place of $\mathbb{C}$ and get product type hyperbolic valued functions. Further it is to be noted that a function $\varphi_{\mathscr{D}} : \mathscr{A} = \mathscr{A}_1 e + \mathscr{A}_2 e^\dagger \subseteq \mathscr{D} \longmapsto \mathscr{D}$ can also be written as a product type function defined by

$$\varphi_{\mathscr{D}}(x_1 e + x_2 e^\dagger) = \varphi_1(x_1)e + \varphi_2(x_2)e^\dagger$$

if there exixts $\varphi_1 : \mathscr{A}_1 = \pi_1(\mathscr{A}) \longmapsto \mathbb{R}$ and $\varphi_2 : \mathscr{A}_2 = \pi_2(\mathscr{A}) \longmapsto \mathbb{R}$ and $\pi_k : \mathscr{D} \to \mathbb{R}$ are the projection onto the coordinate axis in $\mathbb{R}$.

The concept of Riemann $\mathscr{D}$-Integral for bicomplex Functions(hyperbolic functions) was developed in a very recent paper [21] by Juan, Cesar and Shapiro. We shall use this concept to develop an $\mathscr{D}$-integral representation for product type convex function.

**Definition 3.3.** *[21] Let $\Phi : [X,Y]_{\mathscr{D}} \to \mathscr{D}$ be a product type function. Then the $\mathscr{D}$-integral of the function $\Phi$ is defined by*

$$lim_{\mathscr{B}} Sum(\Phi) = \int_{[X,Y]_{\mathbb{D}}} \Phi(Z) dZ \wedge dZ^\dagger \tag{3.2}$$

*where $lim_{\mathscr{B}} Sum(\Phi)$ is the $\mathscr{D}$-integral sum of $\Phi$.*

For more details on $\mathscr{D}$-integral sum one can refer to [21]

**Theorem 3.4.** *[21] If a product type function $\Phi : [X,Y]_{\mathscr{D}} \to \mathscr{D}$ is $\mathscr{D}$-integrable then*

$$\int_{[X,Y]_{\mathbb{D}}} \Phi(Z) dZ \wedge dZ^\dagger = \int_{x_1}^{y_1} \phi_1(\zeta_1) d\zeta_1 e + \int_{x_2}^{y_2} \phi_2(\zeta_2) d\zeta_2 e^\dagger \tag{3.3}$$

*where $\Phi(a_1 e + b_1 e^\dagger) = \phi_1(a_1)e + \phi_2(a_2)e^\dagger$ is a product type function.*

Using the concept of product type function defined above we shall now define product type convex function.

**Definition 3.5.** *A function $\varphi_{\mathscr{D}} : \mathscr{D} \to \overline{\mathscr{D}}^{+}$ is said to be a $\mathscr{D}$-convex function if for every $X, Y \in \mathscr{D}$ with $0 \leq' \alpha \leq' 1$ , we have that*

$$\varphi_{\mathscr{D}}(\alpha X + (1-\alpha)Y) \leq' \alpha \varphi_{\mathscr{D}}(X) + (1-\alpha)\varphi_{\mathscr{D}}(Y).$$

**Example 3.6.** *Let $\varphi_{\mathscr{D}} : \mathscr{D} \to \overline{\mathscr{D}}^{+}$ be defined by $\varphi_{\mathscr{D}}(X) = |X|_k$ for every $X \in \mathscr{D}$. Then clearly $\varphi_{\mathscr{D}}$ is a $\mathscr{D}$-convex function.*

Note that if $\varphi_{\mathscr{D}} : \mathscr{D} \to \overline{\mathscr{D}}^{+}$ is a function, then $\varphi_{\mathscr{D}}(X) \in \mathscr{D}^{+}$ and it can be written as

$$\varphi_{\mathscr{D}}(X) = \psi_1(X) + \psi_2(X)k = \varphi_{\mathscr{D}_1}(X)e + \varphi_{\mathscr{D}_2}(X)e^{\dagger},$$

where $\psi_1, \psi_2$ are functions satisfying $(\psi_1)^2 - (\psi_2)^2 \geq 0$ and $(\psi_1) \geq 0$ and $\phi_{\mathscr{D}_1}, \phi_{\mathscr{D}_2}$ are the idempotent components given by

$$\varphi_{\mathscr{D}_1} = \psi_2 + \psi_1 \geq 0 \text{ and } \varphi_{\mathscr{D}_2} = \psi_1 - \psi_2 \geq 0.$$

