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Modelling of an M/M/2 production inventory system with multiple vacation

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Abstract
The paper analyzes an M/M/2 multiple vacations production inventory system with two heterogeneous servers. Server 2 avails multiple vacation, whereas the other one continues static even when the system is vacant. At a time one unit of order arrive from customers randomly. We model the system according to (s, S) policy and the products are manufactured one at a time. The arrival of demands is according to a Poisson process. The duration of the vacation time of server 2 is exponentially distributed. Once the stock position goes down to s, the manufacturing procedure is started and it ceases at the point when the stock position is at the largest extent S. The time gap between the replenishment of two successive items is also distributed exponentially with rate β . Matrix Geometric Method is employed to derive the steady state solution. Several measures of performance in the steady state are derived. An appropriate cost function is constructed and numerical experiments are conducted to obtain the optimum value of the cost function for parameter values.

Keywords
Heterogeneous server, Matrix Geometric Method, Multiple vacation, Cost analysis.

AMS Subject Classification
60K25, 90B05, 91B70.

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Contents

1. Introduction

The queueing system with vacations has been studied extensively and has practical applications in production inventory models, data communication systems, designing computer networks, etc. Vacations to the servers may due to server failures, lack of work, or some other secondary jobs being assigned to the servers. If we allow the servers to avail vacations, the vacation time can be utilized for performing some other jobs, which will improve the profitability of an organization. We consider the concept of heterogeneity of service that allows the customers to receive the different quality of service and it can be seen in organizations like banks, hospitals, railway stations, manufacturing systems, etc. Krishnamoorthy and Viswanath[1] analyzed a production inventory model with service time and vacation to the server.They considered the Markovian production scheme and the arrival of customers are according to a Markovian Arrival Process (MAP). Relevant performance measures and the impact of correlation in the manufacturing process on different performance measures are discussed.

Three production inventory systems were detailed by Krishnamoorthy and Jose [2] in which the loss and retrial of customers are considered. They compared the three systems using Matrix Analytic Method. The queueing system with two heterogeneous servers was analyzed by Krishnamoorthy and Sreenivasan[3]. They discussed the system in which one server always ready to serve whereas the other one was a vacationing server. A numerical example is provided to illustrate the characteristics of the system. They focused on the busy period analysis of the system and long run waiting time in the queue. Another remarkable study was made by Krishna Kumar and Pavai Madheswari[4] on a Markovian queue. They discussed the system when both servers are heterogeneous and choose multiple vacations for want of customers waiting in the queue. The Markovian queue with a working vacation to the heterogeneous servers was devised by Sridhar and Allah Pitchai[5]. When there are no customers both servers go for a vacation and after this vacation, server 1 is always available. During the period of vacation of server 2, if server 1 busy, then server 2 returns from vacation and start servicing at a rate lower than the normal rate. Then after the vacation period, when the customers are staying in the queue for service facility, server 2 is busy with the ordinary rate.

For a more detailed study on server vacations, we can refer to the survey paper by Doshy[6]. A numerical solution for a two-stage production and inventory system with the arrival of demands randomly was analyzed by Andrew Junfang Yu and Yuanyuan Dong [7]. In the model, they considered the production rate is constant and to obtain the optimal solution a numerical approach is used. A production inventory model with probabilistic deterioration and the varying production cost was considered by Palanivel and Uthayakumar[8]. Samanta[9] considered a production inventory model with shortages and the items are deteriorating. For quasi-birth-death processes, a logarithmic reduction algorithm was devised by Latouche and Ramaswami[10]. Sennott et al.[11] derived mean drifts and the non-ergodicity of Markov chains. The analysis of a queue with heterogeneous servers, repair of servers, system disaster, and impatience of customers was studied by Sudheesh et al.[12]. Sivakumar[13] considered a retrial demand inventory system and calculated the joint probability distribution of the inventory level, total expected cost and the number of customers in the orbit. Krishnamoorthy et al.[14] analyzed a production inventory system with service time and interruptions. They derived the necessary and sufficient condition for the stability of the system. Back and Moon[15] considered a production inventory model in which service queue is a Markovian and lost sales were discussed. They derived an explicit stationary joint probability in product form and developed a cost model using mean performance measures. For the detailed study of Matrix Analytic Method, one can refer to Neuts[16]. The remaining part of the paper is arranged as follows. Section 2 is devoted to a brief description of the model. Section 3 presents analysis of the model and system stability. In section 4 various measures of performance are calculated. Numerical results in the form of tables and graphs are presented in section 5 followed by the conclusions in section 6.

