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Nonlinear functional integral equation: Existence, global attractivity and positivity of solutions

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Abstract

In this paper, we discuss the powerful tool measure of noncompactness and fixed point theorem of Dhage to study existence and other characteristic such as global attractivity and positivity of solutions of nonlinear functional integral equation.

Keywords

Global asymptotic attractivity, measure of noncompactness, nonlinear functional integral equation, fixed point theorem of Dhage.

AMS Subject Classification

Primary 47H10; Secondary 45G10, 45M10

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1. Introduction

The nonlinear integral equations have applications in potential theory, electromagnetic, antenna synthesis problem, etc (see [6, 15, 18]). The measure of noncompactness is used for the characteristics of the attractivity and asymptotic attractivity of the solutions. There are two approaches for dealing with these characteristic of solutions, namely, classical fixed point theorems involving the hypotheses from analysis and topology, fixed point theorem involving the use of measure of noncompactness The measure of noncompactness is used not only to contain the existence of solution of functional integral equation but also to characterize those solutions in terms of attractivity and positivity on interval. Some of the useful measure of noncompactness in the application to nonlinear integral equations have been discussed in paper of Apell [2]. Recently, Sakure [16] studied the existence of nonlinear Volterra-Hammerstein-Fredholm integral equation using measure of noncompactness. In this paper,we are going to find global attractivity result for nonlinear functional integral equation by using fixed point theorem of Dhage using measure of noncompactness under certain conditions. Our investigations will be situated in the Banach space of real-valued functions defined, continuous and bounded on the right hand real half axis R_+.

2. Preliminaries.

Let E be a Banach space, $P(E)$ a class of subset of E with and $P_p(E)$ denote the class of all nonempty subset of E with the property p. Here p may be p=closed (in the cl), p=bounded (in short bd) , p=relatively compact (in short rcp) etc. $P_{cl}(E)$,$P_{bd}(E)$,$P_{cl,bd}(E)$,$P_{rcp}(E)$ denote the class of closed, bounded, closed and bounded and relatively compact subsets of E respectively.

Definition 2.1. A function $\mu : P_{bd}(E) \to \mathbb{R}_+$ is called a measure of noncompactness, if it satisfies:

1. $\phi \neq \mu^{-1}(0) \subset P_{rcp}(E)$,
2. $\mu(A) = \mu(\overline{A})$, where $\overline{A}$ is closure of A,
3. $\mu(A) = \mu(Conv(A))$, where $Conv(A)$ is convex hull of A,

4. μ is nondecreasing, and
5. if $\{A_n$ is a decreasing sequence of sets in $P_{bd}(E)$ such that $\lim_{n\to\infty}\mu(A_n)=0$, then the limiting set $A_\infty=\lim_{n\to\infty}=\cap_{n=0}^{\infty}\overline{A_n}$ is nonempty.

The different types of measure of noncompactness appear in Akhermov et.al. [1], Appell [2], Banas and Goebel [3] and references given therein.

The family $ker\mu$ is said to be the kernel of measure of noncompactness μ and

$$ker\mu=\{A\in P_{bd}(E)\mid \mu(A)=0\}\subset P_{rcp}(E).$$

The following definition appear in Dhage[9].

Definition 2.2. A mapping $K:E\to E$ is called $\mathscr{D}-set-contraction$ if there exists a continuous nondecreasing function $\phi:\mathbb{R}_+\to\mathbb{R}_+$ such that $\mu(K(A))\leq\phi(\mu(A))$ for all $A\in P_{bd}(E)$ with $K(A)\in P_{bd}(E)$, where $\phi(0)=0$. So0metimes we call the function ϕ to be a $\mathscr{D}-function$ of K on E. In the special case, when $\phi(r)=kr, k>0$, K is called a $k-set-Lipschitz$ mapping and if $k<1$, then K is called a $k-set-contraction$ on E. Further, if $\phi(r)<r$ for $r>0$, then K is called a nonlinear $\mathscr{D}-set-contraction$ on E.

Theorem 2.1. *([7]) Let C be a non-empty, closed, convex and bounded subset of a Banach space E and let $K:C\to C$ be a continuous and nonlinear $\mathscr{D}-set-contraction$. Then K has a fixed point.*

Remark 2.1. Denote $Fix(K)$ by the set all fixed points of the operator K which belongs to C. It is easy to show that the $Fix(K)$ existing in 2.1 belongs to family $ker\mu$. In fact if $Fix(K)\notin ker\mu$, then $\mu(Fix(K))>0$ and $K(Fix(K))=Fix(K)$. Now from nonlinear $\mathscr{D}-set-contraction$ it follows that $\mu(K(Fix(K)))\leq\phi(\mu(Fix(K)))$ which is a contradiction since $\phi(r)<r$ for $r>0$. Hence $Fix(K)\in ker(\mu)$.

Consider the Banach space $BC(\mathbb{R}_+,\mathbb{R})$ consisting of all real functions $x=x(t)$ defined, continuous and bounded on $\mathbb{R}_+$. This space is equipped with the standard supremum norm

$$\|x\|=\sup\{\mid x(t)\mid : t\in\mathbb{R}_+\}$$

Let us fix a nonempty and bounded subset X of the space $BC(\mathbb{R}_+,\mathbb{R})$ and a positive number T. For $x\in X$ and $\varepsilon\geq 0$ denote by $\omega^T(x,\varepsilon)$, the modulus of continuity is

$$\omega^T(x,\varepsilon)=\sup\{\mid x(t)-x(s)\mid : t,s\in[0,T],\mid t-s\mid\leq\varepsilon\}$$

