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A note on the zeros of polar derivative of a polynomial

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Abstract

In [4, 7], Enestrom Kakeya theorem has stated as the following. If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients such that $0<k_0\leq k_1\leq ...\leq k_{n-2}\leq k_{n-1}\leq k_n$ then all zeros of f(z) lies in $|z|\leq 1$. In [1], Aziz and Mahammad, showed that zeros of $f(z)$ satisfies $|z|\geq \frac{n}{n+1}$ are simple, under the same conditions. In this paper, we extend the above result to the polar derivative by relaxing the hypothesis in different ways.

Keywords

Zeros, polynomial, Eneström-Kakeya theorem, polar derivative.

AMS Subject Classification

30C10, 30C15.

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Contents

1. Introduction

Let $f(z)$ be the n^{th} degree polynomial with real coefficients. Let $D_\alpha f(z)$ be the polar derivative of $f(z)$ w.r.t the point α and it is defined by $D_\alpha f(z)=nf(z)+(\alpha-z)f^{'}(z)$.

In this case the degree of $D_\alpha f(z)$ is at most $n-1$ and if α tends to ∞ then it generalize the ordinary derivative

$$i.e \lim_{\alpha\longrightarrow\infty}\frac{D_\alpha f(z)}{\alpha}=f^{'}(z)$$

Regarding the distribution of zeros of $f(z)$, Enestrom Kakeya proved the folowing result.

Theorem 1.1. *Let $f(z)=\sum_{j=0}^{n}k_jz^j$ be the n^{th} degree polynomial with real coefficients such that for some $0<k_0\leq k_1\leq ...\leq k_{n-2}\leq k_{n-1}\leq k_n$ then all zeros of $f(z)$ lies in $|z|\leq 1$.*

Regarding the multiplicity of zeros of $f(z)$, Aziz and Mahammad [1] proved the folowing result

Theorem 1.2. *Let $f(z)=\sum_{j=0}^{n}k_jz^j$ be the n^{th} degree polynomial with real coefficients such that for some $0<k_0\leq k_1\leq ...\leq k_n$ then all zeros of $f(z)$ of modulus greater than or equal to $\frac{n}{n+1}$ are simple.*

Gulzar, Zargar, Akhter [6] have extended the above results to the polar derivatives, there exist some generalizations and extentions of Enestrom kakeya theorem in [2, 3, 5, 8, 9].

In this paper we prove the interesting results by relaxing the hypothesis by replacing b_t *with* $(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

2. Main Results

Theorem 2.1. *Let $f(z)=\sum_{j=0}^{n}k_jz^j$ be the n^{th} degree polynomial with real coefficients. Let α be a real number, $s\geq 1$, $0<\delta\leq 1$ such that for some*

$$sb_n\geq b_{n-1}\geq ...\geq b_4\geq b_3\geq b_2-\delta$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z+s-1|\leq\frac{sb_n-b_2+|b_2|+2\delta}{|b_n|}$$

are simple, where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

Corollary 2.1. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and α be a real number, $s\geq 1$, $0<\delta<1$ such that for some*

$$sb_n\geq b_{n-1}\geq ...\geq b_4\geq b_3\geq b_2-\delta\geq 0$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z+s-1|\leq\frac{sb_n+2\delta}{|b_n|}$$

are simple, where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

Corollary 2.2. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and α be a real number, such that for some*

$$b_n\geq b_{n-1}\geq ...\geq b_4\geq b_3\geq b_2$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\leq\frac{b_n-b_2+|b_2|}{|b_n|}$$

are simple, where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

Corollary 2.3. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and α be a real number, such that for some*

$$b_n\geq b_{n-1}\geq ...\geq b_4\geq b_3\geq b_2\geq 0$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\leq 1$$

are simple, where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

Corollary 2.4. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and $s\geq 1$, $0<\delta\leq 1$ such that for some*

$$sc_n\geq c_{n-1}\geq ...\geq c_4\geq c_3\geq c_2-\delta$$

then all zeros of $D_0 f(z)$ which does not lie in

$$|z+s-1|\leq\frac{sc_n-c_2+|c_2|+2\delta}{|c_n|}$$

are simple, where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

Corollary 2.5. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, such that for some $s\geq 1$, $0<\delta<1$*

$$sc_n\geq c_{n-1}\geq ...\geq c_4\geq c_3\geq c_2-\delta>0$$

then all zeros of $D_0 f(z)$ which does not lie in

$$|z+s-1|\leq\frac{sc_n+2\delta}{|c_n|}$$

are simple, where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

Corollary 2.6. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, such that for some*

