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Bernstein induced one step hybrid scheme for general solution of second order initial value problems

Ojo Ezekiel Olukunle[1]* and M. Okoro Felix[2]

Abstract

In this paper, a Bernstein polynomial with collocation and interpolation techniques were used to develop one step hybrid scheme with one offgrid point for the direct solution of general second order ordinary differential equations. The basic properties of the derived scheme was investigated and found to be of order four(4), zero stable and convergent. The scheme obtained is used to solve some standard initial value problems. From the numerical results obtained, it was revealed that the proposed method performs better than some of the existing methods in the literature.

Keywords

Bernstein polynomial, Collocation; Interpolation, Block method, Zero Stability, Consistency, Region of Absolute stability. **AMS Subject Classification**

26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05.

[1,2] *Department of Mathematics, Ambrose Alli University, Ekpoma, Ekpoma, Nigeria.*

*__Corresponding author__: [1] talk2ojoezekiel@gamil.com; [2]moibiokoro@aauekpoma.edu.ng

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Contents

1. Introduction

In this work, we consider a second order initial value problem of the form

$$y'' = f(x, y(x), y'(x)) \tag{1.1}$$

In order to solve (1.1), the conditions stated below need to be imposed

$$y(x_0) = a, \;\; y'(x_0) = b \tag{1.2}$$

where x_0 is the initial point, y_0 is the solution at x_0, f is a continuous function within the interval of integration and prime indicates differentiation with respect to x, while $y(x)$ is the unknown function to be determined.

Eq. [(1.1) - (1.2)] have been studied in many areas due to their frequent appearance in various applications in physics, engineering, biology and other field, for instance, diffusion reaction process, isothermal gas equilibrium, geophysics, etc. The Exact or approximate solutions of these problems are very importance due to its wide application in science and other fields of research [6].

In recent years, the Bernstein polynomials have gained the attention of many researchers. It has been used to obtained approximate solutions of different differential equations. For example, a method for approximating solutions to differential equations, proposed by Bhatti used Bernstein operational matrix of differentiation[12]. Alshbool et al in [6] solved fractional differential equations (FDEs) with a modified new Bernstein polynomial basis.

Rupa et al in [25], formulate one dimensional linear and nonlinear system of second order boundary value problems (BVPs) using Galerkin weighted residual method with Bernstein and Legendre polynomials as basis functions. Alshbool and Hashim in [7], used Multistage Bernstein polynomials (MB-polynomials) scheme to solve Fractional Order Stiff Systems of differential equations numerically.

Hao et al in [18], proposed a Bernstein polynomials matrices for the numerical solutions of fractional partial differential equations. In [24], Ordokhani and Davae proposed an algorithm based on operational matrix by an expansion of Bernstein polynomials in terms of Legendre polynomials for solving differential equations.

Davaeifar et al in [14], have given solutions of nth order linear Fredholm integro-differential-diference equations subjected to mixed conditions using the Bernstein collocation matrix method. Salih et al, present a Bernstein matrix method to solve the first order nonlinear ordinary differential equations with the mixed non-linear conditions in [27] .

Taiwo and Hassan in [28], developed two approximation methods namely; Iterative Decomposition and Bernstein Polynomial Methods to solve some classes of Singular Initial and Boundary Value Problems.

Ahmed[2], proposed an algorithm for approximating solutions to 2nd-order linear differential equations with polynomial coefficients in B-polynomials and Abbas[26], present a Bernstein operational matrix with collocation method for solving multi-order fractional differential equations.

Aysegul et al in [11] introduced a new method to solve high order linear differential equations with initial and boundary conditions. The method which is numerically based on Bernstein polynomial and depend on collocation method. Also Aysegul et al, have proposed a numerical solution of nonlinear ordinary differential equations based Bernstein collocation method in [10] .

In [30], a numerical method which employs the Bernstein polynomials basis to approximate solution of a parabolic partial differential equation with boundary integral conditions, was proposed by Yousefi et al.

Recently, Ahmad[3] obtained approximate solutions of the calculus of variations problems with B-polynomials operational matrices . Khataybeh et al in [21] solved numerically a class of third-order ordinary differential equations (ODEs) using the operational matrices of Bernstein polynomials method.

