Stability and optimal control analysis of Zika virus with saturated incidence rate

Naba Kumar Goswami[1]* and B. Shanmukha[2]

Abstract
Stability analysis of a non-linear mathematical model is studied and analyzed the transmission dynamics of the Zika virus disease. In our model, the human to human sexual transmission of Zika virus is modeled by considering the saturated incidence rate. This assumption is reasonable as it incorporates the behavioral change of the susceptible individuals and the crowding effect of the infective individuals. The equilibria of the proposed model are obtained and the basic reproduction number (R_0) is computed. The model also exhibits backward bifurcation where the stable disease-free equilibrium coexists with a stable endemic equilibrium, which suggests that the $R_0 < 1$ is not enough to eradicate the disease. The sensitivity analysis of the parameters of the basic reproduction number of the model is presented. The sensitivity analysis is performed to distinguish the main variables that affect the basic reproduction number, which can be regulated to control the transmission dynamics of the Zika. Finally, the optimal control strategies are incorporated into the model and performed a numerical simulation to support our analytical findings.

Keywords
Zika virus, basic reproduction number, bifurcation analysis, stability analysis, sensitivity Analysis.

AMS Subject Classification
34K18,34D20,34D23,37C75,49Q12,49J15,90C31,93C15,93D05.

[1] *Department of Mathematics, PET Research Centre,University of Mysore, Karnataka, India.*
[2] *Department of Mathematics, PES College of Engineering, Mandya, Karnataka, India.*
Corresponding author: [1] nabakrgoswami@gmail.com

1. Introduction

Zika virus (ZIKV) is a major public health challenge in Brazil and Latin American countries recently. ZIKV is primarily spread to the human population by the contact of an infected Aedes genes species female mosquito [1]. Sexual transmission and blood transfusion are also the main cause of the spread of ZIKV [10]. In 2007 to 2018 Zika outbreak is identified in the Island of Yap, Micronesia [3], France Polynesia (42 GBS cases) [1], South Pacific, South American countries, especially in Brazil and Colombia [8]. From October 2013 to April 2014 the largest outbreak was reported in South Pacific, French Polynesia. In February, 2016 the World Health Organization announced Zika outbreak [24] is a Public Health Emergency of International Concern(PHEIC) as it developed Guillain-Barre Syndrome (GBS) [17] and vertical transmission, specifically microcephaly [5]. As a result of GBS and vertical transmission, many people suffered neurological problems in 2016. The first sexual transmission of Zika wears found in France (2016) [1]. During the 2015 out-

break, sexual transmission Zika has been investigated in many countries. It was confirmed that in the year 2016 more than 1,40,000, people [10] have been affected by ZIKA. In Brazil, from October 2015 to February 2016 many people suffered ZIKV including 139 congenital microcephaly cases identified [1]. The symptoms of the disease are very mild and there is no particular medicine to treatment and vaccine for the disease.

Agusto F B et al. [2] proposed a deterministic model to study the vertical transmission of ZIKV disease. Bonyah E et al. [3] proposed and analysis a ZIKV model with simple mass-action type incidence and used different types as a control strategy to reduce the disease. Daozhou G et al. [10] developed a ZIKV epidemic model and computed the basic reproduction number. Moreno V M et al. [14] presented a multi-patch model to see the effect of the role of short-term dispersal dynamics of ZIKV disease. Srivastav A K et al. [21] constructed a new ZIKV model with media impact for the human population and standard mass-action type incidence to reduce the transmission.

In this paper, we have formulated a deterministic model

for ZIKV considering the standard incident type of interaction and saturated incidence rate for the human to human sexual transmission. The study of ZIKV is certainly necessary as at present more than 84 countries [24] people are suffering the disease. This paper is organized as follows: Section 2 formulates the mathematical model; Section 3 analysis of the model and existence of equilibrium; Section 4 discussed the existence of bifurcation of the model; Section 5 present the stability analysis of the model; Section 6 illustrates the numerical simulation and results of the model; Section 7 discussed sensitivity of the parameters of R_0; section 8 demonstrate the numerical simulation results and finally in Section 9 we conclude our paper.

2. The Model

We have proposed a new model for the transmission dynamics of ZIKV with the saturated incidence rate. In the formulation of the proposed model the human population has been classified into four categories such as Susceptible is denoted by (S_h), Infected is denoted by (I_h) and recovered is denoted by (R_h). Similarly, the mosquito population has been classified into two categories such as susceptible is denoted by (S_v) and infected is denoted by (I_v) mosquitoes. There are three types of transmission are possible in the dynamical system between human to human, human to mosquito and mosquito to human. Here, we have incorporated a nonlinear incidence function for the human to human transmission, which consists of a saturated incidence type to reduce the transmission. Also, we have incorporated three types incidence rate [12] such as the bilinear incidence rate of the form $(\beta_1 S_h I_h)$, the saturated incidence rate of the form $\left(\dfrac{\beta_1 S_h I_h}{1+p_1 S_h}\right)$, where (p_1) is a positive constant and another one saturated incidence rate of the form $\left(\dfrac{\beta_1 S_h I_h}{1+p_2 I_h}\right)$, where (p_2) is a positive constant. Some authors have studied in detail this type of nonlinear incidence function [4, 14, 20, 25] in their respective model. Based on the above assumptions we have constructed the following system of model:

$$
\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h - \beta_1\left(\frac{I_h}{1+p_1 S_h+p_2 I_h}\right)S_h - \beta_2\left(\frac{I_v}{N_h}\right)S_h \\
&\quad -\mu_h S_h \\
\frac{dI_h}{dt} &= \beta_1\left(\frac{I_h}{1+p_1 S_h+p_2 I_h}\right)S_h + \beta_2\left(\frac{I_v}{N_h}\right)S_h \\
&\quad -(\gamma_h+\mu_h+\mu_1)I_h \\
\frac{dR_h}{dt} &= \gamma_h I_h - \mu_h R_h \qquad\qquad (2.1)\\
\frac{dS_v}{dt} &= \Lambda_v - \beta_v\left(\frac{I_h}{N_h}\right)S_v - \mu_v S_v \\
\frac{dI_v}{dt} &= \beta_v\left(\frac{I_h}{N_h}\right)S_v - \mu_v I_v.
\end{aligned}
$$

Here Λ_h is the recruitment rate of human population; Λ_v is the recruitment of vector(mosquito) population; β_1 is the
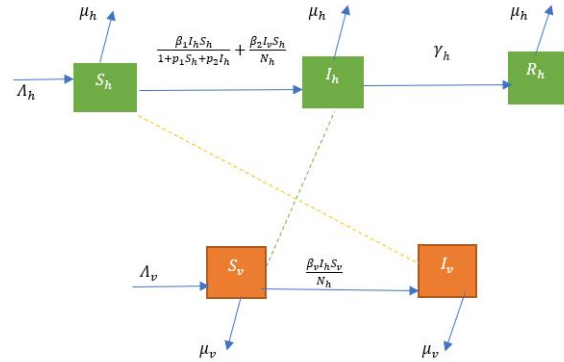

Figure 1. Transmission dynamics of the ZIKV model.

transmission rate between S_h and I_h; β_2 is the transmission rate between S_h and I_v; β_v is the transmission rate between I_h and S_v; μ_h is natural death rate of human population; μ_1 is the natural death rate of human population due to infection; μ_v is the natural death rate of mosquito population; γ_h is the recovery rate of symptomatic infective(human) population; S_h is the susceptible individuals who can suffer the disease but are not yet infective; I_h is the infected individuals one who is suffering the disease and are carriers and R_h is the recovered individuals who has been recovered or removed from the host population by either permanent immunity and temporarily immunity or isolated or dead;

2.1 Positivety and boundedness of the solutions
Consider a feasible region

$$\{\Omega = (S_h,I_h,R_h,S_v,I_v)\in R_+^5 : 0\le N_h\le \frac{\Lambda_h}{\mu_h}, 0\le N_v\le \frac{\Lambda_v}{\mu_v}\}$$

Let $N_h = S_h+I_h+R_h$ and $N_v = S_v+I_v$ are the total population sizes of human and mosquitoes. Then

$$
\begin{aligned}
\frac{dN_h}{dt} &= \Lambda_h - \mu_h N_h - \mu_1 I_h \\
\frac{dN_v}{dt} &= \Lambda_v - \mu_v N_v.
\end{aligned}
$$