Assume that

$$\omega^T(X,\varepsilon)=\sup\{\ \omega^T(x,\varepsilon):x\in X\},$$

$$\omega_0^T(X)=\lim_{\varepsilon\to 0}\omega^T(X,\varepsilon).$$

It is clear that ω_0^T is a measure of noncompactness in the Banach space $C([0,T],\mathbb{R})$ of continuous and real-valued functions defined on a closed and bounded interval $[0,T]$. Now we define

$$\omega_0(X)=\lim_{T\to\infty}\omega_0^T(X).$$

Now, for a fixed number $t\in\mathbb{R}_+$ let us denote

$$X(t)=\{x(t):x\in X\},$$

$$\parallel X(t)\parallel=\sup\{x(t):x\in X\},$$

and

$$\parallel X(t)-c\parallel=\sup\{x(t)-c:x\in X\}.$$

Consider the functions $\mu's$ defined on the family $P_{cl,bd}(X)$ by the formulas

$$\mu_a(X)=\max\{\omega_0(X),\limsup_{t\to\infty}\operatorname{diam}X(t)\},\tag{2.1}$$

$$\mu_b(X)=\max\{\omega_0(X),\limsup_{t\to\infty}\parallel X(t)\parallel\},\tag{2.2}$$

and

$$\mu_c(X)=\max\{\omega_0(X),\limsup_{t\to\infty}\parallel X(t)-c\parallel\}.\tag{2.3}$$

Let $T>0$ be fixed. Then for any $x\in BC(\mathbb{R}_+,\mathbb{R})$ define

$$\delta_T(x)=\sup\{\mid\mid x(t)\mid -x(t)\mid : x\in X\},$$

$$\delta_T(X)=\sup\{\delta_T(x):x\in\}$$

and

$$\delta(X)=\lim_{T\to\infty}\delta_T(X)$$

Define the functions $\mu_{ad}:P_{bd}(E)\to\mathbb{R}_+$ by

$$\mu_{ad}(X)=\max\{\mu_a(X),\delta(X)\},\tag{2.4}$$

for all $X\in P_{cl,bd}(E)$.
Assume that $E=BC(\mathbb{R}_+,\mathbb{R})$ and Ω be a subset of E. Let $K:E\to E$ be an operator and consider the following operator equation in E,

$$Kx(t)=x(t)\tag{2.5}$$

for all $t\in\mathbb{R}_+$.

Definition 2.3. ([8]) The solutions of the equation (2.5) are locally attractive if there exists a closed ball $\overline{\mathscr{B}}_r(x_0)$ in the space $BC(\mathbb{R}_+,\mathbb{R})$ for some $x_0 \in BC(\mathbb{R}_+,\mathbb{R})$ such that arbitrary solutions $x = x(t)$ and $y = y(t)$ of the equation (2.5) belonging to $\overline{\mathscr{B}}_r(x_0) \cap \Omega$ we have that

$$\lim_{t\to\infty}(x(t) - y(t)) = 0. \tag{2.6}$$

In this case when the limit (2.6) is uniform with respect to the set $\overline{\mathscr{B}}_r(x_0) \cap \Omega$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$| x(t) - y(t) | \leq \varepsilon \tag{2.7}$$

for all $x, y \in \overline{\mathscr{B}}_r(x_0) \cap \Omega$ being solution of (2.5) and for $t \geq T$, we will say that solutions of equation (2.5) are uniformly locally attractive on $\mathbb{R}_+$.

Definition 2.4. ([8]) The solution $x = x(t)$ of equation(2.5) is said to be globally attractive if (2.6) holds for each solution $y = y(t)$ of (2.5) on ω. In other words, we may say that the solutions of the equation (2.6) are globally attractive if for arbitrary solutions $x(t)$ and $y(t)$ of (2.5) on Ω, the condition (2.6) is satisfied. In the case when the condition (2.6) is satisfied uniformly with respect to the set Ω, i.e., if for every $\varepsilon > 0$ there exists $T > 0$ such that the inequality (2.7) is satisfied for all $x, y \in \Omega$ being the solutions of (2.5) and for $t \geq T$, we will say that solutions of the equation (2.5) are uniformly globally attractive on $\mathbb{R}_+$.

Definition 2.5. ([8]) A line $y(t) = c$, where c a real number, is called an attractor for a solution $x \in BC(\mathbb{R}_+,\mathbb{R})$ to the equation (2.5) if $\lim_{t\to\infty}[x(t) - c] = 0$ and the solution x to the equation (2.5) is also called asymptotic to the line $y(t) = c$ and the line is an asymptote for the solution x on $\mathbb{R}_+$.

Definition 2.6. ([9]) The solutions of equations (2.5) are said to be globally asymptotic attractive if for any two solutions $x = x(t)$ and $y = y(t)$ of the equation (2.5), the condition (2.6) is satisfied and there is a line which is a common attractor to them on $\mathbb{R}_+$. When the condition (2.6) is satisfied uniformly , i.e., if for every $\varepsilon > 0$ there exists $T > 0$ such that the inequality (2.7) is satisfied for $t \geq T$ and for all x, y being the solution of (2.5) and having a line as common attractor, we will say that solutions of the equation (2.5) are uniformly globally asymptotically attractive on $\mathbb{R}_+$.

Remark 2.2. The concept of global attractivity of solutions are introduced in Hu and Yan [14] and concept of local and global asymptotic attractivity have been presented in Dhage [8] whle concept of uniform local and global attractivity were introduced in Banas and Rzepka [4] and concept of global asymptotic attractivity of solutions are presented in Dhage [9].