In this paper, hybrid one step scheme will be developed for direct solution of general second order initial value problem using Bernstein Polynomial as basis function.

This paper is organization as follows: in Section 2, we introduce the B-polynomials and their properties. Section 3 is a development of the method. Section 4, analysis of the basic properties of the method. Numerical implementation of the scheme in Section 5. Section 6 offers discussion of result. And Section 7 is reserved for conclusion.

2. Bernstein Polynomials and their Properties

Aysegul et al in [11], defined the Bernstein polynomials of degree m on the interval [0, 1], as

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad i = 0,1,...,m \tag{2.1}$$

where the binomial coefficient is

$$\binom{m}{i} = \frac{m!}{i!(m-i)!}$$

There are $(m+1)$ nth degree of Bernstein polynomials. For mathematical convenience, we usually set $B_{i,m} = 0$, if $i < 0$ or $i > m$.

In general, we approximate any function $y(x)$ with the first $(m+1)$ Bernstein polynomials as

$$y(x) = \sum_{i=0}^{m} c_i B_{i,m}(x) = C^T \phi(x), \tag{2.2}$$

where $C^T = [c_0, c_1, ..., c_m]$, are the coefficients to be determined and
$\phi(x) = [B_{0,m}(x), B_{1,m}(x), ..., B_{m,m}]^T$ is the Bernstein polynomial of degree m.
where $\phi(x) = A_m \times T_m(x)$

$$\mathbf{T_m(x)} = \begin{pmatrix} 1 & x & x^2 & \dots & x^n \end{pmatrix}$$

$$\mathbf{A_m} = \begin{pmatrix} b_{0,0} & 0 & 0 & \dots & 0 \\ b_{1,0} & b_{1,1} & 0 & \dots & 0 \\ b_{2,0} & b_{2,1} & b_{2,2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n,0} & b_{n,1} & b_{n,2} & \dots & b_{n,n} \end{pmatrix}$$

so we can convert (2.2) to

$$y(x) = \begin{pmatrix} 1 & x & x^2 & \dots & x^n \end{pmatrix} \begin{pmatrix} b_{0,0} & 0 & 0 & \dots & 0 \\ b_{1,0} & b_{1,1} & 0 & \dots & 0 \\ b_{2,0} & b_{2,1} & b_{2,2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n,0} & b_{n,1} & b_{n,2} & \dots & b_{n,n} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \tag{2.3}$$

Properties of Bernstein Polynomials

1. Positivity property:
   $B_{i,m}(x) > 0$ is hold, for $0 \le x \le 1$
2. Unity partition property:

$$\sum_{k=0}^{m} B_{i,m}(x) = \sum_{k=0}^{m-1} B_{i,m-1}(x) = \sum_{k=0}^{m-2} B_{i,m-2}(x) = \dots = \sum_{i=0}^{1} B_{i,1}(x) = 1$$

3. Recursive relation property:
   $B_{i,m}(x) = (1-x)B_{i,m-1}(x) + xB_{i-1,m-1}(x)$
4. First derivatives of the generalized Bernstein basis polynomials:
   $\frac{d}{dx} B_{i,m}(x) = m(B_{i-1,m-1}(x) - B_{i,m-1}(x))$

The proofs of the property can be found in [19].

3. Development of the method

Our aim in this section is to find an approximate solution of Eq. (1.1), expressed in the Bernstein polynomial form:

$$y(t) = \sum_{k=0}^{c+i-1} a_k B_{k,n}(t), \tag{3.1}$$

where c and i are number of distinct collocation and interpolation points respectively and $B_{k,n}(t)$ is the Bernstein Polynomial derived from the recursive relation

$$B_{k,n}(t) = (1-t)B_{k,n-1}(t) + tB_{k-1,n-1}(t) \tag{3.2}$$

Differentiating (3.1) twice and substituting into (1.1) gives:

