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Common fixed point theorems for three self maps of a complete S-metric space

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Abstract
In this present paper we prove a common fixed point theorem for three self maps of a S-metric space which satisfy certain conditions.

Keywords
S-metric space, Compatible mappings, Fixed point, Associated sequence of a point relative to three self maps, contractive modulus.

AMS Subject Classification
54H25,47H10.

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Contents

1. Introduction

In an attempt to generalize metric space Gahler [2,3] introduced the notion of 2-metric spaces while B.C.Dhage [1] initiated the notion of D- metric spaces.Subsequently several researchers have proved that most of their claims are not valid.As probable modification to D- metric spaces,very recently Shaban Sedghi, Nabi Shobe and Haiyun Zhou [4] introduced D^*- metric spaces. In 2006 Zead Mustafa and Brailey Sims [7] have initiated G- metric spaces, while Shaban Sedghi, Nabi Shobe and Abdelkrm Aliouche [5] considered S-metric spaces in 2012.Of these three generalizations, the S-metric space seen evinced interest in many researchers.
The purpose of this paper is to prove a common fixed point theorem for three self maps of a S-metric space.Also as a consequence, we prove a common fixed point theorem for three self maps of a complete S-metric space. Further we show that a common fixed point theorem for three self maps of a metric space proved by S.L.Singh and S.P.Singh ([6] pp 1584-1586) follows as a particular case of our theorem.

Now we recall some basic definitions and lemmas required in the sequel in section 2 and establish main results in section 3.

2. Preliminaries

Definition 2.1. *Let X be a non empty set.By S-metric we mean a function $S : X^3 \to [0,\infty)$ which satisfies the following conditions for each $x,y,z,w \in X$*
(a) $S(x,y,z) \geq 0$
(b) $S(x,x,y) = 0$ if and only if $x = y = z$.
(c) $S(x,y,z) \leq S(x,x,w)+S(y,y,w)+S(z,z,w)$
In this case (X,S) is called a S-metric space

Example 2.2. *Let $X = \mathbb{R}$ and $S : \mathbb{R}^3 \to [0,\infty)$ be defined by $S(x,y,z) = |y+z-2x|+|y-z|$ for $x,y,z \in \mathbb{R}$, then (X,S) is a S-metric space.*

Example 2.3. *Let (X,d) be a metric space.*
Define $S_d : X^3 \to [0,\infty)$ by $S_d(x,y,z) = d(x,y)+d(y,z)+d(z,x)$ then S_d is a S-metric on X and we call this as the S-metric induced by d.

Remark 2.4. *It is shown ([5], Lemma 2.5) in a S-metric space that*
$S(x,x,y) = S(y,y,x)$ for all $x,y \in X$. Also we need the following notions given in [6].

Definition 2.5. *Let (X,S) be an S-metric space. Let $x \in X$ and $r > 0$, then the open ball with centre at x and radius r is given by $B(x,r) = \{y \in X : S(y,y,x) < r\}$*

Remark 2.6. *Let (X,S) be an S-metric space and $A \subset X$.*
(1) It has been proved in [5] that $B(x,r)$ is an open set in X and that the topology generated by the open balls as a basis is a topology called the topology induced by the S-metric on X.
(2) If for every $x \in A$, there exists a $r > 0$ such that $B_s(x,r) \subset A$, then the subset A is called an open subset of X.
(3) A sequence $\{x_n\}$ in X said to converge to x if $S(x_n,x_n,x) \to 0$; that is for each $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $S(x_n,x_n,x) < \varepsilon$ and we write this by $\lim_{n\to\infty} x_n = x$ in this case.
(4)A sequence $\{x_n\}$ in X is called a Cauchy sequence if to each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x) < \varepsilon$ for each $n,m \geq n_0$.
(5) In [5] it has been proved that if $\{x_n\}$ is a sequence in S-metric space (X,S) that converges to x is unique and that $\{x_n\}$ is a Cauchy sequence.
(6) An S-metric space (X,S) is said to be complete if every Cauchy sequence in it converges.