$$c_n\geq c_{n-1}\geq ...\geq c_4\geq c_3\geq c_2\geq 0$$

then all zeros of $D_0 f(z)$ which does not lie in

$$|z|\leq 1$$

are simple, where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

Corollary 2.7. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and $s\geq 1$, $0<\delta<1$ such that for some*

$$sc_n\geq c_{n-1}\geq ...\geq c_4\geq c_3\geq c_2-\delta$$

then all zeros of $D_0 f(z)$ which does not lie in

$$|z+s-1|\leq\frac{sc_n-c_2+|c_2|+2\delta}{|c_n|}$$

are simple, where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

Remark 2.1.

1. *By substituting $b_t\geq 0$ in Theorem 2.1, then it gives Corollary 2.1.*
2. *By substituting $s=1$, $\delta=0$ in Theorem 2.1, then it gives Corollary 2.2.*
3. *By substituting $\delta=0$, $s=1$ and $b_t\geq 0$ in Theorem 2.1, then it gives Corollary 2.3.*
4. *By substituting $\alpha=0$ in Theorem 2.1, then it gives Corollary 2.4.*
5. *By substituting $\alpha=0$ and $c_t\geq 0$ in Theorem 2.1, then it gives Corollary 2.5.*
6. *By substituting $\alpha=0$, $s=1$, $\delta=0$ and $c_t\geq 0$ in Theorem 2.1, then it gives Corollary 2.6.*
7. *By substituting $\delta=0$, $s=1$, $\alpha=0$ in Theorem 2.1, then it gives Corollary 2.7.*

Theorem 2.2. *Let $f(z)=\sum_{j=0}^{n}k_jz^j$ be the n^{th} degree polynomial with real coefficients, and α be a real number, $\delta>0$, $0<r\leq 1$ such that for some*

$$rb_n\geq b_{n-1}\geq ...\geq b_4\geq b_3\geq b_2-\delta$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\leq\frac{|b_n|+r(b_n-|b_n|)-b_2+|b_2|+2\delta}{|b_n|}$$

are simple, where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

Corollary 2.8. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is a polynomial of degree n with real coefficients, and α be a real number, $\delta>0$, $0<r\leq 1$ such that for some*

$$rb_n \geq b_{n-1} \geq ... \geq b_4 \geq b_3 \geq b_2-\delta \geq 0$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{|b_n|+r(b_n-|b_n|)+2\delta}{|b_n|}$$

are simple, where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ $for \quad t=2,3,4,...,n$

Corollary 2.9. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and α be a real number, such that for some*

$$b_n \geq b_{n-1} \geq ... \geq b_4 \geq b_3 \geq b_2$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{b_n-b_2-|b_2|}{|b_n|}$$

are simple, where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ $for \quad t=2,3,4,...,n$

Corollary 2.10. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and $\delta>0$, $0<r\leq 1$ such that for some*

$$rc_n \geq c_{n-1} \geq ... \geq c_4 \geq c_3 \geq c_2-\delta$$

then all zeros of $D_0 f(z)$ which does not lie in

$$|z| \leq \frac{|c_n|+r(c_n-|c_n|)-c_2+|c_2|+2\delta}{|c_n|}$$

are simple, where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ $for \quad t=2,3,4,...,n$

Corollary 2.11. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and $\delta>0$, $0<r\leq 1$ such that for some*

$$rc_n \geq c_{n-1} \geq ... \geq c_4 \geq c_3 \geq c_2-\delta \geq 0$$

then all zeros of $D_0 f(z)$ which does not lie in

$$|z| \leq \frac{|c_n|+r(c_n-|c_n|)+2\delta}{|c_n|}$$

are simple, where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ $for \quad t=2,3,4,...,n$

Corollary 2.12. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, such that for some*

$$c_n \geq c_{n-1} \geq ... \geq c_4 \geq c_3 \geq c_2$$

then all zeros of $D_0 f(z)$ which does not lie in

$$|z| \leq \frac{c_n-c_2-|c_2|}{|c_n|}$$

are simple, where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ $for \quad t=2,3,4,...,n$

Remark 2.2.