$$f(x,y(x),y'(x)) = \sum_{k=0}^{c+i-1} a_k B''_{k,n}(t). \tag{3.3}$$

We consider a grid point of steplength one and off step point at $x = x_{n+\frac{1}{2}}$. Collocating (3.3) at points $x = x_n$, $x_{n+\frac{1}{2}}$ and x_{n+1}, interpolating (3.1) at $x = x_n$ and $x_{n+\frac{1}{2}}$ give a system of five equations which are solved using Gaussian elimination method to obtained the parameters $a'_j s, j = 0,1,\ldots 4$. The parameters $a'_j s$ obtained are then substituted back into (3.1) to give the continuous hybrid one step method of the form;

$$y(x) = \alpha_0 y_n + \alpha_{\frac{1}{2}} y_{n+\frac{1}{2}} + h^2\left[\beta_0 f_n + \beta_1 f_{n+1} + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}}\right], \tag{3.4}$$

where

$$\alpha_0(t) = 1-2t,$$
$$\alpha_{\frac{1}{2}}(t) = 2t$$
$$\beta_0(t) = h^2\left[-\frac{7t}{48} + \frac{t^2}{2} - \frac{t^3}{2} + \frac{t^4}{6}\right] \tag{3.5}$$
$$\beta_{\frac{1}{2}}(t) = h^2\left[-\frac{t}{8} + \frac{2t^3}{3} - \frac{t^4}{3}\right]$$
$$\beta_1(t) = h^2\left[\frac{t}{18} - \frac{t^3}{6} + \frac{t^4}{6}\right]$$

Evaluating (3.4) at $t = 1$ gives rise to:

$$y_{n+1} = -y_n + 2y_{n+\frac{1}{2}} + \frac{h^2}{48}\left[f_n + 10f_{n+\frac{1}{2}} + f_{n+1}\right] \tag{3.6}$$

Also differentiating (3.4), where $\frac{dt}{dx} = \frac{1}{h}$ give rise to

$$y'(x) = \alpha'_0 y_n + \alpha'_{\frac{1}{2}} y_{n+\frac{1}{2}} + h^2\left[\beta'_0 f_n + \beta'_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta'_1 f_{n+1}\right], \tag{3.7}$$

where

$$\alpha'_0 = -\frac{2}{h}$$
$$\alpha'_{\frac{1}{2}} = \frac{2}{h}$$
$$\beta'_0(t) = h\left[\frac{2t^3}{3} - \frac{3t^2}{2} + t - \frac{7}{48}\right] \tag{3.8}$$
$$\beta'_{\frac{1}{2}}(t) = h\left[-\frac{4t^3}{3} + 2t^2 - \frac{1}{8}\right]$$
$$\beta'_1(t) = h\left[\frac{2t^3}{3} - \frac{t^2}{2} + \frac{1}{48}\right]$$

On evaluating (3.7) at $t = 0$, $\frac{1}{2}$ and 1, we have the following discrete methods

$$hy'_n - 2y_{n+\frac{1}{2}} + 2y_n = \frac{h^2}{48}\left[-7f_n - 6f_{n+\frac{1}{2}} + f_{n+1}\right] \tag{3.9}$$

$$hy'_{n+\frac{1}{2}} - 2y_{n+\frac{1}{2}} + 2y_n = \frac{h^2}{48}\left[3f_n + 10f_{n+\frac{1}{2}} - f_{n+1}\right] \tag{3.10}$$

$$hy'_{n+1} - 2y_{n+\frac{1}{2}} + 2y_n = \frac{h^2}{48}\left[f_n + 26f_{n+\frac{1}{2}} + 9f_{n+1}\right] \tag{3.11}$$

The methods derived in equation (3.6), (3.9) to (3.11) will be combined and implemented as a block to solve numerical examples in section 5.

4. Analysis of the Basic Properties of the Method

In this section, we analyze the derived scheme by determining the order and error constant, consistency, zero stability and region of absolute stability of the scheme.