2. Description of the model

The study focuses on an M/M/2 multiple vacation and production inventory system with two heterogeneous servers. One server is taking multiple vacations while the other remains in the network even when the system is empty. Assume that the inventory system consists of a single manufacturer, two distributors, and customers. Orders from the buyers arrive at the retailer randomly, which is single unit at a time. Items are manufactured one after another according to (s,S) policy. Inter arrival times of demand are assumed to be a Poisson process with rate λ. Two severs, server 1 and server 2, gives service of different dimension to the customers on an FCFS basis. The service rates of server 1 and server 2 are μ_1 and μ_2 respectively. It is assumed that server 1 is ever present, but on the other hand, server 2 continues with multiple vacations whenever no customer is in the queue for service or no items in the inventory. The vacation period of server 2 is exponentially distributed with a rate of θ. Having the server 2 vacated, if no demand is made or there is no stock of items, server 2 takes another vacation else it comes back to service. If the servers are free and the inventory level is positive all the arriving customers get service immediately. On the arrival of a stock level at a position say $s>0$ determined earlier, the manufacturing process gets started and it ceases when the stock level reaches back to the maximum inventory level S. In order to replenish the items, a random amount of lead time is required and lead time follows an exponential distribution with rate β.

3. Analysis of the model

The presumptions and the notations applied in the model are listed below.

Presumptions

1. order from customers is according to a Poisson process with rate λ.
2. service rates of server 1 and 2 are exponentially distributed with parameter μ_1 and μ_2.
3. the time duration of the inclusion of two consecutive items to the inventory is exponential with rate β.
4. duration of vacation of server 2 is exponentially distributed with parameter θ.

Notations

$N(t)$: Number of customers in the system at time t.

$I(t)$: Inventory level at time t.

$$C(t) : \begin{cases} 0 \text{ if server 2 is on vacation} \\ 1 \text{ if server 2 is busy} \end{cases}$$

$$J(t) : \begin{cases} 0 \text{ if the production process is an OFF mode} \\ 1 \text{ if the production process is ON mode} \end{cases}$$

$\mathbf{e}$: $(1,1,...,1)'$, column vector of 1's of size $(S+1)$

Then $\{(N(t),C(t),J(t)),I(t)),t \geq 0\}$ is a continuous time Markov chain on the state space $\{(0,0,0,k),s+1 \leq k \leq S\} \cup \{(0,1,0,k),0 \leq k \leq S-1\} \cup \{(i,0,j,k),i \geq 1,j=0,1,s+1 \leq k \leq S\} \cup \{(i,1,j,k),i \geq 1,j=0,1,0 \leq k \leq S-1\}$. Now the generator matrix of the process is given by

$$Q = \begin{bmatrix} A_{00} & A_{01} & 0 & 0 & 0 & \dots \\ A_{10} & A_{11} & A_0 & 0 & 0 & \dots \\ 0 & A_2 & A_1 & A_0 & 0 & \dots \\ 0 & 0 & A_2 & A_1 & A_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where the blocks A_{00},A_{01},A_{10},A_{11},A_0,A_1, andA_2 are given by

$$[A_{01}]_{uv} = \begin{cases} \lambda, & \text{if } 1 \leq u \leq S-s, v=u \\ \lambda, & \text{if } S-s+1 \leq u \leq 2S-s, \\ & \quad 2(S-s)+1 \leq v \leq 3S-2s \\ 0, & \text{otherwise} \end{cases}$$

$$[A_{00}]_{uv} = \begin{cases} -\lambda, & \text{if } 1 \leq u \leq S-s, v=u \\ \beta, & \text{if } u=2S-s, v=S-s \\ -(\lambda+\beta), & \text{if } S-s+1 \leq u \leq 2S-s, v=u \\ \beta, & \text{if } S-s+1 \leq u \leq 2S-s-1, \\ & \quad v=u+1 \\ 0, & \text{otherwise} \end{cases}$$