Definition 2.7. A solution x of the equation (2.5) is called locally ultimately positive if there exists a closed ball $\overline{\mathscr{B}}_r(x_0)$ in $BC(\mathbb{R}_+,\mathbb{R}$ for some $x_0 \in BC(\mathbb{R}_+,\mathbb{R}$ such that $x \in \overline{\mathscr{B}}_r(x_0)$ and

$$\lim_{t\to\infty}[| x(t) | - x(t)] = 0. \tag{2.8}$$

When the limit (2.8) is uniform with respect to the solution set of the operator equation (2.5),i.e., when for each $\varepsilon > 0$ there exist $T > 0$ such that

$$|| x(t) || - | x(t) | \leq \varepsilon \tag{2.9}$$

for all x being solutions of (2.5) and for $t \geq T$, we will say that solutions of equation (2.5) are uniformly locally ultimately positive on $\mathbb{R}_+$.

Definition 2.8. A solution $x \in C(\mathbb{R}_+,\mathbb{R})$ of the equation (2.5) is called globally ultimate positive if (2.8) is satisfied. When the limit (2.8) is uniform with respect to the solution set of the operator equation (2.5) in $C(\mathbb{R}_+,\mathbb{R})$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that (2.9) is satisfied for all x being solutions of (2.5) and for $t \geq T$, we will say that solutions of equation (2.5) are uniformly globally ultimate positive on $\mathbb{R}_+$.

Remark 2.3. The global attractivity and global asymptotic attractivity implies the local attractivity and local asymptotic attractivity, respectively, of the solutions for the operator equation (2.5) on $\mathbb{R}_+$. Similarly, global ultimate positivity implies local ultimate positivity of the solutions for the operator equation (2.5) on unbounded intervals. The converse of the above two statements may not be true.

3. Attractivity and Positivity of Solutions

In this section, we will investigate the following functional integral equation (in short FIE)

$$\begin{aligned} x(t) &= h(t) + f(t, x(\alpha_1(t)), x(\alpha_2(t))) \\ &\quad + \int_0^{\beta(t)} k(t,s) g(t, s, x(\gamma_1(s)), x(\gamma_2(s))) ds \end{aligned} \tag{3.1}$$

for all $t \in \mathbb{R}_+$, $h : \mathbb{R}_+ \to \mathbb{R}$, $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\alpha_1, \alpha_2, \beta, \gamma_1, \gamma_2 : \mathbb{R}_+ \to \mathbb{R}_+$.

By a solution of FIE (3.1), we mean a function $x \in C(\mathbb{R}_+,\mathbb{R})$ that satisfies FIE (3.1) where $C(\mathbb{R}_+,\mathbb{R})$ is the space of continuous real valued functions on $\mathbb{R}_+$.

When $k(t,s) \equiv 1$ for all $t \in \mathbb{R}_+$, the FIE (3.1) reduces to functional integral equation

$$\begin{aligned} x(t) &= h(t) + f(t, x(\alpha_1(t)), x(\alpha_2(t))) \\ &\quad + \int_0^{\beta(t)} g(t, s, x(\gamma_1(s), x(\gamma_2(s))) ds \end{aligned} \tag{3.2}$$

for $t \in \mathbb{R}_+$. The integral equation (3.2) has been studied in Dhage [9] for global asymptotic attractivity and positivity of the solutions via measure of noncompactness.

When $k(t,s) \equiv 1$ and $\alpha_1(t) = \gamma + 1(t)$ for $t \in \mathbb{R}_+$, then FIE (3.1) reduces to FIE

$$x(t) = h(t) + f(t, x(t), x(\alpha_2(t))) + \int_0^{\beta(t)} g(t,s,x(s),x(\gamma_2(s)))ds. \tag{3.3}$$

The integral equation (3.3) has been studied in Dhage [8] for global attractivity and global asymptotic attractivity of the solutions via classical hybrid fixed point theorem.
Consider the following functional integral equation

$$x(t) = f(t,,x(\alpha(t)) + \int_0^{\beta(t)} g(t,s,x(\gamma(s)))ds \tag{3.4}$$

for $t \in \mathbb{R}_+$ is special case of functional integral equation (3.1). The integral equation (3.4) has been studied in Banas and Dhage [5].
Consider the following functional integral equation

$$x(t) = f(t,,x(t) + \int_0^{t} g(t,s,x(t))ds \tag{3.5}$$

for $t \in \mathbb{R}_+$ is special case of functional integral equation (3.1). The integral equation (3.5) has been studied in Banas and Rzepka [4, 5].

Therefore, our FIE (3.1) is more general and so, the attractivity and positivity results of this paper include the attractivity and positivity results for all the above mentioned functional integral equations.

Global attractivity of solutions

The FIE (3.1) will be considered under the following assumptions:
($\mathbf{K_0}$) The functions $\alpha_1, \alpha_2, \beta, \gamma_1, \gamma_2 : \mathbb{R}_+ \to \mathbb{R}$ are continuous.
($\mathbf{K_1}$) The function $h : \mathbb{R}_+ \to \mathbb{R}$ is continuous and bounded.
($\mathbf{K_2}$) The function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there is bound $l : \mathbb{R}_+ \to \mathbb{R}$ with bound L and a positive constant M such that

$$| f(t,x_1,x_2) - f(t,y_1,y_2) | \leq \frac{l(t) \max\{| x_1 - x_2 |, | y_1 - y_2 |\}}{M + \max\{| x_1 - x_2 |, | y_1 - y_2 |\}}$$

for $t \in \mathbb{R}_+$ and for all $x_1, x_2, y_1, y_2 \in \mathbb{R}_+$. Moreover assume that $L \leq M$.
($\mathbf{K_3}$) The function $t \mapsto f(t,0,0)$ is bounded on $\mathbb{R}_+$ with

$$F_0 = \sup\{| f(t,0,0) | : t \in \mathbb{R}_+\}.$$

($\mathbf{K_4}$) The function $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is continuous and there is a positive real number N such that

$$| k(t,s) | \leq N$$

($\mathbf{K_5}$) The function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous such that there is a continuous map $b : \mathbb{R}_+ \times \mathbb{R}_+$ such that

$$| g(t,s,x,y) | \leq b(t,s)$$

for $t, s \in \mathbb{R}_+$. Moreover, we assume that

$$\lim_{t \to 0} \int_0^{\beta(t)} b(t,s)ds = 0.$$

Theorem 3.1. *Under the assumptions ($\mathbf{K_0}$)- ($\mathbf{K_5}$), the FIE (3.1) has atleast one solution in the space $BC(\mathbb{R}_+, \mathbb{R})$. Moreover solutions of FIE (3.1) are globally uniformly attractive on $\mathbb{R}_+$.*