Note that both the idempotent components $\varphi_{\mathscr{D}_1}$ and $\varphi_{\mathscr{D}_2}$ are both real valued with hyperbolic domain $\mathscr{D}$ which does not serves our purpose..Thus in order to make hyperbolic convex function as an idempotent decomposed type of function where each of the component functions will be functions from subset of real to real. We have the following theorem.
Further in order to reduce the lengthy equations and make equations more simpler. We shall give some short notations as follows $\mathbf{I}\varphi_i = \varphi_i(\lambda_i x_i + (1-\lambda_i)y_i)$, $\mathbf{IE}\varphi_i = \lambda_i \varphi_i(x_i) + (1-\lambda_i)\varphi_i(y_i)$ ,$\mathbf{IE}\varphi_{ij}^{(n)} = \lambda_i^{(n)}\varphi_i(x_i)e + \lambda_j^{(n)}\varphi_j(x_j)e^{\dagger}$ for $n = 0,1$ and $\mathbf{I\tilde{E}}\varphi_{ij}^{(n)} = (1-\lambda_i)^{(n)}\varphi_i(y_i)e + (1-\lambda_j)^{(n)}\varphi_j(y_i)e^{\dagger}$ for $n = 0,1$.
Now we shall prove the following theorem.

**Theorem 3.7.** *Let $\mathscr{A}$ be a product type subset of $\mathscr{D}$ and $\varphi_1 : \mathscr{A}_1 \to \mathbb{R}$ and $\varphi_2 : \mathscr{A}_2 \to \mathbb{R}$ be two $\mathbb{R}$- convex functions. Then the hyperbolic valued product type function $\varphi_{\mathscr{D}} : \mathscr{A} \subseteq \mathscr{D} \to \mathscr{D}$ defined by*

$$\varphi_{\mathscr{D}}(X) = \mathbf{IE}\varphi_{12}^{(0)} \;\; \forall \;\; X = x_1 e + x_2 e^{\dagger} \in \mathscr{D} \tag{3.4}$$

*is a $\mathscr{D}$-convex function.*

*Proof.* For this let $X, Y \in \mathscr{A}$, $0 \leq \lambda_1 \leq 1$ and $0 \leq \lambda_1 \leq 1$ be such that $0 \leq' \lambda \leq' 1$ where $\lambda = \lambda_1 e + \lambda_2 e^{\dagger}$. Now since $\varphi_1 : \mathscr{A}_1 \to \mathbb{R}$ and $\varphi_2 : \mathscr{A}_2 \to \mathbb{R}$ are $\mathbb{R}$-convex function and so we have for $x_i = \pi_i(X)$ and $y_i = \pi_i(Y)$ for $i = 1,2$

$$\mathbf{I}\varphi_1 \leq \mathbf{IE}\varphi_1 \;\text{ and }\; \mathbf{I}\varphi_2 \leq \mathbf{IE}\varphi_2 \tag{3.5}$$

Multiplying the first part of the equation in (3.5) by $e$ and the second part of the equation in (3.5) by $e^{\dagger}$ and adding we get

$$\mathbf{I}\varphi_1 e + \mathbf{I}\varphi_2 e^{\dagger} \leq' \mathbf{IE}\varphi_1 e + \mathbf{IE}\varphi_2 e^{\dagger} \tag{3.6}$$

So that

$$\begin{aligned} \varphi_{\mathscr{D}}(\lambda X + (1-\lambda)Y) &= \mathbf{I}\varphi_1 e + \mathbf{I}\varphi_2 e^{\dagger} \\ &\leq' \mathbf{IE}\varphi_1 e + \mathbf{IE}\varphi_2 e^{\dagger} \\ &\leq' \mathbf{IE}\varphi_{12}^{(1)} + \mathbf{I\tilde{E}}\varphi_{12}^{(1)} \\ &= \lambda \mathbf{IE}\varphi_{12}^{(0)} + (1-\lambda)\mathbf{I\tilde{E}}\varphi_{12}^{(0)} \\ &= \lambda \varphi_{\mathscr{D}}(X) + (1-\lambda)\varphi_{\mathscr{D}}(Y) \end{aligned}$$

Thus we have

$$\varphi_{\mathscr{D}}(\lambda X + (1-\lambda)Y) \leq' \lambda \varphi_{\mathscr{D}}(X) + (1-\lambda)\varphi_{\mathscr{D}}(Y)$$

This proves that $\varphi_{\mathscr{D}}$ is a hyperbolic valued convex function. □