Definition 2.7. *Let (X,S) be an S-metric space, If there exists sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ then $\lim_{n\to\infty} S(x_n,x_n,y_n) = S(x,x,y)$, then we say that $S(x,y,z)$ is continuous in x and y.*

It is well known now that the commutativity of maps is generalized as follows

Definition 2.8. *If g and f are self maps of a S-metric space (X,S) such that for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = t$ for some $t \in X$ we have $\lim_{n\to\infty} S(gfx_n, gfx_n, fgx_n) = 0$ then g and f are said to be compatible*

Trivially commuting self maps of a S-metric space are compatible but not conversely. For example

Example 2.9. *Let $X = [0,1]$ and*
$S(x,y,z) = |x-y| + |y-z| + |z-x|$ for $x,y,z \in X$.
Defining $f : X \to X, g : X \to X$ by $gx = \frac{x^2}{2}$ and $fx = \frac{x^2}{3}$ for $x \in X$ then it is easy to see that g,f are compatible but not commutative.

Lemma 2.10. *Let (X,d) be any metric space and S_d be the S-metric induced by d. For any sequence $\{x_n\}$ in (X,S_d), is a Cauchy sequence if and only if $\{x_n\}$ is a Cauchy sequence in (X,d).*

Proof. First observe that $d(x,y) \leq S_d(x,x,y) \leq 2d(x,y)$ for all $x,y \in X$. Now the lemma follows immediately in view of the above inequality □

Corollary 2.11. *Let (X,d) be any metric space and S_d be the S-metric on X. Then (X,S_d) is a complete if and only if (X,d) is complete*

Proof. Follows from Lemma 2.10 □

Definition 2.12. *If f,g and h be self maps of a non empty set X such that $f(X) \cup g(X) \subseteq h(X)$, then for any $x_0 \in X$,there is a sequence $\{x_n\}$ in X such that $fx_{2n} = hx_{2n+1}, gx_{2n+1} = hx_{2n+2}$ for $n \geq 0$ then $\{x_n\}$ is called an associated sequence of x_0 relative to three self maps f,g and h.*

The existence of an associated sequence of x_0 relative to f,g and h is ensured.
In fact, if $x_0 \in X$ then $fx_0 \in f(X)$ and $f(X) \subseteq h(X)$ imply that there is a $x_1 \in X$ such that $fx_0 = hx_1$. Now $gx_1 \in g(X)$ and $g(X) \subseteq h(X)$ imply that there is a $x_2 \in X$ with $gx_1 = hx_2$. Again $fx_2 \in f(X)$ and $f(X) \subseteq h(X)$ then we get $x_3 \in X$ with $fx_2 = hx_3$ and $gx_3 \in g(X)$, $g(X) \subseteq h(X)$ gives $gx_3 = hx_4$ for some $x_4 \in X$. Repeating this process, alternatively using the fact $f(X) \subseteq h(X)$ and $g(X) \subseteq h(X)$ we can find a sequence $\{x_n\}$ with $fx_{2n} = hx_{2n+1}$ and $gx_{2n+1} = h_{2n+2}$ for $n \geq 0$.
It may be noted that for a given point $x_0 \in X$ there may be more than one sequence $\{x_n\}$ with the above condition. For example

Example 2.13. *Suppose $X = \mathbb{R}$ with $S(x,y,z) = |x-y| + |y-z| + |z-x|$ for $x,y,z \in X$. Define self maps $f : X \to X$, $g : X \to X$ and $h : X \to X$ by $fx = gx = \frac{x^2}{3}$ and $h(x) = x^2$. Then as explained above we get a sequence $\{x_n\}$ with $fx_{2n} = hx_{2n+1}$ and $gx_{2n+1} = h_{2n+2}$ for $n \geq o$ where each x_n has two choices viz $\frac{x_0}{(\sqrt{3})^n}$ or $\frac{-x_0}{(\sqrt{3})^n}$ for $n \geq 0$. Hence to each $x_0 \in X$, there are infinitely many associated sequences $\{x_n\}$.*

Definition 2.14. *A mapping $\phi : [0,\infty) \to [0,\infty)$ is said to be a contractive modulus if $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$*

Example 2.15. *The mapping $\phi : [0,\infty) \to [0,\infty)$ defined by $\phi(t) = ct$ where $0 \leq c < 1$ is a contractive modulus.*

3. Main Results

Theorem 3.1. *Suppose f,g and h be three self maps of S-metric space (X,S) satisfying the conditions*

(i) $f(X) \cup g(X) \subseteq h(X)$

(ii) $S(fx,fx,gy) \leq \phi\Big(\lambda(x,y)\Big)$ *for all $x,y \in X$ where ϕ is an upper semi continuous contractive modulus and*
$\lambda(x,y) = \max\{S(hx,hx,hy), S(fx,fx,hx), S(gy,gy,hx), \frac{1}{2}[S(fx,fx,hy) + S(gy,gy,hx)]\}$

(iii) Either (f,h) or (g,h) is compatible pair and

(iv) h is continuous
Further if

(v) There is a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of x_0 relative to the three self maps such that the sequence $fx_0, gx_1, fx_2, gx_3, \cdots fx_{2n}, gx_{2n+1}, \cdots$ converge to some point $z \in X$.