Clearly, whenever $N_h > \dfrac{\Lambda_h}{\mu_h}$, $N_v > \dfrac{\Lambda_v}{\mu_v}$, $\dfrac{dN_h}{dt}<0$ and $\dfrac{dN_v}{dt}<0$. Here, $\dfrac{dN_h}{dt}$ and $\dfrac{dN_v}{dt}$ are bounded by $(\Lambda-\mu N_h)$ and $(\Lambda_v - \mu_v N_v)$. By using the standard comparison theorem as described in, [13], we get

$$0\le N_h(t)\le \frac{\Lambda_h}{\mu_h}(1-e^{-\mu^t})+N_h(0)e^{-\mu^t}$$

and

$$0\le N_v(t)\le \frac{\Lambda_v}{\mu_v}(1-e^{-\mu_v^t})+N_v(0)e^{-\mu_v^t}$$

Then

$$N_h(t) \le \frac{\Lambda_h}{\mu_h}, N_v(t) \le \frac{\Lambda_v}{\mu_v} \ \ if \ \ N_h(0) \le \frac{\Lambda}{\mu}, N_v(0) \le \frac{\Lambda_v}{\mu_v}$$

Hence the the biological feasible region

$$\{\Omega = (S_h, I_h, R_h, S_v, I_v) \in R_+^5 : 0 \le N_h \le \frac{\Lambda_h}{\mu_h}, 0 \le N_v \le \frac{\Lambda_v}{\mu_v}\}$$

is positively invariant and bounded by $(\Lambda_h - \mu_h N_h)$ and $(\Lambda_v - \mu_v N_v)$.

As $N_h = S_h + I_h + R_h$ and $N_v = S_v + I_v$, we werite the system of eqs. (2.1) in the following form for further analysis:

$$\frac{dN_h}{dt} = \Lambda_h - \mu_h N_h - \mu_1 I_h$$

$$\frac{dI_h}{dt} = \frac{\beta_1(N_h - I_h - R_h)I_h}{1 + p_1(N_h - I_h - R_h) + p_2 I_h} + \frac{\beta_2(N_h - I_h - R_h)I_v}{N_h}$$
$$- (\gamma_h + \mu_h + \mu_1)I_h$$

$$\frac{dR_h}{dt} = \gamma_h - \mu_h R_h \qquad (2.2)$$

$$\frac{dS_v}{dt} = \Lambda_v - \mu_v N_v$$

$$\frac{dI_v}{dt} = \beta_v \frac{(N_v - I_v)I_h}{N_h} - \mu_v I_v.$$

3. Existence of Equilibria

3.1 Disease-free equilibrium and The Basic Reproduction Number

The disease-free equilibrium point $E_0 = (N_h^0, I_h^0, R_h^0, N_v^0, I_v^0) = \left(\frac{\Lambda_h}{\mu_h}, 0, 0, \frac{\Lambda_v}{\mu_v}, 0 \right)$ for the model (2.2).

We find the basic reproduction number R_0 by following the next generation matrix method as described in [6, 9]. Following the same notations as in [6, 9] we get:

$$\mathscr{F} = \left(\begin{array}{c} \frac{\beta_1(N_h - I_h - R_h)I_h}{1 + p_1(N_h - I_h - R_h) + p_2 I_h} + \frac{\beta_2(N_h - I_h - R_h)I_v}{N_h} \\ \beta_v \frac{(N_v - I_v)I_h}{N_h} \end{array} \right)$$

and

$$\mathscr{V} = \left(\begin{array}{c} (\gamma_h + \mu_h + \mu_1)I_h \\ \mu_v I_v \end{array} \right)$$

F= Jacobian of $\mathscr{F}$ at $E_0 = \left(\begin{array}{cc} \frac{\beta_1 N_h^0}{1 + p_1 N_h^0} & \beta_2 \\ \frac{\beta_v N_v^0}{N_h^0} & 0 \end{array} \right)$

and V = Jacobian of $\mathscr{V}$ at $E_0 = \left(\begin{array}{cc} \gamma_h + \mu_h + \mu_1 & 0 \\ 0 & \mu_v \end{array} \right)$

and it follows that

$$FV^{-1} = \left(\begin{array}{cc} \frac{\beta_1 N_0^v}{(1 + p_1 N_0^h)(\gamma_h + \mu_h + \mu_1)} & \frac{\beta_2}{\mu_v} \\ \beta_v \frac{N_0^v}{N_0^v(\gamma_h + \mu_h + \mu_1)} & 0 \end{array} \right),$$

The spectral radius of the matrix $\rho(FV^{-1})$ is called the basic reproduction number R_0 and it follows that:

$$R_0 = \frac{\frac{\beta_1 N_h^0 d_1}{(1 + p_1 N_h^0)} + \sqrt{(\frac{\beta_1 N_h^0 d_1}{(1 + p_1 N_h^0)})^2 + \frac{4\beta_2 \beta_v N_v^0 d_1}{\mu_v N_h^0}}}{2}$$
$$= \frac{R_1 + \sqrt{(R_1)^2 + 4R_2}}{2}$$

where,

$$R_1 = \frac{\beta_1 N_h^0 d_1}{(1 + p_1 N_h^0)}, R_2 = \frac{4\beta_2 \beta_v N_v^0 d_1}{\mu_v N_h^0}, d_1 = \frac{1}{\gamma_h + \mu_h + \mu_1}$$

Here R_1 denotes the basic reproduction due to human to human transmission by ignoring the transmission due to vectors. Similarly, R_2 denotes the basic reproduction due to interactions with vectors in the absence of human to human transmission. The reproduction number R_0 gives the average number of infected individuals generated by the one infected in a fully susceptible population and for our model it is given by above expression of R_0.

3.2 The Endemic Equilibrium

For the system (2.2), we get the endemic equilibrium point as $E_1 = (N_h^*, I_h^*, R_h^*, N_v^*, I_v^*)$,
where

$$N_h^* = \frac{\Lambda_h - \mu_1 I h^*}{\mu_h},$$

$$R_h^* = \frac{\gamma_h I_h^*}{\mu_h},$$

$$N_v^* = \frac{\Lambda_v}{\mu_v},$$

$$I_v^* = \frac{\Lambda_h \mu_h \beta_v I_h^*}{\mu_h \mu_v \beta_v I_h^* + \mu_v^2(\Lambda_v - \mu_1 I_h^*)}, \ provided \ \frac{\Lambda_v}{\mu_v} > I_h^*$$

For I_h^*, we substituting the value of $N_h^*, R_h^*, N_v^*, I_v^*$ in the equilibrium $\frac{dI_h^*}{dt}$, we get the following equation

$$g(I_h) = \frac{(\Lambda_h - A)\beta_1(\Lambda_h - \mu_1 I_h)}{\mu_h + p_1 \Lambda_h - p_1 A I_h + p_2 \mu_h I_h}$$
$$+ \frac{(\Lambda_h - A)\beta_2 \mu_h \mu_v \Lambda_v}{\beta_2 \mu_h \mu_v I_h + (\Lambda_h - \mu_1 I_h)\mu_v^2}$$
$$- (\Lambda_h - \mu_1 I_h)(\mu_1 + \mu_h + \gamma_h) = 0$$

$$g(0) = \Lambda_h \left[\frac{\beta_1 \Lambda_h}{\mu_h + p_1 \Lambda_h} + \frac{\beta_2 \mu_h \mu_v \Lambda_v}{\Lambda_h \mu_v^2} \right] - \Lambda_h(\mu_1 + \mu_h + \gamma_h) > 0$$

$$g\left(\frac{\Lambda_h}{\mu_1}\right) = -\Lambda_h\left(\frac{\mu_h+\gamma_h}{\mu_1}\right)\left(\frac{\beta_2\mu_1\Lambda_v}{\Lambda_h\mu_v}\right) < 0$$

$$g\left(\frac{\Lambda_h}{\mu_1+\mu_h+\gamma_h}\right) = -\Lambda_h(\mu_h+\gamma_h) < 0$$