**Theorem 3.8.** *Let $\varphi_{\mathscr{D}} : \mathscr{A} \subseteq \mathscr{D} \to \mathscr{D}$ be a product type $\mathscr{D}$-convex function. Then the two $\mathbb{R}$-functions $\varphi_1 : \mathscr{A}_1 \to \mathbb{R}$ and $\varphi_2 : \mathscr{A}_2 \to \mathbb{R}$ where $\mathscr{A}$ is a product type subset of $\mathscr{D}$ such that*

$$\varphi_{\mathscr{D}}(X) = \mathbf{IE}\varphi_{12}^{(0)} \;\; \forall \;\; X = x_1 e + x_2 e^{\dagger}. \tag{3.7}$$

*are $\mathbb{R}$-convex functions.*

*Proof.* Suppose that $\varphi_{\mathscr{D}}$ be a $\mathscr{D}$-convex function.Then we shall proof that $\varphi_1$ and $\varphi_2$ are $\mathbb{R}$-convex. For this suppose that $0 \leq \lambda_1 \leq 1$ and $0 \leq \lambda_2 \leq 1$ be such that $0 \leq' \lambda \leq' 1$ where $\lambda = \lambda_1 e + \lambda_2 e^{\dagger}$. Now since $\varphi_{\mathscr{D}}$ is a $\mathscr{D}$-convex function implies that

$$\varphi_{\mathscr{D}}(\lambda X + (1-\lambda)Y) \leq' \lambda \varphi_{\mathscr{D}}(X) + (1-\lambda)\varphi_{\mathscr{D}}(Y)$$

Since $\varphi_{\mathscr{D}}$ is a product type function, we have by expanding

$$\begin{aligned} \mathbf{I}\varphi_1 e + \mathbf{I}\varphi_2 e^{\dagger} &\leq' \mathbf{IE}\varphi_{12}^{(1)} + \mathbf{I\tilde{E}}\varphi_{12}^{(1)} \\ &= \mathbf{IE}\varphi_1 e + \mathbf{IE}\varphi_2 e^{\dagger} \end{aligned}$$

So that we have the following equation

$$\mathbf{I}\varphi_1 e + \mathbf{I}\varphi_2 e^{\dagger} \leq' \mathbf{IE}\varphi_1 e + \mathbf{IE}\varphi_2 e^{\dagger} \tag{3.8}$$

Now first multiplying equation (3.8) by $e$ we get

$$\mathbf{I}\varphi_1 e \leq' \mathbf{IE}\varphi_1 e \tag{3.9}$$

which implies that

$$\varphi_1(\lambda_1 x_1 + (1-\lambda_1)y_1) \leq \lambda_1 \varphi_1(x_1) + (1-\lambda_1)\varphi_1(y_1) \tag{3.10}$$

Similarly multiplying equation 3.8 by $e^{\dagger}$ we find that

$$\mathbf{I}\varphi_2 e^{\dagger} \leq' \mathbf{IE}\varphi_2 e^{\dagger} \tag{3.11}$$

so that

$$\varphi_2(\lambda_2 x_2 + (1-\lambda_2)y_2) \leq \lambda_2 \varphi_2(x_2) + (1-\lambda_2)\varphi_2(y_2) \tag{3.12}$$

□

Thus equation (3.10) and (3.12) proves that $\varphi_1$ and $\varphi_2$ are $\mathbb{R}$-convex functions.

**Corollary 3.9.** *A product type function $\varphi_{\mathscr{D}} : \mathscr{D} \to \mathscr{D}$ defined by*

$$\varphi_{\mathscr{D}}(X) = \mathbf{IE}\varphi_{12}^{(0)} \;\; \forall\, X = x_1 e + x_2 e^{\dagger}, \tag{3.13}$$

*is $\mathscr{D}$-convex function iff each of the function $\varphi_1 : \mathscr{A}_1 \to \mathbb{R}$ and $\varphi_2 : \mathscr{A}_2 \to \mathbb{R}$ where $\mathscr{A}$ is a product type subset of $\mathscr{D}$ are $\mathbb{R}$-convex function.*

We call such type of $\mathscr{D}$-convex functions as the **Product Type Hyperbolic convex functions**

**Example 3.10.** *Let $\varphi_1 : \mathscr{A}_1 \to \mathbb{R}$ and $\varphi_2 : \mathscr{A}_2 \to \mathbb{R}$ be defined by*

$$\varphi_1(x_1) = \log x_1^{x_1} \;\; and \;\; \varphi_2(x_2) = \log x_2^{x_2}$$

*on $(0,\infty)$ be $\mathbb{R}$-functions. We know that $\varphi_1$ and $\varphi_2$ are $\mathbb{R}$ convex functions. Now if we take*

$$\varphi_{\mathscr{D}}(X) = \mathbf{IE}\varphi_{12}^{(0)} = \log x_1^{x_1} e + \log x_2^{x_2} e^{\dagger}.$$

*Then it is clearly easy to show that $\varphi_{\mathscr{D}}$ is a product type hyperbolic convex function.*

The proof of the following theorem is straightforward.