$$[A_{10}]_{uv} = \begin{cases} \mu_1, & \text{if } 2 \leq u \leq S-s, v=u-1 \\ \mu_1, & \text{if } u=1, v=S+1 \\ (\mu_1+\mu_2), & \text{if } S-s+2 \leq u \leq 2(S-s), \\ & \quad 1 \leq v \leq S-s \\ (\mu_1+\mu_2), & \text{if } u=S-s+1, v=S+1 \\ \mu_1, & \text{if } 2(S-s)+2 \leq u \leq 3S-2s, \\ & \quad S-s+1 \leq v \leq 2S-s-1 \\ (\mu_1+\mu_2), & \text{if } 3S-2s+2 \leq u \leq 4S-2s, \\ & \quad S-s+1 \leq v \leq 2S-s-1 \\ 0, & \text{otherwise} \end{cases}$$

$$[A_{11}]_{uv} = \begin{cases} -(\lambda+\mu_1), & \text{if } 1 \leq u \leq S-s, v=u \\ -(\lambda+\mu_1+\mu_2), & \text{if } S-s+1 \leq u \leq 2S-2s, \\ & \quad v=u \\ -(\lambda+\beta), & \text{if } u=2S-2s+1, v=u \\ -(\lambda+\mu_1+\beta), & \text{if } 2S-2s+2 \leq u \leq 3S-2s, \\ & \quad v=u \\ \beta, & \text{if } 2S-2s+1 \leq u \leq 3S-2s-1, \\ & \quad v=u+1 \\ -(\lambda+\beta), & \text{if } u=3S-2s+1, v=u \\ \beta, & \text{if } u=3S-2s, v=S-s \\ \nabla, & \text{if } 3S-2s+2 \leq u \leq 4S-2s, v=u \\ \beta, & \text{if } 3S-2s+1 \leq u \leq 4S-2s-1, \\ & \quad v=u+1 \\ \beta, & \text{if } u=4S-2s, v=2S-2s \\ 0, & \text{otherwise} \end{cases}$$

$$[A_0]_{uv} = \begin{cases} \lambda, & \text{if } 1 \leq u \leq 4S-2s, v=u \\ 0, & \text{otherwise} \end{cases}$$

$$[A_2]_{uv} = \begin{cases} \mu_1, & \text{if } 2 \leq u \leq S-s, v=u-1 \\ \mu_1, & \text{if } u=1, v=2S-s+1 \\ (\mu_1+\mu_2), & \text{if } S-s+2 \leq u \leq 2S-2s, v=u-1 \\ (\mu_1+\mu_2), & \text{if } u=S-s+1, v=3S-s+1 \\ \mu_1, & \text{if } 2S-2s+2 \leq u \leq 3S-2s, \\ & \quad v=u-1 \\ (\mu_1+\mu_2), & \text{if } 3S-2s+2 \leq u \leq 4S-2s, \\ & \quad v=u-1 \\ 0, & \text{otherwise} \end{cases}$$

$$[A_1]_{uv} = \begin{cases} -(\lambda+\mu_1+\theta), & \text{if } 1 \leq u \leq S-s, v=u \\ \theta\,, & \text{if } 1 \leq u \leq S-s, \\ & \quad S-s+1 \leq v \leq 2S-2s \\ -(\lambda+\mu_1+\mu_2), & \text{if } S-s+1 \leq u \leq 2S-2s, v=u \\ -(\lambda+\beta+\theta), & \text{if } u=2S-2s+1, v=u \\ \beta, & \text{if } 2S-2s+1 \leq u \leq 3S-2s-1, \\ & \quad v=u+1 \\ \theta, & \text{if } 2S-2s+1 \leq u \leq 3S-2s, \\ & \quad v=S+u \\ \beta, & \text{if } u=3S-2s, v=S-s \\ \beta, & \text{if } u=4S-2s, v=2S-2s \\ \Delta, & \text{if } 2(S-s+1) \leq u \leq 3S-2s, v=u \\ -(\lambda+\beta), & \text{if } u=3S-2s+1, v=u \\ \nabla & \text{if } 3S-2s+2 \leq u \leq 4S-2s, v=u \\ \beta, & \text{if } 3S-2s+1 \leq u \leq 4S-2s-1, \\ & \quad v=u+1 \\ 0, & \text{otherwise} \end{cases}$$