Then z is the unique common fixed point for f,g and h.

Before proving the main theorem, we establish a lemma which is noteworthy.

Lemma 3.2. *Suppose f, g and h be three self maps of S-metric space (X, S) satisfying the conditions (i),(ii),(iv) and (v) of the Theorem 3.1. Then for the associated sequence $\{x_n\}$ of x_0 relative to f, g and h we have*
(a) $\lim_{n\to\infty} \lambda(hx_{2n}, x_{2n+1}) = S(z,z,hz)$ if (f,h) is compatible
(b) $\lim_{n\to\infty} \lambda(x_{2n}, hx_{2n+1}) = S(z,z,hz)$ if (g,h) is compatible

Proof. Since by (v), each of the sequence fx_{2n} and gx_{2n+1} converges to $z \in X$ and since $fx_{2n} = hx_{2n+1}$ and $gx_{2n+1} = h_{2n+2}$ for $n \geq 0$, we have

$$fx_{2n}, gx_{2n+1}, hx_{2n+1}, hx_{2n+2} \to z \text{ as } n \to \infty \tag{3.1}$$

Now since h is continuous, we have

$$hfx_{2n} \to hz, \qquad h^2x_{2n} \to hz \text{ as } n \to \infty \tag{3.2}$$

(a) If the pair (f,h) is compatible, we have

$$\lim_{n\to\infty} S(hfx_{2n}, hfx_{2n}, fhx_{2n}) = 0 \tag{3.3}$$

since $fx_{2n}, hx_{2n} \to z$ as $n \to \infty$ by 3.1
Now, in view of 3.2 and 3.3, we get

$$fhx_{2n} \to hz \text{ as } n \to \infty \tag{3.4}$$

Also from (ii) we have

$$\begin{aligned}
&\lambda(hx_{2n}, x_{2n+1}) \\
&= \max\{S(h^2x_{2n}, h^2x_{2n}, hx_{2n+1}), S(fhx_{2n}, fhx_{2n}, h^2x_{2n}), \\
&\quad S(gx_{2n+1}, gx_{2n+1}, hx_{2n+1}), \\
&\frac{1}{2}[S(fhx_{2n}, fhx_{2n}, hx_{2n+1}) + S(gx_{2n+1}, gx_{2n+1}, h^2x_{2n})]\}
\end{aligned}$$

So that, in view of Remark 2.4 , we have

$$\begin{aligned}
&\lim_{n\to\infty} \lambda(hx_{2n}, x_{2n+1}) \\
&= \max\{S(hz,hz,z), S(hz,hz,hz), S(z,z,z)), \\
&\qquad \frac{1}{2}[S(hz,hz,z) + S(z,z,hz)]\} \\
&= S(z,z,hz)
\end{aligned}$$

Proving part (a) of the lemma
(b) If the pair (g,h) is compatible, we have by 3.1

$$\lim_{n\to\infty} S(hgx_{2n+1}, hgx_{2n+1}, ghx_{2n}) = 0 \tag{3.5}$$

Also since h is continuous, we have again by 3.1, that

$$h^2x_{2n+1} \to hz \text{ and } hgx_{2n+1} \to hz \text{ as } n \to \infty \tag{3.6}$$

Now, in view of 3.5 and 3.6, we get

$$ghx_{2n+1} \to hz \text{ as } n \to \infty \tag{3.7}$$

Now, from (ii) we have

$$\lambda(x_{2n}, hx_{2n+1}) = \max\{S(hx_{2n}, hx_{2n}, h^2x_{2n+1}), \tag{3.8}$$
$$S(fx_{2n}, fx_{2n}, hx_{2n}), S(ghx_{2n+1}, ghx_{2n+1}, \tag{3.9}$$
$$h^2x_{2n+1}), \frac{1}{2}[S(fx_{2n}, fx_{2n}, h^2x_{2n+1}) + \tag{3.10}$$
$$S(ghx_{2n+1}, ghx_{2n+1}, hx_{2n})]\} \tag{3.11}$$