Clearly, $g(I_h)$ is negative in the interval $\dfrac{\Lambda_h}{\mu_1+\mu_h+\gamma_h} < I_h < \dfrac{\Lambda_h}{\mu_1}$. Therefore, in the interval $\dfrac{\Lambda_h}{\mu_1+\mu_h+\gamma_h} < I_h < \dfrac{\Lambda_h}{\mu_1}$ there is no root, as there is no sign change of $g(I_h)$. Hence, there is one or more root are possible in the interval $0 < I_h < \dfrac{\Lambda_h}{\mu_1+\mu_h+\gamma_h}$ of $g(I_h) = 0$

$$\begin{aligned}
\frac{dg(I_h)}{dI_h} &= -\frac{B\beta_1\mu_1\{\mu_h+p_1(\Lambda_h-AI_h+p_2\mu_hI_h)\}}{(\mu_h+p_1\{(\Lambda_h-AI_h+p_2\mu_hI_h)\})^2}\\
&\quad -\frac{(Bp_2(\mu_1+\mu_h+\gamma_h)+p_2\mu_h}{(\mu_h+p_1\{(\Lambda_h-AI_h+p_2\mu_hI_h)\})^2}\\
&\quad -\left[\frac{BI_h\Lambda_v\beta_2\beta_v\mu_h(\mu_h\mu_v\beta_v-\mu_1\beta_v^2)}{(\mu_h\mu_v\beta_vI_h+(\Lambda_h-\mu_hI_h)\mu_v^2)^2}\right]\\
&\quad -\frac{A\beta_1(\Lambda_h-\mu_1I_h)}{\mu_h+p_1\{(\Lambda_h-AI_h+p_2\mu_hI_h\}}\\
&\quad -\frac{A\Lambda_v\beta_2\beta_v\mu_h}{\mu_h\mu_v\beta_vI_h+(\Lambda_h-\mu_hI_h)\mu_v^2}\\
&\quad -(\mu_1+\mu_h+\gamma_h)
\end{aligned}$$

where $A = (\mu_1+\mu_h+\gamma_h)$, $B = \Lambda_h - AI_h$
If $\mu_h\mu_v\beta_v > \mu_1\beta_v^2$ and $\Lambda_h > \mu_hI_h$ then clearly $\dfrac{dg(I_h)}{dI_h} < 0$. So in the interval $0 < I_h < \dfrac{\Lambda_h}{\mu_1+\mu_h+\gamma_h}$ there exists a positive root I_h^* of $\dfrac{dg(I_h)}{dI_h} = 0$. Also it is clear that if $\dfrac{dg(I_h)}{dI_n} < 0$ at $\dfrac{\Lambda_h}{\mu_1+\mu_h+\gamma_h}$ then it must be negative for all I_h in the interval $0 < I_h < \dfrac{\Lambda_h}{\mu_1+\mu_h+\gamma_h}$. Hence we get the positive equilibrium point $E_1 = (N_h^*, I_h^*, R_h^*, N_v^*, I_v^*)$ with respect to the above conditions. But if $\dfrac{dg(I_h)}{dI_h}$ is not negative throughout the interval $0 < I_h < \dfrac{\Lambda_h}{\mu_1+\mu_h+\gamma_h}$, then there is a probability of getting two or more roots the equation $g(I_h) = 0$. Hence the phenomenon of backward bifurcation exist for any type of mosquito-borne disease model with respect to the endemic equilibrium points for $R_0 < 1$.

4. Analysis of Backward Bifurcation

Let us consider the following change of variables $N_h = x_1, I_h = x_2, R_h = x_3, N_v = x_4, I_v = x_5$. Also further by using vector notation $X = (x_1, x_2, x_3, x_4, x_5)^T$, our system (2.2) can be formulated.

$$\frac{dX}{dt} = F(x), \text{ where } F = (f_1, f_2, f_3, f_4, f_5)^T$$

$$\begin{aligned}
x_1' &= \Lambda_h - \mu_hx_1 - \mu_1x_2\\
x_2' &= \frac{\beta_1(x_1-x_2-x_3)x_2}{1+p_1(x_1-x_2-x_3)+p_2x_2} + \beta_2(x_1-x_2-x_3)\frac{x_5}{x_1}\\
&\quad -(\mu_1+\mu_h+\gamma_h)x_2\\
x_3' &= \gamma_hx_2 - \mu_hx_3 \qquad\qquad\qquad (4.1)\\
x_4' &= \Lambda_v - \mu_vx_4\\
x_5' &= \beta_v(x_4-x_5)\frac{x_2}{x_1} - \mu_vx_5
\end{aligned}$$

Consider the case $R_0 = 1$. Suppose, further, that $\beta_1 = \beta_1^*$ is chosen as a bifurcation parameter. Solving for $\beta_1 = \beta_1^*$ from $R_0 = 1$ gives

$$\beta_1^* = \frac{(\mu_h+p_1\Lambda_h)(\mu_1+\mu_h+\gamma_h)}{\Lambda_h}\left(1-\frac{\mu_h\beta_2\beta_v\Lambda_v}{\mu_v^2\Lambda_h(\mu_1+\mu_h+\gamma_h)}\right)$$

The Jacobian of the above system at disease-free equilibrium point is given by

$$J(\beta_1) = \begin{pmatrix}
-\mu_h & -\mu_1 & 0 & 0 & 0\\
0 & m_{22} & 0 & 0 & \beta_2\\
0 & \gamma_h & -\mu_h & 0 & 0\\
0 & 0 & 0 & -\mu_v & 0\\
0 & \beta_v\dfrac{x_4}{x_1} & 0 & 0 & -\mu_v
\end{pmatrix}$$

Where, $x_1 = \dfrac{\Lambda_h}{\mu_h}, x_4 = \dfrac{\Lambda_v}{\mu_v}, m_{22} = \dfrac{\beta_2x_1}{1+p_1x_1} - (\mu_1+\mu_h+\gamma_h)$.
According to Castillo-Chavez and Song [7], we use the center manifold theory and analyze it, which is shown below

Theorem 4.1. *Consider the following general system of ordinary differential equations with a parameter ϕ,*

$$\frac{dx}{dt} = f(x,\phi),$$

$$f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}.$$

and

$$f \in \mathbb{C}^2(\mathbb{R}^n \times \mathbb{R})$$

where 0 is the equilibrium point of the system (i.e. $f(0,\phi)) \equiv 0$ for all ϕ and

(i). $A = D_xf(0,0) = \left(\dfrac{\partial f_i}{\partial x_j}(0,0)\right)$ is the linearization matrix of the system around the equilibrium 0 with f evaluated at 0;
(ii). Zero is the simple eigenvalue of A and other eigenvalues of A has negatives real parts;
(iii). Matrix A has a right eigenvector w and a left eigenvector v corresponding to the zero eigenvalue.

Let f_k be the kth component of f and

$$a_1 = \sum_{k,i,j=1}^{n} v_kw_iw_j\frac{\partial^2 f_k}{\partial x_i\partial x_j}(0,0)$$

$$b_1 = \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(0,0)$$

then the local dynamics of the system around the equilibrium point 0 is totally determined by the signs a_1 and b_1.

1. $a_1 > 0$, $b_1 > 0$. When $\phi < 0$ with $|\phi| \ll 1$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium; when $0 < \phi \ll 0$, 0 is unstable and there exists a negative and locally asymptotically stable equilibrium;

2. $a_1 < 0$, $b_1 < 0$. When $\phi < 0$ with $|\phi| \ll 1$, 0 is unstable; when $0 < \phi \ll 0$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium;

3. $a_1 > 0$, $b_1 < 0$. When $\phi < 0$ with $|\phi| \ll 1$ is unstable, and there exists a locally asymptotically stable negative equilibrium; when $0 < \phi \ll 0$, 0 is stable, and a positive unstable equilibrium appears;

4. $a_1 < 0$, $b_1 > 0$. When ϕ changes from negative to positive, 0 changes its stability from stable to unstable. Correspondingly a negative unstable equilibrium becomes positive and locally asymptotically stable.