Proof. Consider the operator K defined on the space $BC(\mathbb{R}_+, \mathbb{R})$ such that

$$\begin{aligned} Kx(t) &= h(t) + f(t, x(\alpha_1(t)), x(\alpha_2(t))) \\ &\quad + \int_0^{\beta(t)} k(t,s) g(t,s,x(\gamma_1(s)), x(\gamma_2(s)))ds \end{aligned} \tag{3.6}$$

By our assumptions, the function $Kx(t)$ is continuous for any function of $x \in BC(\mathbb{R}_+, \mathbb{R})$.
For arbitrarily fixed $t \in \mathbb{R}_+$,

$$\begin{aligned} |Kx(t)| &= \Big| h(t) + f(t, x(\alpha_1(t)), x(\alpha_2(t))) \\ &\quad + \int_0^{\beta(t)} k(t,s) g(t,s,x(\gamma_1(s)), x(\gamma_2(s)))ds \Big| \\ &\leq ||h|| + |f(t, x(\alpha_1(t)), x(\alpha_2(t))) - f(t,0,0)| \\ &\quad + |f(t,0,0)| + \int_0^{\beta(t)} |k(t,s)||g(t,s,x(\gamma_1 s), x(\gamma_2(s)))|ds \\ &\leq ||h|| + \frac{L \max\{|x(\alpha_1(t))|, |x(\alpha_2(t))|\}}{M + \max\{|x(\alpha_1(t))|, |x(\alpha_2(t))|\}} \\ &\quad + |f(t,0,0)| + N \int_0^{\beta(t)} b(t,s)ds \\ &\leq ||h|| + \frac{L||x||}{M + ||x||} + F_0 + Nv(t) \\ &\leq ||h|| + \frac{L||x||}{M + ||x||} + F_0 + NV \\ &\leq ||h|| + L + F_0 + NV \\ ||K(x)|| &\leq ||h|| + L + F_0 + NV \end{aligned} \tag{3.7}$$

for all $x \in BC(\mathbb{R}_+, \mathbb{R})$. This means that the operator K transforms the space $BC(\mathbb{R}_+, \mathbb{R})$ into itself. From (3.7), we obtain the operator K transforms continuously the space $BC(\mathbb{R}_+, \mathbb{R})$ into the closed ball $\overline{B}_r(0)$, where $r = ||h|| + L + F_0 + NV$. Therefore the existence of the solution for FIE (3.1) is global in nature. We will consider the operator $K : \overline{B}_r(0) \to \overline{B}_r(0)$.
Now we will show that the operator K is continuous on ball $\overline{B}_r(0)$. Let $\varepsilon > 0$ be arbitrary and take $x, y \in \overline{B}_r(0)$ such that $||x - y|| \leq \varepsilon$, then

$$\begin{aligned} |(Kx)(t) &- (Ky)(t)| \leq |f(t, x(\alpha_1(t)), x(\alpha_2(t))) \\ &\quad - f(t, y(\alpha_1(t)), y(\alpha_2(t)))| \\ &\quad + \int_0^{\beta(t)} |k(t,s)||g(t,s,x(\gamma_1(s)), x(\gamma_2(s))) \\ &\quad - g(t,s,y(\gamma_1(s)), y(\gamma_2(s)))|ds \\ &\leq \frac{L \max\{|x(\alpha_1(t)) - y(\alpha_1(t))|, |x(\alpha_2(t)) - y(\alpha_2(t))|\}}{M + \max\{|x(\alpha_1(t)) - y(\alpha_1(t))|, |x(\alpha_2(t)) - y(\alpha_2(t))|\}} \\ &\quad + \int_0^{\beta(t)} N2b(t,s)ds \\ &\leq \frac{L||x-y||}{M + ||x-y||} + 2Nv(t) \\ &\leq \varepsilon + 2Nv(t) \end{aligned}$$

from assumption ($\mathbf{K_5}$), there exists $T > 0$ such that $v(t) \leq \varepsilon$ for $t \geq T$. Thus for $t \geq T$, we have

$$|(Kx)(t) - (Ky)(t)| \leq 3\varepsilon. \tag{3.8}$$

Let us assume that $t \in [0,T]$. Then

$$\begin{aligned} \big|(Kx)(t) \quad - \quad & (Ky)(t)\big| \leq \varepsilon + \int_0^{\beta(t)} \big|k(t,s)\big| \\ & \big|g(t,s,x(\gamma_1(s)),x(\gamma_2(s))) \\ & -g(t,s,y(\gamma_1(s)),y(\gamma_2(s)))\big|ds \\ \leq \quad & \varepsilon + N\int_0^{\beta(t)} \omega_r^T(g,\varepsilon)ds \\ \leq \quad & \varepsilon + N\beta_T\omega_r^T \end{aligned} \tag{3.9}$$

where

$$\beta_T = \sup\{\beta(t) : t \in [0,T]\},$$

and

$$\begin{aligned} \omega_r^T \quad = \quad & \sup\{|g(t,s,x_1,x_2) - g(t,s,y_1,y_2)| : t \in [0,T], \\ & s \in [0,\beta_T]], x_1,x_2,y_1,y_2 \in [-r,r], \\ & |x_1 - y_1| \leq \varepsilon, |x_2 - y_2| \leq \varepsilon\}. \end{aligned} \tag{3.10}$$

Obviously we have $\beta_T < \infty$. By uniform continuity of the functions $g(t,s,x,y)$ on the set $[0,T]\times[0,\beta_T]\times[-r,r]\times[-r,r]$, we have $\omega_r^T(g,\varepsilon) \to 0$ as $\varepsilon \to 0$. Now, by (3.9), (3.10) and above established facts we conclude that the operator K maps continuously the closed ball $\overline{B}_r(0)$ into itself.