$\Delta = -(\lambda+\theta+\mu_1+\beta), \nabla = -(\lambda+\beta+\mu_1+\mu_2),$

3.1 System stability

The Markov chain obtained here is a level-independent quasi-birth-death process. For the stability of the Markov chain, it is necessary and sufficient that $\pi A_0 e < \pi A_2 e$, where π is the stationary vector which is unique and satisfies $\pi A = 0$ and $\pi e = 1$, where $A = A_0 + A_1 + A_2$, and e is the column vector of an appropriate order.

3.2 Stability conditions

Theorem 3.1. *The above queueing model is stable if and only if $\rho < 1$ where $\rho = \frac{\lambda}{\mu_1+\mu_2}$.*

Proof. The stability of the Markov chain is proved using Pake's Lemma. Immediately after the service completion epoch of the i^{th} customer, let there be T_i number of customers in the system. Then $T_i : i \in N$ satisfies the equation

$$T_i = \begin{cases} T_{i-1} - 1 + V_i, & \text{if } T_{i-1} \geq 1 \\ V_i, & \text{if } T_{i-1} = 0 \end{cases}$$

where V_i represents the number of entries during the service of the i^{th} customer. Then the Markov chain $\{T_i : i \in N\}$ is irreducible and aperiodic. According to Pake's lemma an aperiodic Markov chain is ergodic, if there exist an $\varepsilon > 0$ such that the mean drift $\psi_j = E[(T_{(i+1)} - T_i)/T_i = j]$ is finite for all j$\in N$ and $\psi_j \leq -\varepsilon$ for all j$\in N$ except perhaps for a finite number. The Mean drift obtained here as

$$\psi_j = \begin{cases} -1+\rho, & \text{if } j \geq 1 \\ \rho, & \text{if } j = 0 \end{cases}$$

The Markov chain $\{T_i : i \in N\}$ is ergodic if $\rho < 1$ and hence the sufficient condition is satisfied. For obtaining the necessary condition, we assume that $\rho \geq 1$. Using the theorem in Sennot, which states that $\{T_i : i \in N\}$ non ergodic if the Kaplan'condition : $\psi_j < \infty$, for $j \geq 0$ and there exist a j_0 such that $\psi_j \geq 0$, for $j \geq j_0$ is satisfied. Here Kaplan's condition is fulfilled when $\rho \geq 1$ and hence the Markov chain $\{T_i : i \in N\}$ is not ergodic. □

3.3 Steady State Probability Vector

Our objective is to compute the stationary probability vector $\mathbf{X} = (x_0, x_1, \dots)$ of Q from the system of equations $\mathbf{X}Q = 0$. Neuts developed a matrix geometric solution to solve this problem that there exists a positive matrix R such that $x_i = x_{(i-1)} * R$ for (i = 2,3,4...). If we know the sub vectors x_0 and x_1 and the rate matrix R then the remaining sub vectors of the stationary distribution can be calculated. R can be computed from $R^2 A_2 + RA_1 + A_0 = 0$. The rate matrix R is given by $R = -A_0(A_1)^{-1} - R^2 A_2 (A_1)^{-1}$. This leads to the successive substitution procedure derived by Nuets , $R_0 = 0, R_{k+1} = -A_0(A_1)^{-1} - R_k^2 A_2 (A_1)^{-1}$, $k = 0,1,2,\dots$. Nuets proved that the sequences of matrices $R_k, (k = 0,1,2,\dots)$ is nondecreasing and converges to the rate matrix R. The process came into an end once successive differences are less than a specified tolerance criterion. To find x_0 and x_1, from $\mathbf{X}Q = 0$ we get

$$\left.\begin{aligned} x_0 A_{00} + x_1 A_{10} = 0 \\ x_0 A_{01} + x_1 A_{11} + x_2 A_2 = 0 \end{aligned}\right\} \tag{3.1}$$