4.1 Order and Error constant

Definition 4.1. *The one-step implicit hybrid block linear method and the associated linear difference operator are said to have order p if $C_0 = C_1 = C_2 = C_3 = \ldots = C_p = C_{p+1}$ and $C_{p+2} \neq 0$ see [23] for details*

According to Fatunla [16], we expand (3.6), (3.9) to (3.11) using Taylor's series and combining the coefficient of the like terms in h^n, the following result are obtained

Table 1. Order and Error Constants of additional methods.

method(eqn no)	Order	Error Constant
2.6	4	$-\frac{1}{15360}$
2.9	3	$-\frac{1}{720}$
2.10	3	$\frac{7}{5760}$
2.11	3	$-\frac{1}{720}$

4.2 Consistency of the Scheme

Definition 4.2. *A numerical method is said to be consistent, if it has order greater than one ($p \geq 1$) see [22] for details*

hence our methods are consistent

4.3 Zero Stability

Definition 4.3. *According to Lambert[23] a method is said to be Zero stable if no roots of the first characteristic polynomial $\rho(z)$ has modulus greater than one, and if every root of the modulus one has multiplicity not greater than one, $|z| \leq 1$ and is simple.*

Therefore our numerical schemes in 3.6, 3.9, 3.10 and 3.11 are zero stable.

4.4 Convergence of the method

Definition 4.4. *The necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable. see [13] for details*

Hence our methods are convergent.

4.5 Region of Absolute Stability of the method

Definition 4.5. *A method is said to be absolutely stable within a given interval if for a given h, all roots z_s of the characteristic polynomial $\pi(z,h) = \rho(z) + h^2\sigma(z) = 0$, satisfies $|z| < 1, s = 1,2,\ldots,n$, where $h = \lambda^2 h^2$ and $\lambda = \frac{\partial f}{\partial y}$.*

We adopted the boundary locus method to determine the stability interval of our main method(3.6).

Table 2:The boundaries of the region of absolute stability of the method

θ^0	0	30	60	90	120	150	180
$\vec{h}(\theta)$	0.0000	−0.2885	−1.0963	−2.4633	−4.3636	−6.7651	−9.6000

The region of absolute stability of the method is between (-9.60 ,0.00)

5. Numerical implementation of the scheme

In this section, we test effectiveness and validity of our newly derived scheme by applying it to some second order differential equations. All numerical calculations and programs are carried out with the aid of Maple 16 software.

Example 1

We consider a moderately stiff problem [[1],[4]]:

$$y'' = y', y(0) = 0, y'(0) = -1, \quad h = 0.1$$

whose exact solution is $y(x) = 1 - \exp(x)$.

Table 3: Numerical result for Example 1 with $h = 0.1$

x	Exact	Numerical	Error in the Proposed Method$p = 3$
0.1	−0.10517091807565	−0.105170902716915	$1.5358735 \times 10^{-08}$
0.2	−0.22140275816017	−0.221402724212121	$3.3948049 \times 10^{-08}$
0.3	−0.34985880757600	−0.349858751298410	$5.6277590 \times 10^{-08}$
0.4	−0.49182469764127	−0.491824614712792	$8.2928478 \times 10^{-08}$
0.5	−0.64872127070013	−0.648721156137452	$1.14562678 \times 10^{-07}$
0.6	−0.82211880039051	−0.822118648456907	$1.51933603 \times 10^{-07}$
0.7	−1.01375270747048	−1.01375251157245	$1.9589803 \times 10^{-07}$
0.8	−1.22554092849247	−1.22554068106298	$2.4742949 \times 10^{-07}$
0.9	−1.45960311115695	−1.45960280352360	$3.0763335 \times 10^{-07}$
1.0	−1.71828182845905	−1.71828145069524	$3.7776381 \times 10^{-07}$

Table 4: Comparison of the error for Example 1.