Now, letting $n \to \infty$ in 3.8 and using the continuity of $S(x,y,z)$ in x and y ,3.1, 3.6, 3.7 we get

$$\begin{aligned}
&\lim_{n\to\infty} \lambda(x_{2n}, hx_{2n+1}) \\
&= \max\{S(z,z,hz), S(z,z,z), S(hz,hz,hz)), \\
&\qquad \frac{1}{2}[S(z,z,hz) + S(hz,hz,z)]\} \\
&= S(z,z,hz)
\end{aligned}$$

Proving part (b) of the lemma □

Proof of the Theorem 3.1
In this section we first prove the existence of a common fixed point in one of the two cases of the condition (iii) and the other case follows similarly with appropriate changes. Here we prove in case the pair (f,h) is compatible. Now from (ii), we have

$$S(fhx_{2n}, fhx_{2n}, gx_{2n+1}) \leq \phi\Big(\lambda(hx_{2n}, x_{2n+1}\Big) \tag{3.12}$$

in which on letting $n \to \infty$ and using Lemma 3.2 and the continuity of $S(x,y,z)$ in x and y we get

$$S(hz,hz,z) \leq \phi\Big(S(hz,hz,z)\Big) \tag{3.13}$$

And this leads to a contradiction if $hz \neq z$. Therefore $hz = z$. Again from the condition (ii), we have

$$S(fz,fz,gx_{2n+1}) \leq \phi\Big(\lambda(z, x_{2n+1})\Big) \tag{3.14}$$

But

$$\begin{aligned}
\lambda(z, x_{2n+1}) = \max\{&S(hz,hz,hx_{2n+1}) \\
&S(fz,fz,hz), S(gx_{2n+1}, gx_{2n+1}, hx_{2n+1})), \\
&\frac{1}{2}[S(fz,fz,hx_{2n+1}) + S(gx_{2n+1}, gx_{2n+1}, hz)]\}
\end{aligned}$$

In which on letting $n \to \infty$, we find

$$\begin{aligned}
&\lim_{n\to\infty} \lambda(z, x_{2n+1}) = \max\{S(hz,hz,z), S(fz,fz,z), \\
&\qquad S(z,z,z), \frac{1}{2}[S(fz,fz,z) + S(z,z,z)]\} \\
&= \max\{0, S(fz,fz,z), 0, \frac{1}{2}[S(fz,fz,z) + 0]\} \\
&= S(fz,fz,z)
\end{aligned}$$

Now, letting $n \to \infty$ in 3.14, we get by the upper semicontinuity of ϕ, that

$$S(fz, fz, z) \leq \phi\left(S(fz, fz, z)\right) \tag{3.15}$$

which leads to a contradiction if $fz \neq z$. Therefore $fz = z$.
Now, again from the condition (ii), we have

$$S(fx_{2n}, fx_{2n}, gz) \leq \phi\left(\lambda(x_{2n}, z)\right) \tag{3.16}$$

But

$$\begin{aligned}\lambda(x_{2n}, z) = \max\{&S(hx_{2n}, hx_{2n}, hz),\\ &S(fx_{2n}, fx_{2n}, hx_{2n}), S(gz, gz, hz)),\\ &\frac{1}{2}[S(fx_{2n}, fx_{2n}, hz) + S(gz, gz, hx_{2n})]\}\end{aligned}$$

so that

$$\begin{aligned}\lim_{n\to\infty} \lambda(x_{2n}, z) &= \max\{S(z, z, hz), S(z, z, hz), S(gz, gz, z),\\ &\qquad \frac{1}{2}[S(z, z, z) + S(gz, gz, z)]\}\\ &= S(gz, gz, z) = S(z, z, gz)\end{aligned}$$

Now, letting $n \to \infty$ in 3.16, we get by the upper semicontinuity of ϕ, that

$$S(z, z, gz) \leq \phi\left(S(z, z, gz)\right) \tag{3.17}$$

and this leads to a contradiction if $gz \neq z$. Therefore $gz = z$.
Hence $z = fz = gz = hz$
Showing that z is a common fixed point of f,g and h.
We now prove the uniqueness of the common fixed point.If possible let z' be another common fixed point of f,g and h. Then from condition (ii), we have

$$S(z, z, z') = S(fz, fz, gz') \leq \phi\left(\lambda(z, z')\right) \tag{3.18}$$

where

$$\begin{aligned}&\lambda(z, z')\\ &= \max\{S(hz, hz, hz'), S(fz, fz, hz), S(gz', gz', hz')),\\ &\qquad \frac{1}{2}[S(fz, fz, hz') + S(gz', gz', hz)]\}\\ &= \max\{S(z, z, z'), 0, 0, \frac{1}{2}[S(z, z, z') + S(z', z', z)]\}\\ &= S(z, z, z')\end{aligned}$$