Replacing x_2 with $x_1 * R$ we get the homogeneous equations

$$\left.\begin{aligned} x_0 A_{00} + x_1 A_{10} = 0 \\ x_0 A_{01} + x_1 [A_{11} + RA_2] = 0 \end{aligned}\right\} \tag{3.2}$$

The normalizing equation is

$$x_0 \mathbf{e} + \{x_1[(I-R)^{-1}]\}\mathbf{e} = 1\} \tag{3.3}$$

The boundary probabilities x_0, x_1, and the probabilities x_i, for $i \geq 2$ can be obtained using equations (2), (3) and R.

4. System Performance Measures

We partition the components of x_i as
$x_i = \{y_{i,0,0,s+1}, \dots, y_{i,0,0,S}, y_{i,0,1,s+1} \dots, y_{i,0,1,S},$
$y_{i,1,0,0}, \dots, y_{i,1,0,S-1}, y_{i,1,1,0}, \dots, y_{i,1,1,S-1}\}$ for $(i \geq 1)$
$x_0 = \{y_{0,0,0,s+1}, \dots, y_{0,0,0,S}, y_{0,1,0,0} \dots, y_{0,1,0,S-1}\}$

Now we derive the system performance measures under steady state.

1. Expected inventory level, E_I, is given by

$$E_I = \sum_{j=0}^{1} \sum_{k=s+1}^{S} \sum_{i=0}^{\infty} k y_{i,0,j,k} + \sum_{j=0}^{1} \sum_{k=1}^{S-1} \sum_{i=0}^{\infty} k y_{i,1,j,k}$$

2. Expected number of customers in the system, E_{CS}, is given by

$$E_{CS} = (\sum_{i=1}^{\infty} i x_i) e$$

$$= x_1 e + \{x_1 R[(I-R)^{-1} + (I-R)^{-2}]\} e$$

3. Expected reorder level, E_{RO}, is given by

$$E_{RO} = \mu_1 \sum_{i=1}^{\infty} y_{i,0,0,s+1} + (\mu_1 + \mu_2) \sum_{i=1}^{\infty} y_{i,0,1,s+1}$$

4. Fraction of time production process is ON is given by

$$E_{PON} = \sum_{i=0}^{\infty} \sum_{j=0}^{1} \sum_{k=0}^{S-1} y_{i,1,j,k} + \sum_{k=0}^{S-1} y_{0,1,0,k}$$

5. Expected number of departures after completing service is given by

$$E_{DS} = \mu_1 \sum_{i=1}^{\infty} \sum_{k=s+1}^{S} y_{i,0,0,k} + \mu_1 \sum_{i=2}^{\infty} \sum_{k=0}^{S-1} y_{i,1,0,k} + (\mu_1 + \mu_2) \sum_{i=1}^{\infty} \left(\sum_{k=s+1}^{S} y_{i,0,1,k} + \sum_{k=0}^{S-1} y_{i,1,1,k} \right)$$

5. Cost Analysis

Define the expected total cost of the system per unit per unit time is given by

$$ETC = (C + (S-s)c_1)E_{RO} + c_2E_I + c_3E_{CS} + c_4E_{DS}$$

where,

C : Fixed cost/unit/unit time

c_1 : Procurement cost perunit per unit time

c_2 : Holding cost of inventory per unit per unit time

c_3 : Holding cost of customers per unit per unit time

c_4 : Cost due to service per unit per unit time

5.1 Numerical Results

Here some numerical results are provided to show the variations in various performance measures under the changes in the value of one parameter at a time keeping the other parameter values constant.