4.1 Eigenvalues of $J_{\beta_1^*}$

It can easily seen that the Jacobian with $\beta_1 = \beta_1^*$ of the linearized system has a simple zero eigenvalue and the other eigenvalues have negative real parts. Hence, the center manifold theorem can be used to analyze the dynamics of the syster (4.1) near $\beta_1 = \beta_1^*$. For the case when $R_0 = 1$, using the technique in [7] it can shown taht the matrix $J_{\beta_1^*}$ has a right eigenvector (corresponding to the zero eigenvalue) given by $w = [w_1, w_2, w_3, w_4, w_5]^T$, where

$$w_1 = -\frac{\mu_1}{\mu_h}, w_2 = 1, w_3 = \frac{\gamma_h}{\mu_h}, w_4 = 0, w_5 = \frac{\beta_v \Lambda_h \Lambda_v}{\mu_h \mu_v^2}$$

Similarly, the matrix $J_{\beta_1^*}$ has a left eigenvector (corresponding to the zero eigenvalue), denoted by $v = [v_1, v_2, v_3, v_4, v_5]^T$,where

$$v_1 = 0, v_2 = 1, v_3 = 0, v_4 = 0, v_5 = \frac{\beta_2}{\mu_v}$$

4.2 Computation of a_1

For the system (4.1), the associated non-zero partial derivatives are given by

$$\frac{\partial^2 f_2}{\partial x_1^2} = \frac{-2p_1}{(1+p_1 x_1)^2}; \quad \frac{\partial^2 f_2}{\partial x_1 \partial x_2} = \frac{\beta_1}{(1+p_1 x_1)^2};$$

$$\frac{\partial^2 f_2}{\partial x_2^2} = \frac{-2\beta_1(1+p_1 x_2)}{(1+p_1 x_1)^2}; \quad \frac{\partial^2 f_5}{\partial x_2 \partial x_5} = \frac{-\beta_v}{x_1}$$

$$\frac{\partial^2 f_2}{\partial x_2 \partial x_3} = \frac{-\beta_1}{(1+p_1 x_1)^2}; \quad \frac{\partial^2 f_5}{\partial x_1 \partial x_2} = \frac{-\beta_v x_4}{x_1}$$

It follows from the above expressions that

$$\begin{aligned}
a_1 &= v_2 \left[w_1^2 \left(\frac{-2p_1}{(1+p_1 x_1)^2} \right) + 2w_1 w_2 \left(\frac{\beta_1}{(1+p_1 x_1)^2} \right) \right] \\
&\quad - v_2 \left[2\beta_1 \left(\frac{w_2^2(1+p_1 x_2) + w_2 w_3}{(1+p_1 x_1)^2} \right) \right] \\
&\quad + v_5 \left[2w_1 w_2 \left(\frac{-\beta_v x_4}{x_1} \right) + 2w_2 w_5 \left(\frac{-\beta_v}{x_1} \right) \right] \\
&= \frac{2\mu_h}{(\mu_h + p_1 \Lambda_h)^2} \left[\frac{p_1 \mu_1^2}{\mu_h} - \beta_1(\mu_1 + \mu_h + \Lambda_h p_2 + 2\gamma_h) \right] \\
&\quad + \frac{2\beta_2 \beta_v \Lambda_v}{\mu_v^2} \left(\mu_1 - \frac{\beta_v}{\mu_v} \right)
\end{aligned}$$

4.3 Computation of b_1

For the system (4.1), the associated non-zero partial derivatives are given by

$$\frac{\partial^2 f_2}{\partial x_2 \partial \beta_1} = \frac{x_1}{1+p_1 x_1}$$

It follows from the above expressions that

$$b_1 = v_2 w_2 \left(\frac{x_1}{1+p_1 x_1} \right) = \frac{\Lambda_h}{\mu_h + p_1 \Lambda_h} > 0$$

Clearly, the coefficient b_1 is positive and according to the Theorem (4.1), the sign of the cofficient a_1, which decides the local dynamics of the model aroud the disease-free equilibrim for $\beta_1 = \beta_1^*$.

5. Stability Analysis

Theorem 5.1. *If $R_0 < 1$, the disease-free equilibrium E_0 is locally asymptotically stable otherwise it is unstable.*

The Jacobian matrix of the model (2.2) at disease-free equilibrium point $E_0 = (N_h^0, 0, 0, N_v^0, 0)$ is given by:

$$J_0 = \begin{pmatrix} -\mu_h & -\mu_1 & 0 & 0 & 0 \\ 0 & m_{22} & 0 & 0 & \beta_2 \\ 0 & \gamma_h & -\mu_h & 0 & 0 \\ 0 & 0 & 0 & -\mu_v & 0 \\ 0 & \beta_v \frac{N_v^0}{N_h^0} & 0 & 0 & -\mu_v \end{pmatrix}$$

where,

$$m_{22} = \beta_1 \left(\frac{N_h^0}{1+p_1 N_v^0} \right) - (\mu_1 + \mu_h + \gamma_h)$$

The three eigenvalues of the above matrix J_0 are clearly negarive, that is, $-\mu_h, -\mu_h$ and $-\mu_v$ and the remaining two eigenvalues are the roots of the following characteristics equation:

$$\lambda^2 + A_1 \lambda + A_2 = 0$$

Where,

$$A_1 = \mu_v - m_{22} = \mu_v - (1 - R_1)(\gamma_h + \mu_h + \mu_1) \ for \ R_1 < 1$$

$$A_2 = -\left(\mu_v m_{22} + \beta_2 \beta_v \frac{N_v^0}{N_h^0}\right)$$

$$= -\mu_v \left[\beta_1 \left(\frac{N_h^0}{1 + p_1 N_v^0}\right) - (\mu_1 + \mu_h + \gamma_h)\right] - \beta_2 \beta_v \frac{N_v^0}{N_h^0}$$

$$= \mu_v (\mu_1 + \mu_h + \gamma_h)[1 - (R_1 + R_2)] \ for \ (R_1 + R_2) < 1$$

It is observed that $R_1 < 1$ and $(R_1 + R_2) < 1$. If $A_1 > 0, A_2 > 0$ then both the eigenvalues of the quadratic equation will have negative real part. Hence our system (2.2) is stable if $R_0 < 1$. Hence the disease-free equilibrium E_0 is locally asymptotically stable.

Theorem 5.2. *The disease-free equilibrium E_0 is globally asymptotically stable under some restriction of parameters if $R_0 < 1$, otherwise it is unstable.*

Proof. Using the comparison theorem [13], we proved the theorem. The equations of exposed and infected compartment of the system (2.1) we re-wrire as follows.

$$\begin{pmatrix} \dot{I}_h \\ \dot{I}_v \end{pmatrix} = (F_1 - V_1)\begin{pmatrix} I_h \\ I_v \end{pmatrix} - \left(\begin{matrix} \left(\frac{\beta_1 I_h}{K} + \frac{\beta_2 I_v}{N_h}\right)(S_h^0 - S_h) \\ \beta_v \left(\frac{I_h}{N_h}\right)(S_v^0 - S_v) \end{matrix}\right)$$

where, $K = 1 + p_1 S_h + p_2 I_h$,

$$F_1 = \begin{pmatrix} \frac{\beta_1 N_h^0}{1 + p_1 N_h^0} & \beta_2 \\ \frac{\beta_v N_v^0}{N_h^0} & 0 \end{pmatrix} \ and \ V_1 = \begin{pmatrix} A & 0 \\ 0 & \mu_v \end{pmatrix}$$

As $S_h^0 > S_h$ and $S_v^0 > S_v$, we get

$$\begin{pmatrix} \dot{I}_h \\ \dot{I}_v \end{pmatrix} \le (F_1 - V_1)\begin{pmatrix} I_h \\ I_v \end{pmatrix}$$

Here,

$$F_1 - V_1 = \begin{pmatrix} \beta_1 \left(\frac{N_h^0}{1 + p_1 N_v^0}\right) - (\mu_1 + \mu_h + \gamma_h) & \beta_2 \\ \beta_v \frac{N_v^0}{N_h^0} & -\mu_v \end{pmatrix}$$

The associate characteristic equation of the matrix $(F_1 - V_1)$ is obtain by

$$\lambda^2 + B_1\lambda + B_2 = 0$$

Where,

$$B_1 = \mu_v - (\gamma_h + \mu_h + \mu_1)(1 - R_1)$$
$$B_2 = \mu_v(\mu_1 + \mu_h + \gamma_h)[1 - (R_1 + R_2)]$$

Here it is observed that $R_1 < 1$ and $(R_1 + R_2) < 1$ whenever $R_0 < 1$. If $B_1 > 0, B_2 > 0$ then both the eigenvalues of $(F_1 - V_1)$ will have negative real part. Hence our model (2.2) is stable if $R_0 < 1$. So $(I_h, I_v) \to (0,0)$ as $t \to \infty$. According to comparison theorem [13] it follows that $(I_h, I_v) \to (0,0)$ and $N_h^* \to \frac{\Lambda_h}{\mu_h}$ as $t \to \infty$. Hence $(S_h, I_h, R_h, I_v, S_v) \to E_0$ as $t \to \infty$. Hence E_0 is globally asymptotically stable for $R_0 < 1$, if $B_1 > 0, B_2 > 0$

Theorem 5.3. *When $R_0 > 1$, then endemic equilibrium (E_1) is locally asymptotically stable under some conditions, otherwise it is unstable.*