x	Error in [4] $p = 4$	Error in [29] $p = 4$	Error in[20] $p = 6$	Error in [1] $p = 3$	Error in our method $p = 4$
0.2	0.5372×10^{-05}	3.2672×10^{-04}	8.17176×10^{-07}	1.25×10^{-07}	3.3948×10^{-08}
0.3	0.6247×10^{-05}	2.2156×10^{-03}	3.10356×10^{-06}	3.250×10^{-07}	5.6278×10^{-08}
0.4	0.1517×10^{-05}	4.8571×10^{-03}	6.56957×10^{-06}	6.424×10^{-07}	8.2928×10^{-08}
0.5	0.1001×10^{-04}	9.0977×10^{-03}	1.14380×10^{-05}	1.099×10^{-06}	1.1456×10^{-07}
0.6	0.2970×10^{-04}	1.4391×10^{-02}	1.79656×10^{-05}	1.7213×10^{-06}	1.5193×10^{-07}
0.7	0.5916×10^{-04}	2.1438×10^{-02}	2.64474×10^{-05}	2.538×10^{-06}	1.9590×10^{-07}
0.8	0.1002×10^{-03}	2.9899×10^{-02}	3.72222×10^{-05}	3.583×10^{-06}	2.4743×10^{-07}
0.9	0.1550×10^{-03}	4.0301×10^{-02}	5.06788×10^{-05}	4.896×10^{-06}	3.0763×10^{-07}
1.0	0.2259×10^{-03}	5.2552×10^{-02}	6.72615×10^{-05}	6.522×10^{-06}	3.7776×10^{-07}

Example 2

We consider a highly stiff problem [[1], [4]]:

$$y'' + 1001y' + 1000y = 0, y(0) = 1, y'(0) = -1, \quad h = 0.05$$

whose exact solution is $y(x) = \exp(-x)$.

Table 5: Numerical result for Example 2 with $h = 0.05$

x	Exact	Numerical	Error
0.1	0.951229424500714	0.951229424396234	1.04480×10^{-10}
0.2	0.904837418035960	0.904837417831670	2.04290×10^{-10}
0.3	0.860707976425058	0.860707976126546	2.98512×10^{-10}
0.4	0.818730753077982	0.818730752691421	3.86561×10^{-10}
0.5	0.778800783071405	0.778800782603302	4.68103×10^{-10}
0.6	0.740818220681718	0.740818220138724	5.42994×10^{-10}
0.7	0.704688089718713	0.704688089107476	6.11237×10^{-10}
0.8	0.670320046035639	0.670320045362696	6.72943×10^{-10}
0.9	0.637628151621773	0.637628150893473	7.28300×10^{-10}
1.0	0.606530659712633	0.606530658935081	7.77552×10^{-10}

Table 6:Comparison of the error Example 2

x	Error in [4]	Error in [1]	Error in Proposed Scheme
0.1	$1.0886170 \times 10^{-10}$	2.22×10^{-08}	1.04480×10^{-10}
0.2	$2.0752355 \times 10^{-10}$	1.250×10^{-07}	2.04290×10^{-10}
0.3	$2.8642155 \times 10^{-10}$	3.254×10^{-07}	2.98512×10^{-10}
0.4	$3.4842440 \times 10^{-10}$	6.428×10^{-07}	3.86561×10^{-10}
0.5	$3.9603265 \times 10^{-08}$	1.0993×10^{-06}	4.68103×10^{-10}
0.6	$4.3142434 \times 10^{-07}$	1.7209×10^{-06}	5.42994×10^{-10}
0.7	$4.5649384 \times 10^{-07}$	2.538×10^{-06}	6.11237×10^{-10}
0.8	$4.7288495 \times 10^{-07}$	3.583×10^{-06}	6.72943×10^{-10}
0.9	$4.8202237 \times 10^{-07}$	4.896×10^{-06}	7.28300×10^{-10}
1.0	$4.8513832 \times 10^{-07}$	6.522×10^{-06}	7.77552×10^{-10}

Example 3

We consider an inhomogeneous problem [1] :

$$y'' = -100y + 99\sin(x), \quad y(0) = 1, y'(0) = 11, h = \frac{1}{320}$$

which has a solution of the form $y(x) = \cos(10x) + \sin(10x) + sin(x)$.