Therefore 3.18 gives

$$S(z, z, z') \leq \phi\left(S(z, z, z')\right) \tag{3.19}$$

which leads to a contradiction if $z \neq z'$. Hence z is the unique common fixed point of f,g and h.
Hence the Theorem 3.1 is completely proved.

Theorem 3.3. *Suppose (X, S) is a S-metric space satisfying the conditions (i) to (iv)of Theorem 3.1. Further if*
(v)' (X, S) is complete
Then f, g and h have a unique common fixed point $z \in X$.

Before proving the main theorem, we establish an essential lemma.

Lemma 3.4. *Suppose (X, S) is a S-metric space (X, S) and f, g and h be three self maps of X such that*

(i) $f(X) \cup g(X) \subseteq h(X)$

(ii) $S(fx, fx, gy) \leq c\left(\lambda(x, y)\right)$ *for all $x, y \in X$ where*
$0 \leq c < \frac{1}{2}$ *and*
$\lambda(x, y) = \max\{2S(hx, hx, hy), S(fx, fy, hx), S(gy, gy, hx),$
$\frac{1}{2}[S(fx, fx, hy) + S(gy, gy, hx)]\}$ *and*

(iii) (X, S) *is complete*

Then for any $x_0 \in X$ and for any of its associated sequence $\{x_n\}$ relative to the three self maps, the sequence $fx_0, gx_1, fx_2, gx_3, \cdots fx_{2n}, gx_{2n+1}, \cdots$ converge to some point $z \in X$.

Proof. Suppose f, g and h be self maps of a S-metric space (X, S) for which condition (i) and (ii) hold.
Let $x_0 \in X$ and $\{x_n\}$ be an associated sequence of x_0 relative to the three self maps.Then since $fx_{2n} = hx_{2n+1}$ and $gx_{2n+1} = hx_{2n+2}$ for $n \geq 0$.
Note that $\lambda(x_{2n}, x_{2n+1}) = \max\{2S(hx_{2n}, hx_{2n}, hx_{2n+1}),$
$S(fx_{2n}, fx_{2n}, hx_{2n}), S(gx_{2n+1}, gx_{2n+1}, hx_{2n+1}),$
$\frac{1}{2}[S(fx_{2n}, fx_{2n}, hx_{2n+1}) + S(gx_{2n+1}, gx_{2n+1}, hx_{2n})]\}$
$= \max\{2S(hx_{2n}, hx_{2n}, hx_{2n+1}), S(hx_{2n+1}, hx_{2n+1}, hx_{2n}),$
$S(hx_{2n+2}, hx_{2n+2}, hx_{2n+1}), \frac{1}{2}[S(hx_{2n+1}, hx_{2n+1}, hx_{2n+1}) +$
$S(hx_{2n+2}, hx_{2n+2}, hx_{2n})]\}$
$= \max\{2S(hx_{2n}, hx_{2n}, hx_{2n+1}), S(hx_{2n+2}, hx_{2n+2}, hx_{2n+1}),$
$\frac{1}{2}S(hx_{2n+2}, hx_{2n+2}, hx_{2n})\}$
andsince
$\frac{1}{2}S(hx_{2n+2}, hx_{2n+2}, hx_{2n})$
$\leq S(hx_{2n+2}, hx_{2n+2}, hx_{2n+1}) + \frac{1}{2}S(hx_{2n}, hx_{2n}, hx_{2n+1})$
and $\alpha + \beta \leq 2\max(\alpha, \beta)$ for any $\alpha \geq 0, \beta \geq 0$,
we get
$\frac{1}{2}S(hx_{2n+2}, hx_{2n+2}, hx_{2n})$
$\leq 2\max\{S(hx_{2n+2}, hx_{2n+2}, hx_{2n+1}), \frac{1}{2}S(hx_{2n}, hx_{2n}, hx_{2n+1})\}$
$= \max\{2S(hx_{2n+2}, hx_{2n+2}, hx_{2n+1}), S(hx_{2n}, hx_{2n}, hx_{2n+1})\}$
It follows that
$\lambda(x_{2n}, x_{2n+1}) \leq \max\{2S(hx_{2n}, hx_{2n}, hx_{2n+1}),$
$2S(hx_{2n+2}, hx_{2n+2}, hx_{2n+1})\}$
Now by (ii) and **??** we have