The Jacobian matrix of the model (2.2) at endemic equilibrium point $E_1 = (N_h, I_h, R_h, N_v, I_v)$ is obtained as follows:

$$J_1 = \begin{pmatrix} -\mu_h & -\mu_1 & 0 & 0 & 0 \\ k_{21} & k_{22} & k_{23} & 0 & k_{25} \\ 0 & \gamma_h & -\mu_h & 0 & 0 \\ 0 & 0 & 0 & -\mu_v & 0 \\ k_{51} & k_{52} & 0 & k_{54} & k_{55} \end{pmatrix}$$

where,

$$k_{21} = \frac{(1 + p_2 I_h^*)\beta_1 I_h^*}{(1 + p_1(N_h^* - I_h^* - R_h^*) + p_2 I_h^*)^2} + \frac{\beta_2 I_v^*(I_h^* + R_h^*)}{(N_h^*)^2}$$

$$k_{22} = -\frac{\beta_1\{(1 + p_1(N_h^* - I_h^* - R_h^*)}{(1 + p_1(N_h^* - I_h^* - R_h^*) + p_2 I_h^*)^2}$$
$$+ \frac{p_2 I_h^*\} + p_1\beta_2 I_h^*(N_h^* - I_h^* - R_h^*)(-p_1 + p_2)}{(1 + p_1(N_h^* - I_h^* - R_h^*) + p_2 I_h^*)^2} - \frac{\beta_2 I_v^*}{N_h^*}$$
$$- (\mu_1 + \mu_h + \gamma_h)$$

$$k_{23} = -\left[\frac{(1 + p_2 I_h^*)\beta_1 I_h^*}{(1 + p_1(N_h^* - I_h^* - R_h^*) + p_2 I_h^*)^2} + \frac{\beta_2 I_v^*}{N_h^*}\right]$$

$$k_{25} = \beta_2\left(1 - \frac{(I_h^* + r_h^*)}{N_h^*}\right)$$

$$k_{51} = -\beta_v\left(\frac{I_h^*(N_v^* - I_v^*)}{(N_h^*)^2}\right)$$

$$k_{52} = \beta_v\left(\frac{N_v^* - I_v^*}{(N_h^*)^2}\right)$$

$$k_{54} = \beta_v\frac{I_h^*}{N_h^*}; k_{55} = -\left(\beta_v\frac{I_h^*}{N_h^*} + \mu_v\right)$$

Clearly, two eigenvalues of the matrix J_1 are negative such as $-\mu_h, -\mu_v$ and the remaining three roots possible to determined by the following cubic equation:

$$\lambda^3 + X\lambda^2 + Y\lambda + Z = 0$$

where,

$$X = (\mu_h - k_{22} - k_{55})$$
$$Y = \{k_{22}k_{55} - \mu_h(k_{22} + k_{55} - k_{22}k_{55}) - \gamma_h k_{23} - \mu_1 k_{21}\}$$
$$Z = (\mu_h k_{22}k_{55} + \gamma_h k_{23}k_{55} - \mu_1 k_{21}k_{55} + \mu_1 k_{51}k_{25})$$

336

According to the Routh-Hurwitz Conditions, the three roots of the cubic equation will have negative roots or roots with negative real parts, if

$$X > 0, \ Y > 0, \ Z > 0, \ XY - Z > 0$$

$$
\begin{aligned}
AB - C \ = \ & -\mu_h^2(k_{22} + 55) - \mu_h(k_{22}^2 + k_{55}^2 - \gamma_h k_{23}) \\
& -\mu_1 k_{21} + k_{22}k_{55} - k_{22}^2 K_{55} \\
& +k_{22}k_{55} - k_{22}k_{55}^2) + \mu_1(k_{21}k_{22} + k_{25}k_{51}) \\
& -k_{22}k_{55}(k_{22} + k_{55} - \mu_h^2) \\
& +\gamma_h k_{22}k_{55} - k_{23}k_{25}k_{52}
\end{aligned}
$$

Thus, the endemic equilibrium point E_1 of the system (2.2) is locally asymptotically stable.

6. Sensitivity Analysis

For the model system, the parameters $\beta_1, \beta_2, \beta_v$ are the main parameters which regulate the basic reproduction number R_0. The algebraic representations of the sensitivity index of R_0 to the parameters $\beta_1, \beta_2, \beta_v$ as described in [22, 23], we get:

$$
\frac{\partial R_0}{\partial \beta_1} \ = \ \frac{N_h^0 d_1}{2(1 + p_1 N_h^0)} + \frac{\beta_1 N_h^0 d_1}{2(1 + p_1 N_h^0)}\sqrt{B}
$$

$$
\frac{\partial R_0}{\partial \beta_2} \ = \ \frac{\beta_v N_v^0 d_1}{\mu_v N_h^0}\sqrt{B}
$$

$$
\frac{\partial R_0}{\partial \beta_v} \ = \ \frac{\beta_2 N_v^0 d_1}{\mu_v N_h^0}\sqrt{B}
$$

$$
where \ B \ = \ \frac{1}{(\frac{\beta_1 N_h^0 d_1}{(1 + p_1 N_h^0)})^2 + \frac{4\beta_2 \beta_v N_v^0 d_1 d_2}{N_h^0}}
$$

The above mathematics expression $\frac{\partial R_0}{\partial \beta_1}, \frac{\partial R_0}{\partial \beta_2}, \frac{\partial R_0}{\partial \beta_v}$ are positive. Hence we can conclude that the if any of the parameter $\beta_1, \beta_2, \beta_v$ increases, the basic reproduction number R_0 also increases. We compute the elasticities to see the proportional change of effect of the parameters R_0. Elasticity is nothing but the proportional response to a proportional perturbation.

$$
E_{\beta_1} = \frac{\beta_1}{R_0}\frac{\partial R_0}{\partial \beta_1} \ = \ \frac{\beta_1 N_h^0 d_1}{2R_0(1 + p_1 N_h^0)} + \frac{\beta_1^2 N_h^0 d_1}{2(1 + p_1 N_h^0)}\sqrt{B}
$$

$$
E_{\beta_2} = \frac{\beta_2}{R_0}\frac{\partial R_0}{\partial \beta_2} \ = \ \frac{\beta_2 \beta_v N_v^0 d_1}{R_0 \mu_v N_h^0}\sqrt{B}
$$

$$
E_{\beta_v} = \frac{\beta_2}{R_0}\frac{\partial R_0}{\partial \beta_2} \ = \ \frac{\beta_2 \beta_v N_v^0 d_1}{R_0 \mu_v N_h^0}\sqrt{B}
$$

$$
E_{\beta_2} = E_{\beta_v}
$$

It is observed that, from the above expressions E_{β_1}, E_{β_2}, E_{β_v}, are positive. As $E_{\beta_2} = E_{\beta_v}$, we can say that β_2 and β_v will have same influence on basic reproduction number R_0. The minor change in parameters $\beta_1, \beta_2, \beta_v$ will have massive change in R_0. In Figures 2 and 3, we have demonstrated the effect of the parameters $\beta_1, \beta_2, \beta_v$ on R_0.
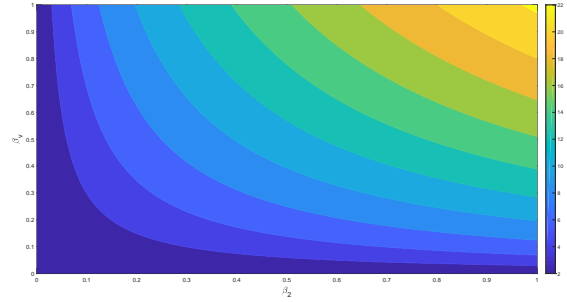
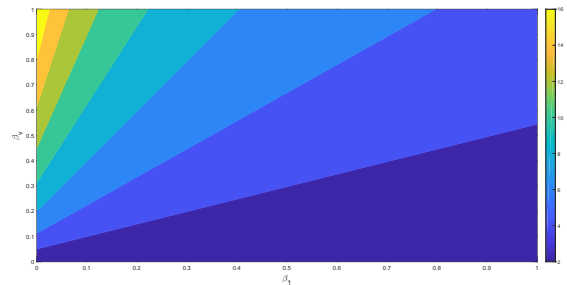

Figure 2. Influence of β_2 and β_v

Figure 3. Influence of β_1 and β_v

7. Numerical Simulation

Here for the simulation of the model system (2.2) is established to support our mathematical findings and we consider all the parameters are in per day basis. To find the stability of disease-free equilibrium, we consider the following set of parameters.

$$\Lambda_h = 2; \ \Lambda_v = 40; \ \gamma_h = 1.436; \ \beta_1 = 0.051; \ \beta_2 = 0.051; \mu_v = 2;$$

$$\beta_v = 0.06; \ p_1 = 0.1; \ p_2 = 0.2; \ \mu_h = 0.047; \ \mu_1 = 0.0523$$

We get $R_0 = 0.9921 < 1$ for above set of parameters and disease-free equilibrium point $E_0(995.51, 0, 0, 298.31, 0)$. The result is illustrated in Figure 4. For endemic equilibrium we consider β_1 from 0.051 to 0.001, Λ_h from 2 to 10 and γ_h from 1.436 to 0.09. Here we found $R_0 = 1.4178 > 1$, and the endemic equilibrium $E_1(85.34, 200.32, 90.15, 987.31, 45.51)$. Which is performed in Figure 5. The variation of I_h for different values of γ_h is shown in Figure 6. In figure 7, we have demonstrated the effect of R_0, β_1 and γ_h.