Further, let us take a nonempty subset X of the ball $\overline{B}_r(0)$. Next, fix arbitrarily $T > 0$ and $\varepsilon > 0$. Let us choose $x \in X$ and $t_1, t_2 \in [0,T]$ with $|t_2 - t_1| \leq \varepsilon$. Without loss of generality we may assume that $t_1 < t_2$. Then

$$\begin{aligned} |(Kx)(t_2) \quad - \quad & (Kx)(t_1)| \leq |h(t_2) - h(t_1)| \\ & +|f(t_2,x(\alpha_1(t_2)),x(\alpha_2(t_2)) - \\ & f(t_1,x(\alpha_1(t_1)),x(\alpha_2(t_1))| \\ & +\left|\int_0^{\beta(t_2)} k(t_2,s)g(t_2,s,x(\gamma_1(s)),x(\gamma_2(s)))ds\right. \\ & \left.+\int_0^{\beta(t_1)} k(t_1,s)g(t_1,s,x(\gamma_1(s)),x(\gamma_2(s)))\right| \\ \leq \quad & \omega^T(h,\varepsilon) + |f(t_2,x(\alpha_1(t_2)),x(\alpha_2(t_2)) \\ & -f(t_2,x(\alpha_1(t_1)),x(\alpha_2(t_1))| \\ & +|f(t_2,x(\alpha_1(t_1)),x(\alpha_2(t_1)) \\ & -f(t_1,x(\alpha_1(t_1)),x(\alpha_2(t_1))| \\ & +\left|\int_0^{\beta(t_2)} k(t_2,s)g(t_2,s,x(\gamma_1(s)),x(\gamma_2(s)))ds\right. \\ & -\int_0^{\beta(t_2)} k(t_2,s)g(t_1,s,x(\gamma_1(s)),x(\gamma_2(s)))ds \\ & +\int_0^{\beta(t_1)} k(t_2,s)g(t_1,s,x(\gamma_1(s)),x(\gamma_2(s)))ds \\ & \left.-\int_0^{\beta(t_1)} k(t_1,s)g(t_1,s,x(\gamma_1(s)),x(\gamma_2(s)))ds\right| \\ \leq \quad & \omega^T(h,\varepsilon) + \mathscr{A} \\ & +\omega_r^T(f,\varepsilon) \\ & +\int_0^{\beta(t_2)} |k(t_2,s)|g(t_2,s,x(\gamma_1(s)),x(\gamma_2(s))) \\ & -g(t_1,s,x(\gamma_1(s)),x(\gamma_2(s)))|ds \\ & +\left|\int_0^{\beta(t_2)} k(t_2,s)g(t_1,s,x(\gamma_1(s)),x(\gamma_2(s)))ds\right. \\ & -\int_0^{\beta(t_2)} k(t_1,s)g(t_1,s,x(\gamma_1(s)),x(\gamma_2(s)))ds \\ & +\int_0^{\beta(t_2)} k(t_1,s)g(t_1,s,x(\gamma_1(s)),x(\gamma_2(s)))ds \\ & \left.-\int_0^{\beta(t_1)} k(t_1,s)g(t_1,s,x(\gamma_1(s)),x(\gamma_2(s)))ds\right| \\ \leq \quad & \omega^T(h,\varepsilon) + \\ & \frac{L\max\{\omega^T(x,\omega^T(\alpha_1,\varepsilon)),\omega^T(x,\omega^T(\alpha_2,\varepsilon))\}}{M+\max\{\omega^T(x,\omega^T(\alpha_1,\varepsilon)),\omega^T(x,\omega^T(\alpha_2,\varepsilon))\}} \\ & +\omega_r^T(f,\varepsilon) + N\int_0^{\beta^T} \omega_r^T(g,\varepsilon)ds \\ & +\int_0^{\beta(t_2)} |k(t_2,s)-k(t_1,s)||g(t_1,s,x(\gamma_1(s)),x(\gamma_2(s)))|ds \\ & +\int_{\beta_{t_1}}^{\beta(t_2)} |k(t_1,s)||g(t_1,s,x(\gamma_1(s)),x(\gamma_2(s)))|ds \\ \leq \quad & \omega^T(h,\varepsilon) + \\ & \frac{L\max\{\omega^T(x,\omega^T(\alpha_1,\varepsilon)),\omega^T(x,\omega^T(\alpha_2,\varepsilon))\}}{M+\max\{\omega^T(x,\omega^T(\alpha_1,\varepsilon)),\omega^T(x,\omega^T(\alpha_2,\varepsilon))\}} \\ & +\omega_r^T(f,\varepsilon) + N\int_0^{\beta_T} \omega_r^T(g,\varepsilon)ds \\ & +\int_0^{\beta_T} \omega_r^T(k,\varepsilon)Vds + N\int_0^{\beta_T} \omega_r^T G_T^r ds \\ \omega^T(KX,\varepsilon) \quad \leq \quad & \omega^T(h,\varepsilon) \\ & +\frac{L\max\{\omega^T(X,\omega^T(\alpha_1,\varepsilon)),\omega^T(X,\omega^T(\alpha_2,\varepsilon))\}}{M+\max\{\omega^T(X,\omega^T(\alpha_1,\varepsilon)),\omega^T(X,\omega^T(\alpha_2,\varepsilon))\}} \\ & +\omega_r^T(f,\varepsilon) + N\int_0^{\beta_T} \omega_r^T(g,\varepsilon)ds \\ & +\int_0^{\beta_T} \omega_r^T(k,\varepsilon)Vds + N\int_0^{\beta_T} G_T^r ds \end{aligned} \tag{3.11}$$