5.2 Tables for computation

Table 1. Effect of variation of λ on various performance measures

λ	E_I	E_{CS}	E_{RO}	E_{DS}
0.1	17.570	0.129	0.143	3.74e-05
0.2	16.700	0.253	0.262	0.0003
0.3	16.005	0.380	0.369	0.0009
0.4	15.399	0.518	0.469	0.0021
0.5	14.826	0.672	0.565	0.0040
0.6	14.248	0.849	0.659	0.0066
0.7	13.639	1.060	0.751	0.0100
0.8	12.994	1.316	0.843	0.0139

$\mu_1 = 0.4, \mu_2 = 1.2, \beta = 2.1, \theta = 4, S = 20, s = 5$

Table 2. Effect of variation of β on various performance measures

β	E_I	E_{CS}	E_{RO}	E_{DS}
2.1	17.524	0.122	0.1436	4.95e-05
2.2	17.594	0.120	0.1428	4.16e-05
2.3	17.657	0.118	0.1420	3.53e-05
2.4	17.716	0.116	0.1411	3.01e-05
2.5	17.770	0.115	0.1403	2.59e-05
2.6	17.819	0.113	0.1395	2.23e-05
2.7	17.865	0.112	0.1387	1.94e-05
2.8	17.908	0.111	0.1380	1.69e-05

$\mu_1 = 0.4, \mu_2 = 1.2, \theta = 4, \lambda = 0.1, S = 20, s = 5$

Table 3. Effect of variation of μ_1 on various performance measures

μ_1	E_I	E_{CS}	E_{RO}	E_{DS}
0.5	17.524	0.122	0.144	4.95e-05
0.6	17.468	0.115	0.144	6.56e-05
0.7	17.401	0.109	0.144	8.65e-05
0.8	17.321	0.103	0.143	0.0001
0.9	17.223	0.097	0.142	0.0001
1.0	17.105	0.092	0.141	0.0001
1.1	16.959	0.087	0.140	0.0002
1.2	16.778	0.083	0.138	0.0003

$\lambda = 0.1, \mu_2 = 1.2, \beta = 2.1, \theta = 4, S = 20, s = 5$

Table 4. Effect of variation of μ_2 on various performance measures

μ_2	E_I	E_{CS}	E_{RO}	E_{DS}
1.2	17.524	0.122	0.144	4.9481e-05
1.3	17.501	0.119	0.146	5.1438e-05
1.4	17.478	0.116	0.149	5.3388e-05
1.5	17.455	0.113	0.151	5.533e-05
1.6	17.432	0.111	0.153	5.726e-05
1.7	17.410	0.109	0.156	5.9179e-05
1.8	17.388	0.107	0.158	6.1085e-05
1.9	17.366	0.106	0.160	6.2976e-05

$\lambda = 0.1, \mu_1 = 0.5, \theta = 4, \beta = 2.1, S = 20, s = 5$

Table 5. Effect of variation of θ on various performance measures

θ	E_I	E_{CS}	E_{RO}	E_{DS}
0.9	17.539	0.12552	0.14395	4.5518e-05
1.0	17.538	0.12513	0.14392	4.5911e-05
1.1	17.536	0.12480	0.14389	4.6251e-05
1.2	17.535	0.12451	0.14387	4.6549e-05
1.3	17.534	0.12425	0.14384	4.6812e-05
1.4	17.533	0.12403	0.14382	4.7047e-05
1.5	17.532	0.12382	0.14381	4.7258e-05

If the arrival rate λ is increased, then the expected inventory level decreases and the expected number of customers in the system increases. There is an increase in the expected inventory level as replenishment rate β increases. Also, there is a slight decrease in the expected number of customers as β increases. The increase in service rates leads to a decrease in the expected inventory level. If the value of θ increased,then there is a slight decrease in the expected inventory level. There is a decrease in the expected number of customers as θ increases.

5.3 Graphical illustrations

In this section, we presented the outcome of model parameters on the expected total cost and system performance measures. Here, the value of the parameters that will contribute the minimum value of the cost function is considered. By fixing the values of all parameters other than λ, from figure 1, it is observed that the cost function is minimum at λ=

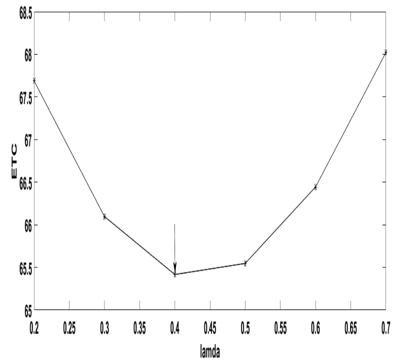

Figure 1. λ vs ETC
$C = 20, c_1 = 65.51, c_2 = 3.6, c_3 = 1.5, c_4 = 1, S = 20, s = 5, \mu_1 = 0.6, \mu_2 = 1.2, \beta = 2.1, \theta = 1.5$