8. Optimal Control Model

In this section, three time-dependent control parameters $u_1(t)$, $u_2(t)$, and $u_3(t)$ are incorporate in model (2.1). With the help of Pontryagin's maximum principle [15] the optimal control theory is used to get the necessary conditions for optimal control strategies to preventing and controlling the spread of the ZIKV. The optimal control parameters and conditions are consider as follows:
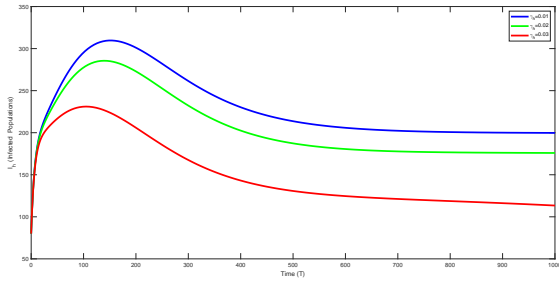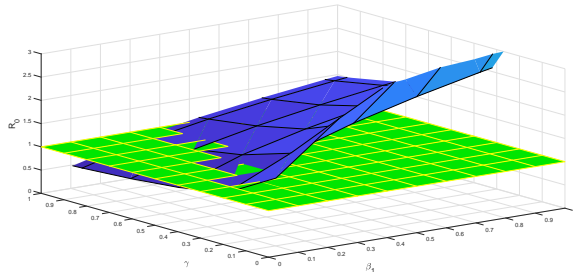(i) the control variable $u_1(t)$ which represents the reductions in the transmission between human to human.
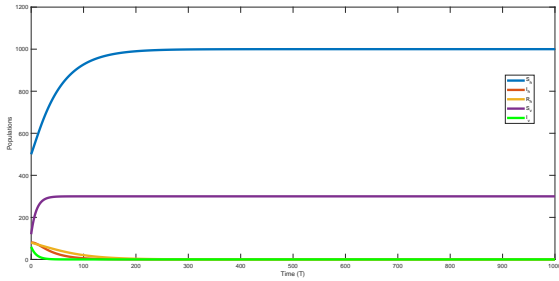
Figure 4. Variation of S_h, I_h, R_h, S_v and I_v with time showing the stability of disease-free equilibrium point.
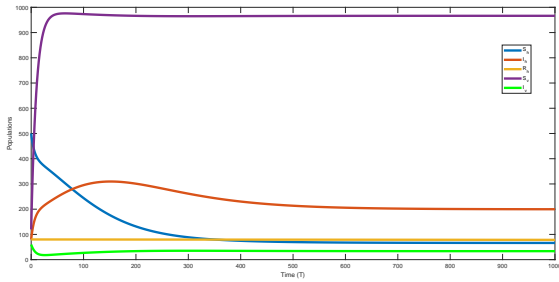
Figure 5. Variation of S_h, I_h, R_h, S_v and I_v with time showing the stability of endemic equilibrium point.

Figure 6. Variation of I_h with time when γ_h is increasing.

Figure 7. Effect of R_0, β_1 and γ_h

(ii) The control variable $u_2(t)$ which represents the use of insecticide-treated bed nets and the use of mosquito repulsive lotions and electronic devices, to reduce mosquito biting rate.
(iii) The control variable $u_3(t)$ corresponds to the additional death rate of mosquitoes due to control efforts.
(iv) If $u_2(t)$ and $u_3(t)$ are equal to zero, then there is no significant effort in these control measures.
(v) If $u_2(t)$ and $u_3(t)$ are equal to one, then the maximum effort being significant.
Based on the above assumptions, the optimal control model as follows:

$$
\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h - (1-u_1)\beta_1 \frac{S_h I_h}{1+p_1 S_h + p_2 I_h} \\
&\quad -(1-u_2)\beta_2 \frac{S_h I_v}{N_h} - \mu_h S_h \\
\frac{dI_h}{dt} &= (1-u_1)\beta_1 \frac{S_h I_h}{1+p_1 S_h + p_2 I_h} \\
&\quad +(1-u_2)\beta_2 \frac{S_h I_v}{N_h} - (\gamma_h + \mu_h + \mu_1)I_h \\
\frac{dR_h}{dt} &= \gamma_h - \mu_h R_h \qquad\qquad (8.1) \\
\frac{dS_v}{dt} &= \Lambda_v - (1-u_2)\beta_v \frac{S_v I_h}{N_h} - (\mu_v + u_3)S_v \\
\frac{dI_v}{dt} &= (1-u_2)\beta_v \frac{S_v I_h}{N_h} - (\mu_v + u_3)I_v.
\end{aligned}
$$

8.1 The Optimal Control Analysis

In this section, we analyze the behavior of the given model by using optimal control theory. The objective functional for fixed time t_f. Following the same notation as described in [20, 21] we get:

$$
J = \int_0^{t_f} \left[P_1 I_h + P_2(S_v + I_v) + \frac{1}{2}P_3 u_1^2 + \frac{1}{2}P_4 u_2^2 + \frac{1}{2}P_5 u_3^2 \right] dt
$$
(8.2)

Here the parameter $P_1 \geq 0$, $P_2 \geq 0$, $P_3 \geq 0$, $P_4 \geq 0$, $P_5 \geq 0$ and they represent the weight constants.
Our main purpose is to determined the control parameters $u_1{}^*$, $u_2{}^*$, $u_3{}^*$ such that

$$
J(u^*) = \min_{u\in\Omega} J(u_1, u_2, u_3), \qquad (8.3)
$$

where the control set Ω is defined as
$\Omega = \{u_1, u_2, u_3 :$ measurable and $0 \leq u_1 \leq 1$, $0 \leq u_2 \leq 1$, $0 \leq u_3 \leq 1\}$ and $t \in [0, t_f]$.
The Lagrangian of this problem is defined as :

$$
\begin{aligned}
L(I_h, S_v, I_v, u_1, u_2, u_3) &= P_1 I_h + P_2(S_v + I_v) + \frac{1}{2}P_3 u_1^2 \\
&\quad + \frac{1}{2}P_4 u_2^2 + \frac{1}{2}P_5 u_3^2
\end{aligned}
$$

For our problem, we formed Hamiltonian $\mathscr{H}$:

$$\mathscr{H} = L(I_h, S_v, I_v, u_1, u_2, u_3) + \lambda_1 \frac{dS_h}{dt} + \lambda_2 \frac{dI_h}{dt} + \lambda_3 \frac{dR_h}{dt}$$
$$+ \lambda_4 \frac{dS_v}{dt} + \lambda_5 \frac{dI_v}{dt}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are the adjoint variables. Now the differential equation corresponding to adjoint variables can be rearranged as

$$\frac{d\lambda_1}{dt} = \mu_h \lambda_1 + (1 - u_1) \frac{\beta_1 (1 + p_2 I_h) I_h}{(1 + p_1 S_h + p_2 I_h)^2} (\lambda_1 - \lambda_2)$$

$$+ (1 - u_2) \frac{\beta_2 I_v}{N_h} (\lambda_1 - \lambda_2)$$

$$+ (1 - u_2) \left[\frac{\beta_2 S_h I_v}{N_h^2} (\lambda_2 - \lambda_1) + \frac{\beta_v S_v I_h}{N_h^2} (\lambda_5 - \lambda_4) \right]$$

$$\frac{d\lambda_2}{dt} = -P_1 + (1 - u_1) \frac{\beta_1 (1 + p_1 S_h) S_h}{(1 + p_1 S_h + p_2 I_h)^2} (\lambda_1 - \lambda_2)$$

$$+ (1 - u_2) \frac{\beta_2 S_h I_v}{N_h^2} (\lambda_2 - \lambda_1) \qquad (8.4)$$

$$+ A\lambda_2 + (1 - u_2) \frac{\beta_v S_v (N_h - I_h)}{N_h^2} (\lambda_4 - \lambda_5)$$

$$\frac{d\lambda_3}{dt} = \mu_h \lambda_3$$

$$+ (1 - u_2) \left[\frac{\beta_2 S_h I_v}{N_h^2} (\lambda_2 - \lambda_1) + \frac{\beta_v S_v I_h}{N_h^2} (\lambda_5 - \lambda_4) \right]$$

$$\frac{d\lambda_4}{dt} = -P_2 + (\mu_v + u_3) \lambda_4 + (1 - u_2) \frac{\beta_v I_h}{N_h} (\lambda_4 - \lambda_5)$$

$$\frac{d\lambda_5}{dt} = -P_2 + (\mu_h + u_3) \lambda_5 + (1 - u_2) \frac{\beta_2 S_h}{N_h} (\lambda_1 - \lambda_2)$$