Table 7: Numerical result for problem 3 with $h = \frac{1}{320}$

x	Exact	Numerical	Error
$\frac{1}{320}$	1.03388166738420	1.03388166734330	4.090×10^{-11}
$\frac{3}{320}$	1.06675678785246	1.06675678777170	8.076×10^{-11}
$\frac{6}{320}$	1.09859628036501	1.09859628024550	1.1951×10^{-10}
$\frac{9}{320}$	1.12937207509627	1.12937207493916	1.5711×10^{-10}
$\frac{12}{320}$	1.15905714081491	1.15905714062147	1.9344×10^{-10}
$\frac{15}{320}$	1.18762551125002	1.18762551102154	2.2848×10^{-10}
$\frac{18}{320}$	1.21505231041716	1.21505231015499	2.6217×10^{-10}
$\frac{21}{320}$	1.24131377687988	1.24131377658543	2.9445×10^{-10}
$\frac{24}{320}$	1.26638728692280	1.26638728659754	3.2526×10^{-10}
$\frac{27}{320}$	1.29025137661388	1.29025137625933	3.5455×10^{-10}

Table 8: Comparison of error of the error Example 3

x	Error in [1]	Error in our Proposed Method
$\frac{1}{320}$	7.9800×10^{-11}	4.090×10^{-11}
$\frac{3}{320}$	8.3780×10^{-10}	8.076×10^{-11}
$\frac{6}{320}$	3.3600×10^{-09}	1.5711×10^{-10}
$\frac{9}{320}$	7.3481×10^{-09}	1.5711×10^{-10}
$\frac{12}{320}$	1.2557×10^{-08}	1.9344×10^{-10}
$\frac{15}{320}$	1.8721×10^{-08}	2.2848×10^{-10}
$\frac{18}{320}$	2.5555×10^{-08}	2.6217×10^{-10}
$\frac{21}{320}$	3.2762×10^{-08}	2.9445×10^{-10}
$\frac{24}{320}$	4.0036×10^{-08}	3.2526×10^{-10}
$\frac{27}{320}$	4.7066×10^{-08}	3.5455×10^{-10}

Example 4

We consider Nonlinear problem [5] :

$$y'' - x(y')^2 = 0, \;\; y(0) = 1, y'(0) = \frac{1}{2}, h = 0.003125$$

The analytical solution of the above problem is given by

$$y(x) = 1 + \frac{1}{2} \ln\left(\frac{2+x}{2-x}\right)$$

Table 9: Comparison of the error of the proposed method with existing literature for problem 4.

x	Error in [8] $p=6$	Error in [9] $p=8$	Error in [5] $p=4$	Error in our scheme $p=4$
0.1	0.8474×10^{-07}	8.7932×10^{-05}	0.817×10^{-06}	3.105×10^{-16}
0.2	0.5372×10^{-05}	3.2672×10^{-04}	8.17176×10^{-07}	6.209×10^{-16}
0.3	0.6247×10^{-05}	2.2156×10^{-03}	3.10356×10^{-06}	9.314×10^{-16}
0.4	0.1517×10^{-05}	4.8571×10^{-03}	6.56957×10^{-06}	1.2420×10^{-15}
0.5	0.1001×10^{-04}	9.0977×10^{-03}	1.14380×10^{-05}	1.5527×10^{-15}
0.6	0.2970×10^{-04}	1.4391×10^{-02}	1.79656×10^{-05}	1.8635×10^{-15}
0.7	0.5916×10^{-04}	2.1438×10^{-02}	2.64474×10^{-05}	2.1745×10^{-15}
0.8	0.1002×10^{-03}	2.9899×10^{-02}	3.72222×10^{-05}	2.4855×10^{-15}
0.9	0.1550×10^{-03}	4.0301×10^{-02}	5.06788×10^{-05}	2.7967×10^{-15}
1.0	0.2259×10^{-03}	5.2552×10^{-02}	6.72615×10^{-05}	3.1080×10^{-15}

6. Discussion of Result

A new one-step hybrid Bernstein method with one off-step points of order 4 and3 is proposed for the direct solution of general second order ordinary differential equations. The main method and the additional methods were obtained from the same continuous method derived via interpolation and collocation procedures. The stability properties and region of the method are also discussed. The methods are then applied in block form as simultaneous numerical integrators over non-overlapping interval. In Tables 4, 6, 8 and 9, we compared the accuracy of proposed method with some existing methods, the proposed method display better accuracy.

7. Conclusion

The one step hybrid method generated in this paper is accurate, efficient and can compete favorably with existing schemes.

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