$$\begin{aligned}&S(hx_{2n+1}, hx_{2n+1}, hx_{2n+2})\\ &= S(fx_{2n}, fx_{2n}, gx_{2n+1})\\ &\leq c\lambda(x_{2n}, x_{2n+1})\\ &\leq 2c\max\{S(hx_{2n}, hx_{2n}, hx_{2n+1}),\\ &\qquad S(hx_{2n+2}, hx_{2n+2}, hx_{2n+1})\}\end{aligned}$$

therefore, in view of Remark 2.4 and the fact $0 < 2c < 1$, we get

$$S(hx_{2n+1}, hx_{2n+1}, hx_{2n+2}) \leq 2cS(hx_{2n}, hx_{2n}, hx_{2n+1}) \quad (3.20)$$

Similarly we can prove that

$$S(hx_{2n}, hx_{2n}, hx_{2n+1}) \leq 2cS(hx_{2n-1}, hx_{2n-1}, hx_{2n}) \quad (3.21)$$

From 3.20 and 3.21 we have for any $m \geq 1$ that

$$S(hx_m, hx_m, hx_{m+1}) \leq 2cS(hx_{m-1}, hx_{m-1}, hx_m) \quad (3.22)$$

which on repeated application yields

$$\begin{aligned} S(hx_m, hx_m, hx_{m+1}) &\leq 2cS(hx_{m-1}, hx_{m-1}, hx_m) \\ &\leq 4c^2 S(hx_{m-2}, hx_{m-2}, hx_{m-1}) \\ &\cdots \\ &\cdots \\ &\leq (2c)^m S(hx_1, hx_1, hx_0) \end{aligned}$$

which imply that the sequence $\{hx_n\}$ and hence $fx_0, gx_1, fx_2, gx_3, \cdots fx_{2n}, gx_{2n+1}, \cdots$ is a Cauchy sequence in the complete metric space (X,S) and therefore converges to a point say $z \in X$, proving the lemma. □

Remark 3.5. *The converse of the above lemma is not true. That is, suppose f, g and h are self maps of a S-metric space (X,S) satisfying conditions (i) and (ii) of Lemma 3.4. Even if for each $x_0 \in X$ and for each associated sequence $\{x_n\}$ of x_0 relative to f, g and h.*
The sequence $fx_0, gx_1, fx_2, gx_3, \cdots fx_{2n}, gx_{2n+1}, \cdots$ converges in X
Then (X,S) need not be complete as shown in the following example.

Example 3.6. *Let $X = [0,1)$ and $d(x,y) = |x-y|$ for $x, y \in X$. Then we know that (X,d) is a metric space,which is not complete. Now if $S_d(x,y,z) = d(x,y) + d(y,z) + d(z,x)$ for $x,y,z \in X$.Then (X,S_d) is a S-metric space and it is not complete by Corollary 2.11*
Now define self maps f, g and h of X by

$$f(x) = g(x) = \begin{cases} \frac{1}{3} & \text{if } x = 0 \\ \frac{11}{20} & \text{if } x \in (0,1) \end{cases}$$

and $h(x) = \frac{3x+1}{4}$ if $x \in [0,1)$
Then $f(X) = g(X) = \{\frac{1}{3}, \frac{11}{20}\}$ and $h(X) = \left[\frac{1}{4}, 1\right)$,
so that $f(X) \cup g(X) \subseteq h(X)$
Also suppose $x, y \in X$. We now prove