where

$$\mathscr{A} = \frac{L\max\{|x(\alpha_1(t_2)) - x(\alpha_1(t_1))|, |x(\alpha_2(t_2)) - x(\alpha_2(t_1))|\}}{M+\max\{|x(\alpha_1(t_2)) - x(\alpha_1(t_1))|, |x(\alpha_2(t_2)) - x(\alpha_2(t_1))|\}}$$

$$\begin{aligned} \omega^T(h,\varepsilon) \quad = \quad & \sup\{|q(t_2) - q(t_1)| \\ & : t_1,t_2 \in [0,T], |t_2 - t_1| \leq \varepsilon\}, \end{aligned}$$

$$\begin{aligned} \omega_r^T(f,\varepsilon) \quad = \quad & \sup\{|f(t_2,x,y) - f(t_1,x,y)| \\ & : t_1,t_2 \in [0,T], |t_2 - t_1| \leq \varepsilon, x,y \in [-r,r]\}, \end{aligned}$$

$$\begin{aligned} \omega_r^T(k,\varepsilon) \quad = \quad & \sup\{|k(t_2,s) - k(t_1,s)| \\ & : t_1,t_2 \in [0,T], s \in [0,\beta_T], |t_2 - t_1| \leq \varepsilon\}, \end{aligned}$$

$$\begin{aligned} \omega_r^T(g,\varepsilon) \quad = \quad & \sup\{|g(t_2,s,x,y) - g(t_1,s,x,y)| \\ & : t_1,t_2 \in [0,T], s \in [0,\beta_T]|, \\ & t_2 - t_1| \leq \varepsilon, x,y \in [-r,r]\}, \end{aligned}$$

$$\begin{aligned} G_r^T \quad = \quad & \sup\{|g(t,s,x,y)| \\ & : t \in [0,T], s \in [0,\beta_T], x \in [-r,r]\}. \end{aligned}$$

From the above estimate, we have

$$\begin{aligned} & \omega^T(KX,\varepsilon) \leq \omega^T(h,\varepsilon) \\ & +\frac{L\max\{\omega^T(X,\omega^T(\alpha_1,\varepsilon)),\omega^T(X,\omega^T(\alpha_2,\varepsilon))\}}{M+\max\{\omega^T(X,\omega^T(\alpha_1,\varepsilon)),\omega^T(X,\omega^T(\alpha_2,\varepsilon))\}} \\ & +\omega_r^T(f,\varepsilon) + N\int_0^{\beta_T} \omega_r^T(g,\varepsilon)ds \\ & +\int_0^{\beta_T} \omega_r^T(k,\varepsilon)Vds + N\int_0^{\beta_T} G_T^r ds \end{aligned} \tag{3.12}$$

By the uniform continuity of the functions h, f, k and g on the sets $[0,T], [0,T]\times[-r,r]\times[-r,r], [0,T]\times[0,\beta_T]$ and $[0,T]\times[0,\beta_T]\times[-r,r]\times[-r,r]$, respectively, we have $\omega^T(h,\varepsilon)\to 0$, $\omega^T(f,\varepsilon)\to 0$, $\omega^T(k,\varepsilon)\to 0$ and $\omega^T(g,\varepsilon)\to 0$. It is obvious that G_T^r is finite and $\omega^T(\alpha_1,\varepsilon)\to 0, \omega^T(\alpha_2,\varepsilon)\to 0, \omega^T(\beta,\varepsilon)\to 0$, as $\varepsilon\to 0$. Thus,

$$\omega_0^T \leq \frac{L\omega_0^T(X)}{M+\omega_0^T(X)} \tag{3.13}$$

For arbitrarily fixed $t\in\mathbb{R}_+$ and $x_1,x_2,y_1,y_2\in X$, we have

$$\begin{aligned}
|(Kx)(t)-(Ky)(t)| &\leq |f(t,x(\alpha_1(t)),x(\alpha_2(t))) \\
&\quad -f(t,y(\alpha_1(t)),y(\alpha_2(t)))| \\
&\quad +\int_0^{\beta(t)}|g(t,s,x(\gamma_1(s)),x(\gamma_2(s))) \\
&\quad -g(t,s,y(\gamma_1(s)),y(\gamma_2(s)))|ds \\
&\leq A+|g(t,s,x(\gamma_1(s)),x(\gamma_2(s))) \\
&\quad -g(t,s,y(\gamma_1(s)),y(\gamma_2(s)))|ds \\
&\leq \frac{L\max\{X(\alpha_1(t)),X(\alpha_2(t))\}}{M+\max\{X(\alpha_1(t)),X(\alpha_2(t))\}} \\
&\quad +2v(t)N \\
diam(KX)(t) &\leq \frac{L\max\{X(\alpha_1(t)),X(\alpha_2(t))\}}{M+\max\{X(\alpha_1(t)),X(\alpha_2(t))\}} \\
&\quad +2v(t)N \\
\limsup_{t\to\infty} diam(KX)(t) &\leq \frac{L\limsup_{t\to\infty} diamX(t)}{M+\limsup_{t\to\infty} diamX(t)}
\end{aligned} \tag{3.14}$$

where

$$A=\frac{L\max\{|x(\alpha_1(t))-y(\alpha_1(t))|,x(\alpha_2(t))-y(\alpha_2(t))|\}}{M+\max\{|x(\alpha_1(t))-y(\alpha_1(t))|,x(\alpha_2(t))-y(\alpha_2(t))|\}}$$