**Theorem 3.11.** *A Product type function $\varphi_{\mathscr{D}} : \mathscr{D} \to \mathscr{D}$ defined by $\varphi_{\mathscr{D}}(X) = \mathbf{IE}\varphi_{12}^{(0)}$ for $X = x_1 e + x_2 e^{\dagger}$ is continuous iff $\varphi_1 : \mathscr{A}_1 \to \mathbb{R}$, $\varphi_2 : \mathscr{A}_2 \to \mathbb{R}$ are continuous.*

**Definition 3.12.** *A hyperbolic valued function $\varphi_{\mathscr{D}} : \mathscr{D} \to \mathscr{D}$ is said to be $\mathscr{D}$-monotonically increasing function iff $\varphi_{\mathscr{D}}(X) \leq' \varphi_{\mathscr{D}}(Y)$ whenever $X \leq' Y$.*

**Theorem 3.13.** *A product type function $\varphi : \mathscr{A} \subseteq \mathscr{D} \to \mathscr{D}$ defined by*

$$\varphi_{\mathscr{D}}(X) = \mathbf{IE}\varphi_{12}^{(0)}, \quad X = x_1 e + x_2 e^{\dagger} \in \mathscr{A}, \tag{3.14}$$

*is $\mathscr{D}$-monotonically increasing function iff each of the function $\varphi_1 : \mathscr{A}_1 \to \mathbb{R}$ and $\varphi_2 : \mathscr{A}_2 \to \mathbb{R}$ where $\mathscr{A}$ is a product type subset of $\mathscr{D}$ are $\mathbb{R}$-monotonically increasing function.*

*Proof.* First suppose that each $\varphi_1$ and $\varphi_2$ are $\mathbb{R}$- monotonically increasing function.To show that $\varphi_{\mathscr{D}}$ is a $\mathscr{D}$-monotonically increasing function. Let $C = c_1 e + c_2 e^{\dagger}$ and $D = d_1 e + d_2 e^{\dagger}$ be two hyperbolic numbers such that $C \leq' D$. This implies that $c_1 \leq d_1$ and $c_2 \leq d_2$. Now since $\varphi_1$ and $\varphi_2$ are $\mathbb{R}$-monotonically increasing function implies that

$$\varphi_1(c_1) \leq \varphi_1(d_1) \;\; \text{and} \;\; \varphi_2(c_2) \leq \varphi_2(d_2) \tag{3.15}$$

Multiplying the first part in equation 3.15 by $e$ and second part by 3.15 by $e^{\dagger}$ and adding we get

$$\varphi_1(c_1)e + \varphi_2(c_2)e^{\dagger} \leq' \varphi_1(d_1)e + \varphi_2(d_2)e^{\dagger}$$

Consequently for $C \leq' D$ we have

$$\varphi_{\mathscr{D}}(C) \leq' \varphi_{\mathscr{D}}(D) \tag{3.16}$$

This $\varphi_{\mathscr{D}}$ is a $\mathscr{D}$-monotonically increasing function. Conversely we can prove by tracing back the proof. □

**Theorem 3.14.** *Let $\varphi_{\mathscr{D}} : (A,B)_{\mathscr{D}} \to \mathscr{D}$ be a product type function. Then $\varphi_{\mathscr{D}}$ is a product type $\mathscr{D}$-convex function iff for each $[C,D]_{\mathscr{D}} \subset (A,B)_{\mathscr{D}}$, $\varphi_{\mathscr{D}}$ has a $\mathscr{D}$-integral representation as*

$$\varphi_{\mathscr{D}}(X) = \varphi_{\mathscr{D}}(C) + \int_{[C,X]_{\mathscr{D}}} \Phi(Z)dZ \wedge dZ^{\dagger} \tag{3.17}$$

*where $\Phi : \mathscr{D} \to \mathscr{D}$ is a $\mathscr{D}$-monotone increasing product type function which is left continuous and has left as well as right derivative at each point of $(A,B)_{\mathscr{D}}$.*

Note that $(A,B)_{\mathscr{D}}$ is a hyperbolic interval defined in [21] with $A = a_1 e + a_2 e^{\dagger}$ and $B = b_1 e + b_2 e^{\dagger}$

