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Note on generating function of higher dimensional bell numbers

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Abstract
In this paper, we study the generating function of the Higher dimensional Bell number, which are arises as dimensions of the class partition algebras an important subalgebra of the tensor product partition algebra $P_k(x) \otimes P_k(y)$, denoted by $P_k(x,y)$.

Keywords
Partition algebra, Bell number, Stirling number, wreath product.

AMS Subject Classification
16S20, 16S50, 05E18.

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Contents

1. Introduction

The partition algebras $P_k(x)$ have been studied independently by Martin and Jones as generalizations of the Temperley-Lieb algebras and the Potts model in statistical mechanics [5]. In 1993, Jones considered the algebra $P_k(n)$, as the centralizer algebra of the symmetric group S_n on $V^{\otimes k}$ (see, [3]).

In this paper, we study the generating function of the Higher dimensional Bell number, which are arises as dimensions of the class partition algebras an important subalgebra of the tensor product partition algebra $P_k(x) \otimes P_k(y)$, denoted by $P_k(x,y)$. The algebras $P_k(n,m)$ are the centralizer algebras of the wreath product $S_m \wr S_n$ on their permutation module $W^{\otimes k}$, where $W = \mathbb{C}^{nm}$, (see, [4]).

2. The action of the wreath product and its orbit

For each positive integer n, let $[n]$ denote the set $\{1,2,...,n\}$. Let $X(m,m') = [m] \times [m']$. We will often abbreviate $X(m,m')$ to X in this article. The set X can be viewed as a disjoint union of m copies of $[m']$:

$$X = X_1 \bigsqcup X_2 \bigsqcup \cdots \bigsqcup X_m,$$

where $X_i = \{(i,j) \mid j \in [m']\}$ for each $i \in [m]$. Consider the action of the direct product of the symmetric group $(S_{m'})^m$ on X given by:

$$(\sigma_1, \sigma_2, ..., \sigma_m)(i,j) = (i, \sigma_i(j)), \text{ where } \sigma_r \in S_{m'},\ r = 1,2,\dots m$$

(i.e.) the ith copy of $(S_{m'})^m$ acts on X_i, and the action of S_m on X by

$$\pi(i,j) = (\pi(i), j), \text{ where } \pi \in S_m.$$

These actions extend to an action of the wreath product of the two symmetric group $G := S_{m'} \wr S_m$ (i.e. the semidirect product of $(S_{m'})^m$ with S_m) on X by

$$\pi(\sigma_1, \sigma_2, ..., \sigma_m)(i,j) = (\pi(i), \sigma_i(j)),$$

for all $\pi(\sigma_1, \sigma_2, ..., \sigma_m) \in S_{m'} \wr S_m$. Our goal is to analyze the number of G-orbits of the diagonal action of G on X^n as a sequence in n.

Definition 2.1. *[Type of a tuple]. Given $x = (x_1, x_2, ..., x_n) \in X^n$, where $x_k = (i_k, j_k)$, let*

$$\mu_i = \#\{j \in [m'] \mid (i_k, j_k) = (i,j) \text{ for some } k \in [n]\},$$

In other words, the number of distict second coordinates of those $x_1, x_2, ..., x_n$ whose first coordinate is i. Let λ be the integer partition obtained by sorting $(\mu_1, \mu_2, ..., \mu_m)$ into weakly decreasing order and discarding the trailing zeroes. Then the partition λ is called the *type* of x.

Example 2.2. *Let $m=3, m'=4$ and $n=5$. Consider the sequence $((2,1),(3,2),(2,3),(3,2),(2,1)) \in X^5$. Here the $\mu_1 = 0, \mu_2 = 2$ and $\mu_3 = 1$. Hence the type of this sequence is $(2,1)$.*

Observe that if λ is the type of an element of X^n, then $|\lambda| \le n$, the parts of λ are bounded above by m' and the number of parts of λ is bounded above by m. In other words, the Young diagram of λ fits inside a $m' \times m$ square. We describe this situation by writing $\lambda \subset m' \times m$.

3. Two dimensional Bell numbers

Definition 3.1. *[Partition Stirling number of the second kind] Given a partition λ such that $\lambda \subset m' \times m$, let $S(n,\lambda)$ denote the number of G-orbits in X^n of type λ. Then the number of G-orbits in X^n of type λ, denoted $S(n,\lambda)$ does not depend on m or m', and is called the partition Stirling number of the second kind.*

A part of the above definition is the assertion that $S(n,\lambda)$ does not depend on m or m'. This will follow from ...