Using measure of noncompactness μ_a,

$$\begin{aligned}
\mu_a(KX) &= \max\{\omega_0(KX),\limsup_{t\to\infty} KX(t)\} \\
&\leq \max\left\{\frac{L\omega_0(X)}{M+\omega_0(X)},\frac{L\limsup_{t\to\infty}X(t)}{M+\limsup_{t\to\infty}X(t)}\right\} \\
&\leq \frac{L\max\{\omega_0(X),\limsup_{t\to\infty}X(t)\}}{M+\max\{\omega_0(X),\limsup_{t\to\infty}X(t)\}} \\
&\leq \frac{L\mu_a(X)}{M+\mu_a(X)}.
\end{aligned} \tag{3.15}$$

Since $L\leq M$,

$$\mu_a(KX)=\phi(\mu_a(X)),$$

where $\frac{Lr}{M+r}$ for $r>0$. Hence we apply Theorem (2.1) to deduce that operator K has a fixed point x in the ball $\overline{B}_r(0)$. Thus x is solution of the FIE (3.1). On taking account that the image of the space $BC(\mathbb{R}_+,\mathbb{R})$ under the operator K is contained in the ball $\overline{B}_r(0)$ because the set $Fix(K)$ of all fixed points of K is contained $\overline{B}_r(0)$. The set $Fix(K)$ contain all solutions of the FIE (3.1. On the other hand, from Remark 2.1 we conclude that the set $Fix(K)$ belongs to the family $ker\mu_a$. Now, taking account the description of sets belonging to $ker\mu_a$, we have that all solutions for the FIE (3.1) are globally uniformly attractive on $\mathbb{R}_+$.

□

Uniform global attractivity and positivity of solutions

To prove next result concerning the asymptotic positivity of the attractive solutions, we need following hypothesis in the sequel.

($\mathbf{K_6}$) The functions h and f satisfy

$$\lim_{t\to\infty}[|h(t)|-h(t)]=0$$

and

$$\lim_{t\to\infty}[|f(t,x,y)|-f(t,x,y)]=0$$

for all $x,y\in\mathbb{R}$.

Theorem 3.2. *Under the hypotheses of Theorem (3.1) and ($\mathbf{K_6}$), the FIE (3.1) has atleast one solution on $\mathbb{R}_+$. Moreover, solutions of the FIE (3.1) are uniformly globally attractive and ultimately positive on $\mathbb{R}_+$.*

Proof. Consider the closed ball $\overline{B}_r(0)$ in the Banach space $BC(\mathbb{R}_+,\mathbb{R})$, where the real number r is given as in the proof of Theorem (3.1) and define a map $K: BC(\mathbb{R}_+,\mathbb{R})\to BC(\mathbb{R}_+,\mathbb{R})$ by (3.1). In proof of Theorem (3.1), we have shown that K is a continuous mapping from the space $BC(\mathbb{R}_+,\mathbb{R})$ from the space $\overline{B}_r(0)$. In particular, K maps $\overline{B}_r(0)$ into itself.

Now we will show that K is a nonlinear-set-contraction with respect to measure μ_{ad} of noncompactness in $BC(\mathbb{R}_+,\mathbb{R})$. For any $x,y\in\mathbb{R}$, we have

$$|x|+|y|\geq|x+y|\geq x+y,$$

therefore

$$\big||x+y|-(x+y)\big|\leq\big||x|+|y|-(x+y)\big|\leq\big||x|-x\big|+\big||y|-y\big|$$

for all $x,y\in\mathbb{R}$.

For any $x\in\overline{B}_r(0)$, we have

$$\begin{aligned}
\big||Kx(t)| &- Kx(t)\big| \leq \big||h(t)|-h(t)\big| \\
&\quad +\big||f(t,x(\alpha_1(t)),x(\alpha_2(t))\big| \\
&\quad -f(t,x(\alpha_1(t)),x(\alpha_2(t))| \\
&\quad +\big||\int_0^{\beta(t)}k(t,s)g(t,s,x(\gamma_1(s)),x(\gamma_2(s)))ds\big| \\
&\quad -\int_0^{\beta(t)}k(t,s)g(t,s,x(\gamma_1(s)),x(\gamma_2(s)))ds \\
&\leq \delta_T(h)+\delta_T(f)+2Nv(t) \\
&\leq \delta_T(h)+\delta_T(f)+2NV_T,
\end{aligned}$$

where $V_T = \sup_{t\geq T} v(t)$. Thus we have

$$\delta_T(X) \leq \delta_T(h) + \delta_T(f) + 2NV_T$$

for all closed $X \subset \overline{B}_r(0)$.
Taking the limit superior as $T \to \infty$, we have

$$\begin{aligned}\limsup_{T\to\infty} \delta_T(X) \quad &\leq \quad \limsup_{T\to\infty} \delta_T(h) \\ &\quad + \limsup_{T\to\infty} \delta_T(f) + 2\limsup_{T\to\infty} NV_T \\ &= 0 \qquad (3.16)\end{aligned}$$

for all closed $X \subset \overline{B}_r(0)$. Hence,

$$\delta(KX) = \lim_{T\to\infty}(X) = 0$$

for all closed $X \subset \overline{B}_r(0)$. By the measure of noncompactness μ_a, we have

$$\begin{aligned}\mu_{ad}(KX) &= \max\{\mu_{ad}(KX), \delta(KX)\} \\ &\leq \max\{\frac{L\mu_a(X)}{M+\mu_a(X)}, 0\} \\ &= \frac{L\mu_a(X)}{M+\mu_a(X)} \\ &\leq \frac{L\mu_{ad}(X)}{M+\mu_{ad}(X)} \qquad (3.17)\end{aligned}$$

since $L \leq M$, therefore we have

$$\mu_{ad}(KX) \leq \phi(\mu_{ad}(X)),$$

where $\phi(r) = \frac{Lr}{M+r}$ for $r > 0$. By Theorem (2.1), the operator K has a fixed point x in the ball $\overline{B}_r(0)$ and x is a solution of FIE (3.1). The image of the space $BC(\mathbb{R}_+, \mathbb{R})$ under the operator K is contained in $\overline{B}_r(0)$ because the set $Fix(K)$ of all fixed points of K is contained $\overline{B}_r(0)$. The set $Fix(K)$ contain all solutions of the FIE (3.1. On the other hand, from Remark 2.1 we conclude that the set $Fix(K)$ belongs to the family $ker\mu_{ad}$. Now, taking account the description of sets belonging to $ker\mu_{ad}$, we have that all solutions for the FIE (3.1) are globally uniformly attractive and ultimately positive on $\mathbb{R}_+$. □