Let the respective optimal values of S_h, I_h, R_h, S_v, I_v be $\widetilde{S_h}$, $\widetilde{I_h}$, $\widetilde{R_h}$, $\widetilde{S_v}$, $\widetilde{I_v}$ and the solutions of the system (8.4) be $\widetilde{\lambda}_1$, $\widetilde{\lambda}_2$, $\widetilde{\lambda}_3$, $\widetilde{\lambda}_4$, $\widetilde{\lambda}_5$.

Theorem 8.1. *There exist optimal controls* $(u_1{}^*, u_2{}^*, u_3{}^*) \in \Omega$ *such that* $J(u_1{}^*, u_2{}^*, u_3{}^*) = \min J(u_1, u_2, u_3)$ *subject to system (8.1).*

Proof. : To prove this theorem we use [15, 19]. Here the state variables and the controls are positive. For this minimizing problem, the necessary convexity of the objective functional in (u_1, u_2, u_3) is satisfied. The control variable set $u_1, u_2, u_3 \in \Omega$ is also convex and closed by the definition. The integrand of the functional $P_1 I_h + P_2 (S_v + I_v) + \frac{1}{2} P_3 u_1{}^2 + \frac{1}{2} P_4 u_2^2 + \frac{1}{2} P_5 u_3^2$ is convex on the control set Ω and the state variables are bounded.
Since there exist optimal controls for minimizing the functional subject to systems (8.1) and (8.4), we use Pontryagin's maximum principle to derive the necessary conditions to find the optimal solutions as follows:

If (x,u) is an optimal solution of an optimal control problem, then there exist a non-trivial vector function $\lambda = \lambda_1, \lambda_2, ..., \lambda_n$ satisfying the following equalities.

$$\frac{dx}{dt} = \frac{\partial H(t,x,u,\lambda)}{\partial \lambda}$$

$$0 = \frac{\partial H(t,x,u,\lambda)}{\partial u}$$

$$\frac{d\lambda}{dt} = -\frac{\partial H(t,x,u,\lambda)}{\partial x}$$

With the help of Pontryagin's maximum principle [19] and theorem (8.1), we prove the following theorem:

Theorem 8.2. *The optimal controls* $(u_1{}^*, u_2{}^*, u_3{}^*)$ *which minimizes J over the region* Ω *given by*

$$u_1{}^* = min\{1, max(0, \widetilde{u}_1)\}$$

$$u_2{}^* = min\{1, max(0, \widetilde{u}_2)\}$$

$$u_3{}^* = min\{1, max(0, \widetilde{u}_3)\}$$

where,

$$\widetilde{u}_1 = \beta_1 \left[\frac{(1 + p_1 S_h^*) S_h^* + (1 + p_2 I_h^*) I_h^*}{P_3 (1 + p_1 S_h^* + p_2 I_h^*)^2} \right] (\lambda_2 - \lambda_1)$$

$$\widetilde{u}_2 = \frac{(\beta_2 S_h^* I_v^*)(\lambda_1 - \lambda_2) + (\beta_v S_h^* I_h^*)(\lambda_4 - \lambda_5)}{P_4 N_h^*}$$

$$\widetilde{u}_3 = \frac{S_v^* \lambda_4 + I_v^* \lambda_5}{P_5}$$

Proof. : Using optimally condition :

$$\frac{\partial \mathscr{H}}{\partial u_1} = 0, \frac{\partial \mathscr{H}}{\partial u_2} = 0, \frac{\partial \mathscr{H}}{\partial u_3} = 0$$

we get,

$$u_1 P_3 + \beta_1 \left[\frac{(1 + p_1 S_h^*) S_h^* + (1 + p_2 I_h^*) I_h^*}{(1 + p_1 S_h^* + p_2 I_h^*)^2} \right] (\lambda_1 - \lambda_2) = 0.$$

This implies,

$$u_1 = \beta_1 \left[\frac{(1 + p_1 S_h^*) S_h^* + (1 + p_2 I_h^*) I_h^*}{A_3 (1 + p_1 S_h^* + p_2 I_h^*)^2} \right] (\lambda_2 - \lambda_1)$$

$$\frac{\partial \mathscr{H}}{\partial u_2} = u_2 P_4 + \frac{\beta_2 S_h^* I_v^*}{N_h^*} (\lambda_2 - \lambda_1) + \frac{\beta_v S_h^* I_h^*}{N_h^*} (\lambda_5 - \lambda_4) = 0$$

This implies,

$$u_2 = \frac{(\beta_2 S_h^* I_v^*)(\lambda_1 - \lambda_2) + (\beta_v S_h^* I_h^*)(\lambda_4 - \lambda_5)}{P_4 N_h^*}$$

And,

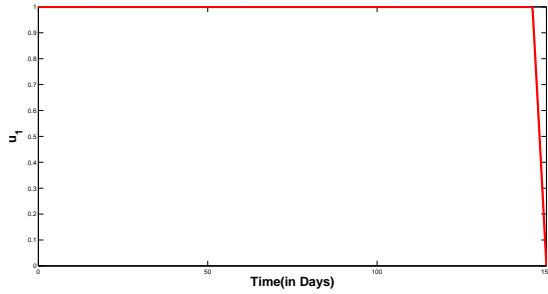$$\frac{\partial \mathscr{H}}{\partial u_3} = u_3 P_5 - (S_v^* \lambda_4 + I_v^* \lambda_5)$$

Figure 8. Control profil of u_1

Figure 9. Control profil of u_2

This implies,

$$u_3 = \frac{S_v^* \lambda_4 + I_v^* \lambda_5}{P_5}$$

,

Again upper and lower bounds for these control are 0 and 1 respectively. i.e. $u_1 = u_2 = u_3 = 0$ if $u_1 < 0, u_2 < 0, u_3 < 0$ and $u_1 = u_2 = u_3 = 1$ if $\widetilde{u_1} > 1$, $\widetilde{u_2} > 1$ and $\widetilde{u_3} > 1$ otherwise $u_1 = \widetilde{u_1}$, $u_2 = \widetilde{u_2}$ and $u_3 = \widetilde{u_3}$. Hence for these controls $u_1{}^*, u_2{}^*, u_3{}^*$ we get optimum value of the function J.

9. Numerical Simulation of Optimal Control

We simulate our optimal control model by keeping the parameters corresponding to stability of endemic equilibrium point E_1 of the model (2.1). With the help of MATLAB the optimal control model is simulated. The weight constants for the optimal control problem are taken as

$$P_1 = 1, P_2 = 1, P_3 = 45, P_4 = 65, P_5 = 75$$

We solve the optimality system (8.1) by iterative method with the help of forward and backward difference approximations [19]. We consider the time interval as [0, 150]. Here in Figure 8, is showing the control profile of u_1, in Figure 9, is showing the control profile of u_2, in Figure 10, is showing the control profile of u_3 and finally in the Figure 11, is plotted to observe the effects of optimal controls for infected human against time with and without optimal control. It is easy to notice that optimal control is more effective in reducing the number of infective is considered period of time. The all three optimal control application is the best control strategy to minimize the number of infective, which will definitely reduce of the spread of ZIKV.