*Proof.* Suppose that the product type function $\varphi_{\mathscr{D}} : (A,B)_{\mathscr{D}} \to \mathscr{D}$ is $\mathscr{D}$-convex function. This means that there exits $\mathbb{R}$-convex function $\varphi_1 : (a_1,b_1) \to \mathbb{R}$ and $\varphi_2 : (a_2,b_2) \to \mathbb{R}$ where $(a_i,b_i) = \pi_i((A,B)_{\mathscr{D}})$ for $i = 1,2$ such that

$$\varphi_{\mathscr{D}}(X) = \mathbf{IE}\varphi_{12}^{(0)} \quad \text{where} \;\; X = x_1 e + x_2 e^{\dagger}$$

By Theorem 1 Page no.7 in [24] we see that $\varphi_1 : (a_1,b_1) \to \mathbb{R}$ and $\varphi_2 : (a_2,b_2) \to \mathbb{R}$ are $\mathbb{R}$-convex iff there exits subintervals $(c_1,d_1) \subset (a_1,b_1)$ and $(c_2,d_2) \subset (a_2,b_2)$ such that

$$\varphi_1(x_1) = \varphi_1(c_1) + \int_{c_1}^{x_1} \phi_1(\zeta_1)d\zeta_1 \;\; \forall \;\; c_1 \leq x_1 \leq d_1$$
$$\varphi_2(x_2) = \varphi_2(c_2) + \int_{c_2}^{x_2} \phi_2(\zeta_2)d\zeta_2 \;\; \forall \;\; c_2 \leq x_2 \leq d_2$$

where each of $\phi_i : \mathbb{R} \to \mathbb{R}$ is a monotone increasing and left continuous function with $\phi_i$ has left as well as right derivative. Now by Thoerem 3.11 and Theorem 3.13 we see that the product type function defined by $\Phi(Y) = \phi_1(y_1)e + \phi_2(y_2)e^{\dagger}$ is also monotone increasing, left continuous and has left as well as right derivative at each point because each of $f_i$ are monotone increasing, left continuous and has left as well as right derivative at each point except at a countable number of points. So that by using Riemann $\mathscr{D}$-integral as defined in [21], we have

$$\begin{aligned}
& \varphi_{\mathscr{D}}(X) \\
= \; & \mathbf{IE}\varphi_{12}^{(0)} \\
= \; & \left[\varphi_1(c_1) + \int_{c_1}^{x_1} \phi_1(\zeta_1)d\zeta_1\right] e + \left[\varphi_2(c_2) + \int_{c_2}^{x_2} \phi_2(\zeta_2)d\zeta_2\right] e^{\dagger} \\
= \; & [\varphi_1(c_1)e + \varphi_2(c_2)e^{\dagger}] + \left[\int_{c_1}^{x_1} \phi_1(\zeta_1)d\zeta_1 e + \int_{c_2}^{x_2} \phi_2(\zeta_2)d\zeta_2 e^{\dagger}\right] \\
= \; & \varphi_{\mathscr{D}}(C) + \int_{[C,X]_{\mathscr{D}}} \Phi(Z)dZ \wedge dZ^{\dagger}
\end{aligned}$$

$$i.e., \;\; \varphi_{\mathscr{D}}(X) = \varphi_{\mathscr{D}}(C) + \int_{[C,X]_{\mathscr{D}}} \Phi(Z)dZ \wedge dZ^{\dagger} \tag{3.18}$$

This completes the proof □

## 4. Product type hyperbolic Young Function

**Definition 4.1.** *A hyperbolic convex function $\varphi_{\mathscr{D}} : \mathscr{D} \to \overline{\mathscr{D}}^+$ is said to be a hyperbolic Young function if it satisfies the following condition*

*(i)* $\varphi_{\mathscr{D}}(0) = 0.$

*(ii)* $\varphi_{\mathscr{D}}(X) = \varphi_{\mathscr{D}}(-X).$

*(ii)* $\lim_{X\to\infty} \varphi_{\mathscr{D}}(X) = +\infty$

*where we assume the convention that* $+\infty = \alpha e + \infty e^\dagger = \infty e + \beta e^\dagger = \infty e + \infty e^\dagger$
*and* $\lim_{x\to\infty} \varphi_{\mathscr{D}}(x)$ *means that the limit must exists along any curve passing through infinity in the hyperbolic plane and must be equal.*

**Theorem 4.2.** *Let $\varphi_1 : \mathbb{R} \to \overline{\mathbb{R}}^+$ and $\varphi_2 : \mathbb{R} \to \overline{\mathbb{R}}^+$ be two real Young function. Then the hyperbolic product type function $\varphi_{\mathscr{D}} : \mathscr{D} \to \overline{\mathscr{D}}^+$ defined by*

$$\varphi_{\mathscr{D}}(X) = \mathbf{IE}\varphi_{12}^{(0)} \ \ \forall\ X = x_1e + x_2e^\dagger \in \mathscr{D}$$