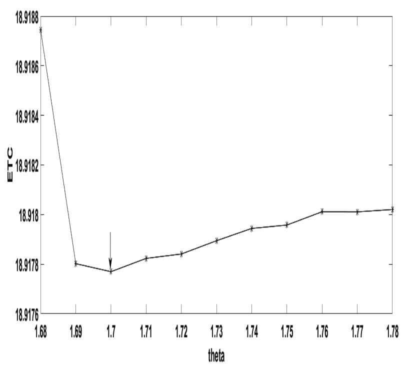

Figure 2. θ vs ETC
$C = 20, c_1 = 231.1, c_2 = 0.11, c_3 = 0.21, c_4 = 45.1, S = 20, s = 5, \mu_1 = 0.6, \mu_2 = 1.2, \beta = 2.1, \lambda = .1$

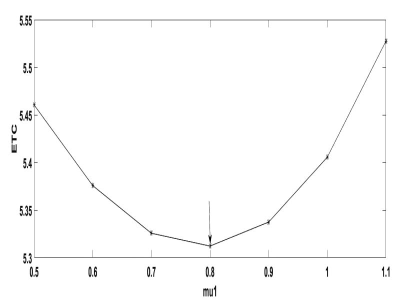

Figure 3. μ_1 vs ETC
$C = 20, c_1 = 333.6, c_2 = 0.1, c_3 = 22.10, c_4 = 5.1, S = 20, s = 5, \theta = 1.5, \mu_2 = 1.2, \beta = 2.1, \lambda = .1$

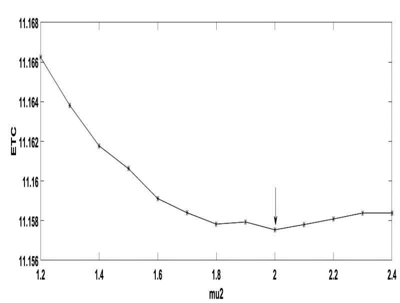

Figure 4. μ_2 vs ETC:$C = 20, c_1 = 353, c_2 = 0.6, c_3 = 1.6, c_4 = 1, S = 20, s = 5, \theta = 1.5, \mu_1 = 0.6, \beta = 2.1, \lambda = 0.1$

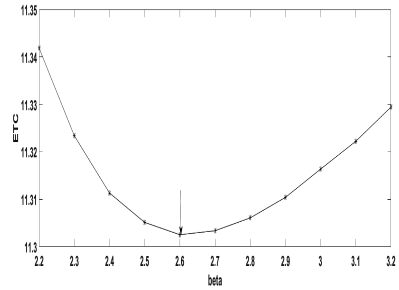

Figure 5. β vs ETC
$C = 20, c_1 = 452.001, c_2 = 0.6, c_3 = 0.005, c_4 = 2.8, S = 20, s = 5, \theta = 1.5, \mu_2 = 1.2, \mu_1 = 0.6, \lambda = .1$

0.4 and its minimum value is 65.5.It can also be seen that the minimum values of the cost function by varying other parameters. As θ increases ETC has a minimum value of 18.9178 at θ = 1.7 (see fig. 4). The minimum values of ETC are 5.3121, 11.1575 and 11.25 at $\mu_1 = 0.8, \mu_2 = 2$, and β =2.6 respectively (see fig.2, fig.3 and fig.5).

6. Concluding remarks

The paper explains a production inventory system with two servers and multiple vacations. The system using Matrix-Analytic Method is also analyzed. Stability condition, Steady-state distributions, System performance measures, and Numerical experiments are also done. For an extension of the present work, one may consider a multi-server production inventory system with the arrival process as MAP (Markovian Arrival Process), service time as a Phase-type distribution, and production process as MPP (Markovian Production Process).

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