Figure 10. Control profil of u_3

10. Conclusion

A deterministic model for transmission dynamics of the Zika virus is designed and analyzed. Here we considered the human to human sexual transmission of ZIKV as a saturated incidence rate. In summary, the study shows the following **(i)** The equilibria of the proposed model and the basic reproduction number (R_0) are computed.
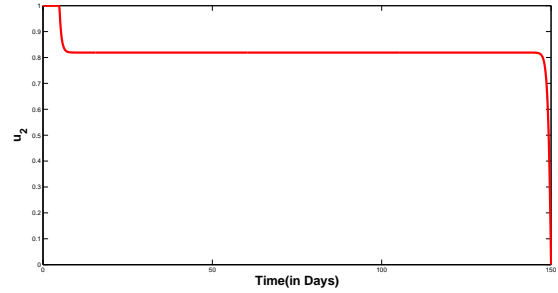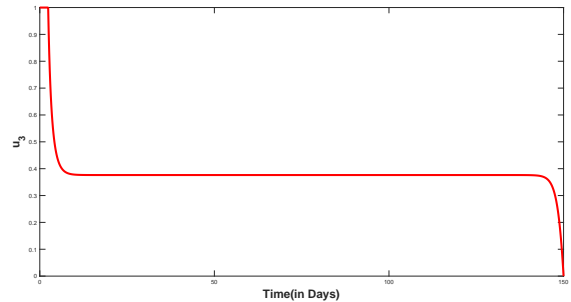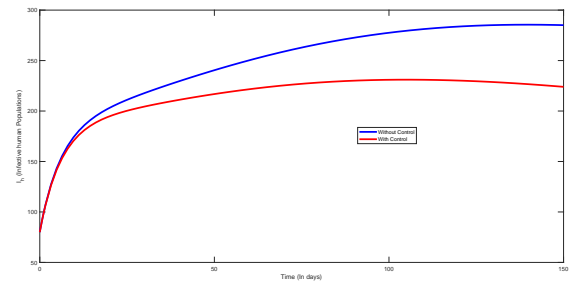

Figure 11. Variation of I_h with and without control

(ii) It is found that for $R_0 < 1$, the disease-free equilibrium point (E_0) is locally asymptotically stable. It also globally asymptotically stable for $R_0 < 1$ under some restriction of parameters.

(iii) The system also exhibits backward bifurcation which suggests that merely reducing R_0 below one is not enough to make disease-free equilibrium to be globally stable.

(iv) The endemic equilibrium point (E_1) also locally asymptotically stable for $R_0 > 1$.

(v) Sensitivity analysis is performed to determine which parameter is more sensitive to basic reproduction number (R_0). The sensitivity of (R_0) with parameters $\beta_1, \beta_2, \beta_v$; it is found that (R_0) is very sensitive to the parameter corresponding to the transmission.

(vi) The model is extended to the optimal control model and is analyzed by using Pontryagin's Maximum Principle. It is observed that the optimal control model gives better effective results to reduce the infection than the model without optimal.

(vii) The numerical simulations are performed to support our analytical findings and to compare our model with the existing model in [11]. Through the numerical simulation, we can conclude that the parameter (γ_h) increases, as a result, the human infective population is decreasing in the equilibrium level. These results will help the policymakers as well as public health for future implications.

References

[1] S. Ahmadi, N. F. Bempong, O. De Santis, D. Sheath, The Role of Digital Technologies in Tackling The Zika Outbreak: A Scoping Review, *J Public Health Emerg*, 20 (2) (2018), 1-15.

[2] F. B. Agusto, S. Bewick and W. F.Fagan, Mathematical Model of Zika Virus with Vertical Transmission, *Infectious Disease Modelling, KeAi*, 2(2)(2017), 244-267.

[3] E. Bonyah, K. O. Okosun, Mathematical Modeling of Zika Virus, *Asian Pac J Trop Dis*, 6 (9) (2016), 673–679.

[4] B. Buonomo, D. Lacitignola, on the Backward Bifurcation of a Vaccination Model with Nonlinear Incidence, *Nonlinear Analysis: Modelling and Control*, 16 (1) (2011), 30-46.

[5] Centers for Disease Controland Prevention (CDC), *CDC Emergency Operations Center Moves to Highest Level of Activation for Zika Response* , Febraury 3, 2016.

[6] C. Castillo-Chavez, Z. Feng and W. Huang, On the Computation of R_0 and Its Role on Global Stability, *Mathematical Approaches for For Emerging and Reemerging Infectious Diseases*,(2002), 229-250.

[7] C. Castillo-Chavez, B. Song, Dynamical Model of Tuber-culosis and Their Applications, *Math Biosci*, 1 (2) (2004), 361–404.

[8] F. Corsica, Zika Virus Transmission From French Polynesia to Brazil, *Emerg. Infect. Diseases* 21 (10) (2015), 68-87

[9] P. V. Driessche and J. Watmough, Reproduction Numbers and Sub-Threshold Endemic Equilibria for Compartmental Models of Disease Transmission, *Mathematical Biosciences*,180 (2002), 29-48.

[10] G. Daozhou, L. Yijun, H. Daihai, C. P. Travis, K. Yang, C. Gerardo and R. Shigui, Prevention and Control of Zika as a Mosquito-Borne and Sexually Transmitted Disease: A Mathematical Modeling Analysis, *Scientific Reports*, 6, 28070, (2016), 1-10.

[11] N. K. Goswami, A. K. Srivastav, M. Ghosh and B. Shanmukha, Mathematical Modeling of Zika Virus Disease with Nonlinear Incidence and Optimal Control, *Journal of Physics: Conf. Series*, 1000 (2018), 012114, 1-17

[12] A. Kaddar, Stability Analysis in A Delayed SIR Epidemic Model with A Saturated Incidence Rate,*Nonlinear Analysis: Modelling and Control*, 15 (3) (2010), 299–306.

[13] V. Lakshmikantham, S. Leela, A. A. Martynyuk, *Stability Analysis of Nonlinear Systems*, Marcel Dekker Inc., (1989), 37-339.

[14] W. Liu, S. A. Levin, Y. Iwasa, Influence of Nonlinear Incidence Rates Upon The Behavior of SIRS Epidemiological Models, *Journal of Mathematical biology*, 23(2) (1986), 187-204.

[15] S. Lenhart, J. T. Workman, *Optimal Control Applied to Biological Models*, CRC Press ,(2007),30-272.

[16] V. M. Moreno, B. Espinoza, D. Bichara, S. A.Holechek, C. Castillo Chavez, Role of Short-term Dispersal on The Dynamics of Zika Virus in an Extreme Idealized Environment, *Infect Dis Model*,2 (2016), 1–14.

[17] E. Oehler, E. Fournier, I. Leparc-Goffart, L Philippe, S Cubizolle, C Sookhareea, S Lastere, F Ghawche, Increase in Cases of Guillain–Barré Syndrome During a Chikungunya Outbreak, French Polynesia, 2014 to 2015, *Eurosurveillance* , 20 (48) (2015), 1-2.

[18] E. Oehler, L. Watrin, P. Larre, I Leparc-Goffart, S Lastere, F Baudouin, H P Mallet, D Musso, F Ghawche, Zika Virus Infection Complicated by Guillain–Barré Syndrome-case Report, French Polynesia, *Euro Surveill*, 19 (2014), 20720, 1-3.

[19] L. S. Pontryagin,V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*, Inter Science Publishers, (1962), 21-186.

[20] S. Ruan, W. Wang, Dynamical Behavior of an Epidemic Model with a Nonlinear Incidence Rate, *Journal of Differential Equations*, 188 (1) (2003), 135-163.

[21] A. K. Srivastav, N. K. Goswami, M. Ghosh and Li Xue-Zhi, Modeling And Optimal Control Analysis of Zika Virus With Media Impact, *Int. J. Dyanm. Control, Springer Nature*, 2 (177) (2018), 1673-1689.

[22] A. K. Srivastav, J. Yang, X. F. Leo and M. Ghosh, Spread

of Zika Virus Disease on Complex Network – A Mathematical Study , *Mathematics and Computers in Simulation* 157 (2019), 15-38.

[23] A. K. Srivastav, M. Ghosh, Assessing The Impact of Treatment on The Dynamics of Dengue Fever: A Case Study of India, *Applied Mathematics and Computation*,362 (2019), 1-17.

[24] World Health Organization (WHO), WHO Statement on the First Meeting of the International Health Regulations (2005) Emergency Committee on Zika Virus And Observed Increase in Neurological Disorder Sandneo Natalmal for Mations, February 1, (2016).

[25] D. Xiao, S. Ruan, Global Analysis of an Epidemic Model with Nonmonotone Incidence Rate, *Mathematical Biosciences*, 208 (2) (2007), 419-429.