Example 3.2. *Take $\lambda = (1^r)$ for some positive integer r. Then $S(n,(1^r))$ is the number of orbits in X^n which have at most one element for each X_i. By using the action of $(S_{m'})^m$, one can assume that each entry $x_k = (i_k, j_k)$ has $j_k = 1$. Thus, the orbit of $((i_1,j_1),(i_2,j_2),\dots,(i_n,j_n))$ of type (1^r) in X^n are completely determined by the S_m-orbit of the sequence $(i_1,i_2,\dots,i_n)$. The number of such orbits is $S(n,r)$, a Stirling number of the second kind; see section 4 of [1]:*

$$S(n,(1^r)) = S(n,r).$$

For a partition λ, let λ^- denote the set of partitions whose Young diagram is obtained by removing one box from the Young diagram of λ. If $\mu \in \lambda^-$, then μ is obtained by subtracting 1 from one of the parts λ_i of λ. Let $b_{\mu\lambda}$ denote the number of times the integer $\lambda_i - 1$ occurs in μ. For example, if $\mu = (4,2,2,2,1)$ and $\lambda = (4,3,2,2,1)$, then μ was obtained from λ by changing 3 to 2. Since 2 occurs three times in $\mu, b_{\mu\lambda} = 3$.

Theorem 3.3. *Let $\lambda \subset [m'] \times [m]$ be a partition. Then for each positive integer n,*

$$S(n,\lambda) = |\lambda| S(n-1,\lambda) + \sum_{\mu \in \lambda^-} b_{\mu\lambda} S(n-1,\mu).$$

Proof. The function $(x_1,\dots,x_n) \mapsto (x_1,\dots,x_{n-1})$ induces a surjective function from $G\backslash X^n \to G\backslash X^{n-1}$. □

Corollary 3.4. *The number of orbits for the diagonal action of G on $X(m,m')^n$ is*

$$B_n^{(2)}(m',m) = \sum_{\lambda \subset [m'] \times [m]} S(n,\lambda).$$

In particular, if $m' \ge n$ and $m \ge n$, then the value of $B_n^{(2)}(m',m)$ does not depend on m' and m. This stable value is given by

$$B_n^{(2)} = \sum_{\lambda} S(n,\lambda).$$

The sum on the right hand side of the above expression is over the set of all partitions of all integers. It ends up being finite because $S(n,\lambda) = 0$ if $|\lambda| > n$.

The numbers $B_n^{(2)}$ are known as two-dimensional Bell numbers (sequence A000258 in the OEIS [6]). By definition, $B_n^{(2)}$ is the number of pairs of set partitions (d,d') of $[n]$ such that d' is finer than d. In order to see this, fix m and m', both at least as large as n. Given $((i_1,j_1),(i_1,j_1),\dots,(i_n,j_n)) \in X(m,m')^n$, let $S_i = \{k \in [n] \mid i_k = i\}$, and further, for each i, let $S_{ij} = \{k \in S_i \mid j_k = j\}$. Then the collection of non-empty S_i's is a set partition d of $[n]$, and the collection of non-empty S_{ij}'s is a set partition d' of $[n]$ which is finer than d. This construction gives rise to a bijection from the set of G-orbits in $X(m,m')^n$ onto the set of pairs of set partitions (d,d') such that d' is finer than d.

We now proceed to derive rational generating functions for the partition Stirling numbers of the second kind and the two-dimensional Bell numbers. For each partition λ, define

$$S_\lambda(t) = \sum_{n=0}^{\infty} S(n,\lambda) t^n.$$

Then Theorem 2.1 gives

$$(1-|\lambda|t)\mathbf{S}_\lambda(t) = t \sum_{\mu \in \lambda^-} b_{\mu\lambda} \mathbf{S}_\mu(t). \tag{3.1}$$

Let λ be a partition of r. Each sequence $\lambda^{(0)},\lambda^{(1)},\dots,\lambda^{(r)}$ of partitions, where $\lambda^{(i-1)} \in \lambda^{(i)-}$ for each $1 \le i \le r$, and $\lambda^{(r)} = \lambda$ can be represented by a unique standard Young tableau T of shape λ. In this tableau, the unique box in $\lambda^{(i)}$ which is not in $\lambda^{(i-1)}$ is filled with the integer i. If T is the tableau corresponding to the sequence $(\lambda^{(0)},\dots,\lambda^{(r)})$, then define

$$b_T = \prod_{i=1}^{r} b_{\lambda^{(i-1)}\lambda^{(i)}}.$$

Iterating the identity (3.1) gives:

$$\mathbf{S}_\lambda(t) = \frac{1}{(1-t)(1-2t)\cdots(1-rt)} \sum_{T \in T(\lambda)} b_T.$$

For each partition λ, define

$$B_\lambda = \sum_{T \in T(\lambda)} b_T. \tag{3.2}$$

Recall that the generating function for Stirling numbers of the second kind is given by

$$\mathbf{S}_r(t) := \sum_{n=0}^{\infty} S(n,r) t^n = \frac{1}{(1-t)(1-2t)\cdots(1-rt)}.$$

As a result we have:

Theorem 3.5. *For every partition λ of r,*

$$\boldsymbol{S}_\lambda(t) = B_\lambda \boldsymbol{S}_r(t).$$

The partition statistic B_λ is nothing but the number of set partitions whose sorted block sizes correspond to the partition λ. This is clear from the relation:

$$B_\lambda = \sum_{\mu \in \lambda^-} b_{\mu\lambda} B_\mu.$$

In the Find Stat database [2], this statistic has identifier St000049. Obviously, the rth Bell number is given by

$$B_r = \sum_{\lambda \vdash r} B_\lambda.$$

Combining this with (3.2), we get an expression for Bell numbers:

$$B_r = \sum_{T \in \mathrm{T}(r)} b_T.$$

Here $\mathrm{T}(r)$ denotes the set of standard Young tableaux of size r. In view of the Robinson-Schensted correspondence, $|\mathrm{T}(r)|$ is the number of permutations whose square is the identity. Since $b_T \geq 1$ for each tableau T, we have

$$B_r \geq \#\{w \in S_r \mid w^2 = 1\}.$$

We are now ready to write down an expression for the rational generating function for the two-dimensional Bell numbers:

Theorem 3.6. *The generating function for two dimensional Bell numbers is given by*

$$\boldsymbol{B}^{(2)}(t) = \sum_{n=0}^{\infty} B_n^{(2)} t^n = \sum_{r=0}^{\infty} B_r \boldsymbol{S}_r(t).$$

4. Higher dimensional Bell numbers

Let $\mathbf{m} = \{m_1, m_2, \ldots, m_k\}$ be a sequence of positive integers. Consider the sequence $\{G_k\}$ of groups defined by

$$G_1 = S_{m_1}; \quad G_k = G_{k-1} \wr S_{m_k}.$$

Let $\{X_k(\mathbf{m})\}$ be a sequence of spaces defined by:

$$X_k = [m_1] \times [m_2] \times \cdots \times [m_k].$$

G_1 acts on X_1 by the standard permutation action. View X_k as the disjoint union of m_k copies of X_{k-1}. Inductively define the action of G_k on X_k as follows: the ith copy of G_{k-1} in G_k acts on $\{i\} \times X_{k-1} \subset X_k$ (the ith copy of X_{k-1} in X_k), and S_{m_k} acts by permuting these m_k copies.

Theorem 4.1. *The set of orbits for the action of G_k on X_k^n are in bijective correspondence with the set of sequences $(b_1, b_2, \ldots, b_k)$ of set partitions of n, where b_r is a refinement of b_{r-1} for each $1 < r \leq k$.*

Proof. Given $(x_1, x_2, \ldots, x_n) \in X^n$, say $x_r = (i_{r_1}, i_{r_2}, \ldots, i_{r_k})$, let b_s be the set partition whose subsets are the non-empty sets among

$$S_{i_1, i_2, \ldots, i_s} = \{r \in [n] \mid i_{rj} = i_j \text{ for } 1 \leq j \leq s\}.$$

By induction on k, one should be able to prove bijection. □

A two-dimensional type is just a partition. Each element of X_2 has a two-dimensional type as in Definition 2.1. For $k > 2$, a k-dimensional type is an unordered multiset of $k-1$-dimensional types. The size $|\tau|$ of $\tau = \{\tau_1, \tau_2, \ldots, \tau_{m_k}\}$ is defined as $\sum_{j=1}^{m_k} |\tau_j|$ of the constituent types. To each sequence we may associate

Definition 4.2. *[Type of a sequence].*
The type of an n-tuple $(x_1, x_2, \ldots, x_n) \in X_k^n$ is defined as the unordered multiset $\tau = \{\tau_1, \tau_2, \ldots, \tau_{m_k}\}$, where τ_j is the type of the subsequence of $x_1, x_2, \ldots, x_n$ consisting of elements of the from $(j, i_2, \ldots, i_k)$, viewed as a sequence in X_{k-1}.

For each k-dimensional type τ, let τ^- denote the set of k-dimensional types ρ all of whose elements $\rho_1, \rho_2, \ldots$ are the same as those of τ, except for exactly one, say ρ_i which lies in τ_i^-. Given $\rho \in \tau^-$, define $b_{\rho\tau}$ to be the number of times ρ_i occurs in ρ, where ρ_i is the element of ρ_i is the element of ρ in which ρ and τ differ.

Let $S(n, \tau)$ denote the number of G_k-orbits in S_k of type τ. Then

$$S(n, \tau) = |\tau| S(n-1, \tau) + \sum_{\rho \in \tau^-} b_{\rho\tau} S(n-1, \rho).$$

Let τ be a k-dimensional type of size r. Let $\mathrm{T}^{(k)}(r)$ be the set of all chains $\tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(r)} = \tau$, where $|\tau^{(j)}| = j$ and $\tau^{(j)} \in \tau^{(j+1)-}$ for $1 \leq j < r$. Given $T \in \mathrm{T}^{(k)}(r)$, define $b_T = \prod_{i=1}^{r-1} b_{\tau^{(i)}\tau^{(i+1)}}$. Then we have:

$$\mathbf{S}_\tau(t) = B_\tau \mathbf{S}_r(t),$$

which is the Stirling numbers of the second kind for Higher dimensional Bell numbers.

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