Uniform global asymptotical attractivity of solutions

Next we prove that the global asymptotic attractivity results the FIE (3.1). We need the following hypotheses in the sequel.
($\mathbf{K_7}$) The function $h : \mathbb{R}_+ \to \mathbb{R}$ is continuous and $\lim_{t\to\infty} h(t) = c$.
($\mathbf{K_8}$) f(t,0,0)= 0 for all $t \in \mathbb{R}_+$, and
($\mathbf{K_9}$) $\lim_{t\to\infty} l(t) = 0$, where the function l is defined as in hypothesis ($\mathbf{K_2}$).

Theorem 3.3. *Under the hypothesis (*$\mathbf{K_0}$*)- (*$\mathbf{K_9}$*), the FIE (3.1) has atleast one solution in the space* $BC(\mathbb{R}_+, \mathbb{R})$*. Moreover, solutions are uniformly globally asymptotically attractive on* $\mathbb{R}_+$.

Proof. Consider the closed ball $\overline{B}_r(0)$ in the space $BC(\mathbb{R}_+, \mathbb{R})$, where the real number r is given as in the proof of Theorem (3.1) and define a map $K : BC(\mathbb{R}_+, \mathbb{R}) \to BC(\mathbb{R}_+, \mathbb{R})$ by (3.1). In proof of Theorem (3.1), we have shown that K is a continuous mapping from the space $BC(\mathbb{R}_+, \mathbb{R})$ from the space $\overline{B}_r(0)$. In particular, K maps $\overline{B}_r(0)$ into itself.

Now we will show that K is a nonlinear-set-contraction with respect to measure μ_c of noncompactness in $BC(\mathbb{R}_+, \mathbb{R})$. For any $x \in \overline{B}_r(0)$, we have

$$\begin{aligned}|Kx(t) - c| &\leq |h(t) - c| + |f(t, x(\alpha_1(t)), x(\alpha_2(t)))| \\ &\quad + \int_0^{\beta(t)} |k(t,s)||g(t,s,x(\gamma_1(s)), x(\gamma_2(s)))|ds \\ &\leq |h(t) - c| + \frac{l(t)\max\{|x(\alpha_1(t)), x(\alpha_2(t))|\}}{M + \max\{|x(\alpha_1(t)), x(\alpha_2(t))|\}} \\ &\quad + 2Nv(t) \\ &\leq |h(t) - c| + \frac{l(t)||x||}{M + ||x||} + 2Nv(t) \\ &\leq |h(t) - c| + \frac{l(t)r}{M+r} + 2Nv(t) \\ &\leq |h(t) - c| + l(t) + 2Nv(t)\end{aligned}$$

for all $t \in \mathbb{R}_+$. Thus we have

$$||Kx(t) - c|| \leq |h(t) - c| + l(t) + 2Nv(t)$$

On taking the limit superior, we have

$$\begin{aligned}\limsup_{t\to\infty} ||Kx(t) - c|| \quad &\leq \quad \limsup_{t\to\infty} |h(t) - c| + \limsup_{t\to\infty} l(t) \\ &\quad + 2N \limsup_{t\to\infty} v(t) \\ &= 0. \qquad (3.18)\end{aligned}$$

Using the measure of noncompactness μ_c, we have

$$\begin{aligned}\mu_c(KX) &= \max\{\omega_0(KX), \limsup_{t\to\infty} ||Kx(t) - c||\} \\ &\leq \max\{\frac{L\omega_0(X)}{M+\omega_0(X)}, 0\} \\ &\leq \frac{L\max\{\omega_0(X), 0\}}{M + \max\{\omega_0(X), 0\}} \\ &= \frac{L\mu_c(X)}{M+\mu_c(X)}. \qquad (3.19)\end{aligned}$$

Since $L \leq M$, therefore we have

$$\mu_c(KX) \leq \phi(\mu_c(X)),$$

where $\phi(r) = \frac{Lr}{M+r}$ for $r > 0$. By Theorem (2.1), the operator K has a fixed point x in the ball $\overline{B}_r(0)$ and x is a solution of FIE (3.1). The image of the space $BC(\mathbb{R}_+, \mathbb{R})$ under the operator K is contained in $\overline{B}_r(0)$ because the set $Fix(K)$ of all fixed points of K is contained $\overline{B}_r(0)$. The set $Fix(K)$ contain all solutions of the FIE (3.1). On the other hand, from Remark

2.1 we conclude that the set $Fix(K)$ belongs to the family $ker\mu_c$. Now, taking account the description of sets belonging to $ker\mu_c$, we have that all solutions for the FIE (3.1) are globally uniformly asymptotically attractive on $\mathbb{R}_+$.

□

4. Conclusions

In this paper, we proved the existence of solution of the nonlinear functional integral equation via Dhage's fixed point theorem and the most powerful tool measure of noncompactness. Also we discuss the qualitative behaviour of solutions such as global attractivity, uniform global attractivity, uniform global asymptotical attractivity and ultimate positivity of the nonlinear functional integral equation.

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