*is a hyperbolic product type Young function.*

*Proof.* Since $\varphi_1$ and $\varphi_2$ are real Young function implies that $\varphi_1(0) = \varphi_2(0) = 0, \varphi_1(-x_1) = x_1$ and $\varphi_2(-x_2) = x_2$ implies that $\varphi_{\mathscr{D}}(0) = \varphi_1(0)e + \varphi_2(0)e^\dagger = 0$ and $\varphi_{\mathscr{D}}(-X) = \varphi_1(-x_1)e + \varphi_2(-x_2)e^\dagger = \varphi_1(x_1)e + \varphi_2(x_2)e^\dagger = \varphi_{\mathscr{D}}(X)$. Further $\lim_{x_1\to\infty} \varphi_1(x_1) = \infty$ and $\lim_{x_2\to\infty} \varphi_2(x_2) = \infty$. We have three cases for $X \to \infty = \alpha e + \infty e^\dagger = \infty e + \beta e^\dagger = \infty e + \infty e^\dagger$

Case I If $X \to \alpha e + \infty e^\dagger$ then $\lim_{X\to\infty} \varphi_{\mathscr{D}}(X) = \lim_{x_1\to\alpha} \varphi_1(x_1)e + \lim_{x_2\to\infty} \varphi_2(x_2)e^\dagger = \gamma_1 e + \infty e^\dagger = \infty$

Case II If $X \to \infty e + \beta e^\dagger$ then $\lim_{X\to\infty} \varphi_{\mathscr{D}}(X) = \lim_{x_1\to\infty} \varphi_1(x_1)e + \lim_{x_2\to\beta} \varphi_2(x_2)e^\dagger = \infty e + \gamma_2 e^\dagger = \infty$

Case III If $X \to \infty e + \infty e^\dagger$ then $\lim_{X\to\infty} \varphi_{\mathscr{D}}(X) = \lim_{x_1\to\infty} \varphi_1(x_1)e + \lim_{x_2\to\infty} \varphi_2(x_2)e^\dagger = \infty e + \infty e^\dagger = \infty$

Thus in each cases we see that $\lim_{X\to\infty} \varphi_{\mathscr{D}}(X) = \infty$ This proves that $\varphi_{\mathscr{D}} : \mathscr{D} \to \overline{\mathscr{D}}^+$ is a product type hyperbolic Young function. □

Let us construct an example in this direction.

**Example 4.3.** *Consider $\varphi_{\mathscr{D}} : \mathscr{D}^+ \to \overline{\mathscr{D}}^+$ defined by $\varphi_{\mathscr{D}}(X) = X^p$ for $p > 1$. Then it is easy to check that $\varphi_{\mathscr{D}}$ is a product type hyperbolic Young function function with*

$$\varphi_{\mathscr{D}}(X) = x_1^p e + x_2^p e^\dagger = X^p, \ \ \text{where} \ \ X = x_1e + x_2e^\dagger.$$

Thus we have the following definition

**Definition 4.4.** *A product type hyperbolic convex function $\varphi_{\mathscr{D}} : \mathscr{D} \to \overline{\mathscr{D}}^+$ is said to be a product type hyperbolic Young function if $\exists$ two real Young functions $\varphi_1 : \mathbb{R} \to \overline{\mathbb{R}}^+$ and $\varphi_2 : \mathbb{R} \to \overline{\mathbb{R}}^+$ such that*

$$\varphi_{\mathbb{D}}(X) = \mathbf{IE}\varphi_{12}^{(0)} \ \ \forall\ X = x_2e + x_2e^\dagger \in \mathscr{D}. \tag{4.1}$$

Next theorem is a $\mathscr{D}$-integral representation for product type hyperbolic Young function.

**Theorem 4.5.** *Let $\varphi_{\mathscr{D}} : \mathscr{D}^+ \to \overline{\mathscr{D}}^+$ be a product type Young function. Then $\varphi_{\mathscr{D}}$ has a $\mathscr{D}$-integral representation as*

$$\varphi_{\mathscr{D}}(X) = \int_{[0,X]_{\mathscr{D}}} \Phi(Z)dZ \wedge dZ^\dagger \ \ \forall\ X \in \mathscr{D}^+ \tag{4.2}$$

*where $\Phi(0) = 0$ and $\Phi : \mathscr{D}^+ \to \overline{\mathscr{D}}^+$ is a product type $\mathscr{D}$-monotone increasing left continuous function.*