$$S_d(fx, fx, gy) \leq c.\lambda(x,y) \text{ for some } c \quad (3.23)$$

case (i): *$x = y = 0$ then*
$S_d(fx, fx, gy) = 0, S_d(hx, hx, hy) = 0, \quad S_d(fx, fx, hx) = \frac{1}{6}$,
$S_d(gy, gy, hy) = \frac{1}{6}, \frac{1}{2}[S_d(fx, fx, hy) + S_d(gy, gy, hx)] = \frac{1}{6}$
so that $\lambda(x,y) = \max\{0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\} = \frac{1}{6}$
Therefore 3.23 holds with c satisfying $0 \leq c < \frac{1}{2}$.
case (ii): *$x = 0$ and $y \neq 0$. Then*
$S_d(fx, fx, gy) = \frac{13}{30}, \quad S_d(hx, hx, hy) = 3y, S_d(fx, fx, hx) = \frac{1}{6}$,
$S_d(gy, gy, hy) = \frac{|15y-6|}{10}$
and $\frac{1}{2}[S_d(fx, fx, hy) + S_d(gy, gy, hx)] = \frac{3|y-1|}{4} + \frac{3}{10}$
and since $\frac{13}{30} \leq c\max\{3y, \frac{1}{6}, \frac{|15y-6|}{10}, \frac{3|y-1|}{4} + \frac{3}{10}\}$ holds for $\frac{13}{90} \leq c < \frac{1}{2}$ is true in this case.
In the other cases of (iii) $x \neq 0, y = 0$ and (iv)$x \neq 0, y \neq 0$ also 3.23 holds.
Thus conditions (i) and (ii) of Lemma 3.4 hold for all these self maps f, g and h.
Now, to prove that if $\{x_n\}$ is an associated sequence of any $x_0 \in X$, then the sequence $fx_0, gx_1, fx_2, gx_3, \cdots fx_{2n}, gx_{2n+1}, \cdots$ converges. We consider the cases if $x_0 = 0$ and $x_0 \neq 0$ separately.
Let $x_0 = 0$ so that $fx_0 = \frac{1}{3}$ and $x_1 \in X$ with $fx_0 = hx_1$ is given by $x_1 = \frac{1}{9}$. Now $x_2 \in X$ with $gx_1 = hx_2$ is $x_2 = \frac{2}{5}$. Now $x_3 \in X$ with $fx_2 = hx_3$ is $x_3 = \frac{2}{5}$.Again $x_4 \in X$ such that $gx_3 = hx_4$ is $x_4 = \frac{2}{5}$.
Thus the sequence $fx_0, gx_1, fx_2, gx_3, \cdots fx_{2n}, gx_{2n+1}, \cdots$ is given by $\frac{1}{3}, \frac{11}{20}, \frac{11}{20}, \frac{11}{20}$, in case $x_0 = 0$. Proceeding in this manner if $x_0 \neq 0$, we get the sequence
$fx_0, gx_1, fx_2, gx_3, \cdots fx_{2n}, gx_{2n+1}, \cdots$ as $\frac{11}{20}, \frac{11}{20}, \frac{11}{20}$.
In any case the sequence $fx_0, gx_1, fx_2, gx_3, \cdots fx_{2n}, gx_{2n+1}, \cdots$ converges to a point $\frac{11}{20} \in X$.
However X is not a complete metric space.

Proof of Theorem 3.3
In view of Lemma 3.4, the condition (v)' of Theorem 3.1 holds in view of (v)', hence the theorem follows from Theorem 3.1

Corollary 3.7. *([6] pp1584-1586) Let f, g and h be self maps of a metric space (X,d) such that*

(i) $f(X) \cup g(X) \subseteq h(X)$

(ii) $d(fx, gy) \leq c\lambda_0(x,y)$ for all $x, y \in X$ where
$\lambda_0(x,y) = \max\{d(hx,hy), d(fx,hx), d(gy,hx), \frac{1}{2}[d(fx,hy) + d(gy,hx)]\}$ and $0 \leq c < 1$

(iii) h is continuous and

(iv) $fh = hf$ and $gh = hg$ Further if

(v) X is complete.

Then f, g and h have a unique common fixed point $x \in X$.

Proof. Given that (X,d) is a metric space satisfying condition (i) to (v) of the corollary. If $S_d(x,y,z) = d(x,y) + d(y,z) + d(z,x)$ then (X,S_d) is a S-metric space.
Also (ii) can be written as $S_d(fx, fy, fy)) \leq c\lambda(x,y)$ for all $x, y \in X$ where

$\lambda(x,y) = \max\{S_d(hx,hy,hy), s_d(fx,fx,hx), S_d(gy,gy,gy),$
$\frac{1}{2}[S_d(fx,fx,hy) + S_d(gy,gy,hx)]\}$
which is same as the condition (ii) of Theorem 3.3. Also since (X,d) is complete, we have (X,S) is complete, by Theorem.

Now, f,g and h are self maps on (X,S) satisfying conditions of Theorem 3.3 and hence the Corollary follows.

□

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