*Proof.* The proof of the theorem follows from Theorem 3.14 □

**Theorem 4.6.** *Let $(\varphi_1, \psi_1)$ and $(\varphi_2, \psi_2)$ be two pair of complementary real Young function. Then the pair $(\varphi_{\mathscr{D}}, \psi_{\mathscr{D}})$ where $\varphi_{\mathscr{D}}$ is a product type hyperbolic Young function defined by $\varphi_{\mathscr{D}} = \varphi_1 e + \varphi_2 e^\dagger$ and $\psi_{\mathscr{D}}$ is a product type hyperbolic Young function defined by $\psi_{\mathscr{D}} = \psi_1 e + \psi_2 e^\dagger$ satisfies*

$$\psi_{\mathscr{D}}(Y) = \sup_{X\in\mathscr{D}^+} \{XY - \varphi_{\mathscr{D}}(X)\} \ \ \ for\ \ X, Y \in \mathscr{D}^+ \tag{4.3}$$

*Proof.* Let $X = x_1e + x_2e^\dagger, Y = y_1e + y_2e^\dagger$. Then since $(\varphi_1, \psi_1)$ and $(\varphi_2, \psi_2)$ are complementary pair of real Young function implies that

$$\psi_1(y_1) = \sup_{x_1\in\mathbb{R}^+} \{x_1y_1 - \varphi_1(x_1)\} \ \text{ and } \ \psi_2(y_2) = \sup_{x_2\in\mathbb{R}^+} \{x_2y_2 - \varphi_2(x_2)\}. \tag{4.4}$$

So that

$$\begin{aligned} \psi_{\mathscr{D}}(Y) &= \psi_1(y_1)e + \psi_2(y_2)e^\dagger \\ &= \sup_{x_1\in\mathbb{R}^+} \{x_1y_1 - \varphi_1(x_1)\} e + \sup_{x_2\in\mathbb{R}^+} \{x_2y_2 - \varphi_2(x_2)\} e^\dagger \\ &= \sup_{X\in\mathscr{D}^+} \left\{(x_1y_1e + x_2y_2e^\dagger) - \mathbf{IE}\varphi_{12}^{(0)}\right\} \\ &= \sup_{X\in\mathscr{D}^+} \{XY - \varphi_{\mathscr{D}}(X)\} \end{aligned}$$

Thus

$$\psi_{\mathscr{D}}(Y) = \sup_{X\in\mathscr{D}^+} \{XY - \varphi_{\mathscr{D}}(X)\} \tag{4.5}$$

□

We call $\psi_{\mathscr{D}}$ as a product type hyperbolic complementary Young function corresponding to $\varphi_{\mathscr{D}}$.

**Theorem 4.7.** *Let $\varphi_{\mathscr{D}}$ be a product type hyperbolic Young function and $\psi_{\mathscr{D}}$ be the product type hyperbolic complementary pair of $\varphi_{\mathscr{D}}$. Then the pair $(\varphi_{\mathscr{D}}, \psi_{\mathscr{D}})$ satisfies the $\mathscr{D}$-Young's inequality*

$$XY \leq' \varphi_{\mathscr{D}}(X) + \psi_{\mathscr{D}}(Y), \;\; X' > 0, Y' > 0 \tag{4.6}$$

*with equality in (4.6) holds if $Y = \Phi_1(X)$ or $X = \Phi_2(Y)$ or $Z = \Phi_3(W)$ or $W = \Phi_4(Z)$ where $X = x_1 e + x_2 e^{\dagger}, Y = y_1 e + y_2 e^{\dagger}, Z = y_1 e + x_2 e^{\dagger}$ and $W = x_1 e + y_2 e^{\dagger}$*

*Proof.* Since $(\varphi_{\mathscr{D}}, \psi_{\mathscr{D}})$ is a complementary pair of hyperbolic Young function implies that the corresponding $(\varphi_1, \psi_1)$ and $(\varphi_2, \psi_2)$ are also two complementary pair of Young function. This means that the pairs $(\varphi_1, \psi_1)$ satisfies $\mathbb{R}$-Young's inequality as

$$x_1 y_1 \leq \varphi_1(x_1) + \psi_1(y_1), \;\; x_1 > 0, y_1 > 0 \tag{4.7}$$

with equality holds if $y_1 = \phi_1(x_1)$ or $x_1 = \phi_2(y_1)$ where $\phi_1 : \mathbb{R}^+ \to \overline{\mathbb{R}}^+$ and $\phi_2 : \mathbb{R}^+ \to \overline{\mathbb{R}}^+$ are $\mathbb{R}$- monotone increasing and left continuous function associated with $\varphi_1$ and $\psi_1$ in its integral representation respectively given on Corollary 2 Page no. 10 [24]. Similarly the other pair $(\varphi_2, \psi_2)$ also satisfies $\mathbb{R}$-Young's inequality as

$$x_2 y_2 \leq \varphi_2(x_2) + \psi_2(y_2), \;\; x_2 > 0, y_2 > 0 \tag{4.8}$$

with equality holds if $y_2 = \phi_3(x_2)$ or $x_2 = \phi_4(y_2)$ where $\phi_3 : \mathbb{R}^+ \to \overline{\mathbb{R}}^+$ and $\phi_4 : \mathbb{R}^+ \to \overline{\mathbb{R}}^+$ are $\mathbb{R}$-monotone increasing and left continuous function associated with $\varphi_2$ and $\psi_2$. Multiplying equation 4.7 by $e$ and equation 4.8 by $e^{\dagger}$ and then adding, we get

$$\begin{aligned} XY &= x_1 y_1 e + x_2 y_2 e^{\dagger} \\ &\leq' (\varphi_1(x_1) + \psi_1(y_1))\, e + (\varphi_2(x_2) + \psi_2(y_2))\, e^{\dagger} \\ &= \mathbf{IE}\varphi_{12}^{(0)} + \left(\psi_1(y_1) e + \psi_2(y_2) e^{\dagger}\right) \\ &= \varphi_{\mathscr{D}}(X) + \psi_{\mathscr{D}}(Y) \end{aligned}$$

So that

$$XY \leq' \varphi_{\mathscr{D}}(X) + \psi_{\mathscr{D}}(Y), \;\; X' > 0, Y' > 0 \tag{4.9}$$

Now for equality to hold in equation (4.9), we have the following cases.

Case I If $y_1 = \phi_1(x_1)$ and $y_2 = \phi_3(x_2)$. Then $Y = y_1 e + y_2 e^{\dagger} = \phi_1(x_1) e + \phi_3(x_2) e^{\dagger}$. So that $Y = \Phi_1(x_1 e + x_2 e^{\dagger}) = \Phi_1(X)$ where $\Phi_1 : \mathscr{D} \to \mathscr{D}$ is a $\mathscr{D}$-monotone increasing left continuous product type function in the $\mathscr{D}$-integral representation of the product type hyperbolic Young function $\varphi_{\mathscr{D}} = \varphi_1 e + \varphi_2 e^{\dagger}$
Hence $Y = \Phi_1(X)$

Case II If $x_1 = \phi_2(y_1)$ and $x_2 = \phi_4(y_2)$.Then $X = x_1 e + x_2 e^{\dagger} = \phi_2(y_1) e + \phi_4(y_2) e^{\dagger}$. So that $X = \Phi_2(y_1 e + y_2 e^{\dagger}) = \Phi_2(Y)$ where $\Phi_2 : \mathscr{D}^+ \to \overline{\mathscr{D}}^+$ is a $\mathscr{D}$-monotone increasing left continuous product type function in the $\mathscr{D}$-integral representation of the product type hyperbolic Young function $\psi_{\mathscr{D}} = \psi_1 e + \psi_2 e^{\dagger}$
Hence $X = \Phi_2(Y)$

Case III If $y_1 = \phi_1(x_1)$ and $x_2 = \phi_4(y_2)$. Then $Z = y_1 e + x_2 e^{\dagger} = \phi_1(x_1) e + \phi_4(y_2) e^{\dagger}$. So that $Z = \Phi_3(x_1 e + y_2 e^{\dagger}) = \Phi_3(W)$, where $\Phi_3 : \mathscr{D}^+ \to \overline{\mathscr{D}}^+$ is a $\mathscr{D}$-monotone increasing left continuous product type function in the $\mathscr{D}$-integral representation of the product type hyperbolic Young function defined by using $\varphi_1$ and $\psi_2$.
Hence $Z = \Phi_3(W)$

Case IV Similarly when $x_1 = \phi_2(y_1)$ and $y_2 = \phi_3(x_2)$ Then $W = \Phi_4(Z)$ where $\Phi_4 : \mathscr{D}^+ \to \overline{\mathscr{D}}^+$ is a $\mathscr{D}$-monotone increasing left continuous product type function in the $\mathscr{D}$-integral representation of the product type hyperbolic Young function defined by using $\psi_1$ and $\varphi_2$
Hence $W = \Phi_4(Z)$

Thus this prove that equality in equation (4.6) holds if $Y = \Phi_1(X)$ or $X = \Phi_2(Y)$ or $Z = \Phi_3(W)$ or $W = \Phi_4(Z)$ for $X' > 0, Y' > 0, Z' > 0$ and $W' > 0$ □

## 5. Conclusion

The concept of the product type hyperbolic convex function and hyperbolic Young functions and its various properties and some other properties still to be proven further may pave way for us to study Product Type Orlicz spaces which considerably uses the concept of Young function.

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