1. *By substituting $b_t\geq 0$ in Theorem 2.2, then it gives Corollary 2.8.*
2. *By substituting $\delta=0$, $r=1$ in Theorem 2.2, then it gives Corollary 2.9.*
3. *By substituting $\delta=0$, $r=1$ and $b_t\geq 0$ in Theorem 2.2, then it gives Corollary 2.3.*
4. *By substituting $\alpha=0$ in Theorem 2.2, then it gives Corollary 2.10.*
5. *By substituting $\alpha=0$ and $c_t\geq 0$ in Theorem 2.2, then it gives Corollary 2.11.*
6. *By substituting $\alpha=0$, $r=1$, $\delta=0$ and $c_t\geq 0$ in Theorem 2.2, then it gives Corollary 2.6.*
7. *By substituting $\delta=0$, $r=1$, $\alpha=0$ in Theorem 2.2, then it gives Corollary 2.12.*

Theorem 2.3. *Let $f(z)=\sum_{j=0}^{n}k_jz^j$ be the n^{th} degree polynomial with real coefficients, and α be a real number, $s\geq 1$, $0<\delta\leq 1$ such that for some*

$$sb_n \leq b_{n-1} \leq ... \leq b_4 \leq b_3 \leq b_2+\delta$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z+s-1| \leq \frac{b_2+|b_2|-sb_n+2\delta}{|b_n|}$$

are simple, where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ $for \quad t=2,3,4,...,n$

Corollary 2.13. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and α be a real number, $s\geq 1$, $0<\delta\leq 1$ such that for some*

$$0 \leq sb_n \leq b_{n-1} \leq ... \leq b_4 \leq b_3 \leq b_2+\delta$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z+s-1| \leq \frac{2b_2-sb_n+2\delta}{|b_n|}$$

are simple, where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ $for \quad t=2,3,4,...,n$

Corollary 2.14. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and α be a real number such that for some*

$$b_n \leq b_{n-1} \leq ... \leq b_4 \leq b_3 \leq b_2$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{b_2+|b_2|-b_n}{|b_n|}$$

are simple, where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ $for \quad t=2,3,4,...,n$

Corollary 2.15. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and α be a real number such that for some*

$$0\le b_n\le b_{n-1}\le\ldots\le b_4\le b_3\le b_2$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\le\frac{2b_2-b_n}{|b_n|}$$

are simple, where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ $for\quad t=2,3,4,\ldots,n$

Corollary 2.16. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and $s\ge 1$, $0<\delta\le 1$ such that for some*

$$sc_n\le c_{n-1}\le\ldots\le c_4\le c_3\le c_2+\delta$$

then all zeros of $D_0f(z)$ which does not lie in

$$|z+s-1|\le\frac{c_2+|c_2|-sc_n+2\delta}{|c_n|}$$

are simple, where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ $for\quad t=2,3,4,\ldots,n$

Corollary 2.17. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and $s\ge 1$, $0<\delta\le 1$ such that for some*

$$0\le sc_n\le c_{n-1}\le\ldots\le c_4\le c_3\le c_2+\delta$$

then all zeros of $D_0f(z)$ which does not lie in

$$|z+s-1|\le\frac{2c_2-sc_n+2\delta}{|c_n|}$$

are simple, where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ $for\quad t=2,3,4,\ldots,n$

Corollary 2.18. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, such that for some*

$$0\le c_n\le c_{n-1}\le\ldots\le c_4\le c_3\le c_2$$

then all zeros of $D_0f(z)$ which does not lie in

$$|z|\le\frac{2c_2-c_n}{|c_n|}$$

are simple, where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ $for\quad t=2,3,4,\ldots,n$

Corollary 2.19. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, such that for some*

$$c_n\le c_{n-1}\le\ldots\le c_4\le c_3\le c_2$$

then all zeros of $D_0f(z)$ which does not lie in

$$|z|\le\frac{c_2+|c_2|-c_n}{|c_n|}$$

are simple, where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ $for\quad t=2,3,4,\ldots,n$

Remark 2.3.

1. *By substituting $b_t\ge 0$ in Theorem 2.3, then it gives Corollary 2.13.*
2. *By substituting $\delta=0$, $s=1$ in Theorem 2.3, then it gives Corollary 2.14.*
3. *By substituting $\delta=0$, $s=1$ and $b_t\ge 0$ in Theorem 2.3, then it gives Corollary 2.15.*
4. *By substituting $\alpha=0$ in Theorem 2.3, then it gives Corollary 16.*
5. *By substituting $\alpha=0$ and $c_t\ge 0$ in Theorem 2.3, then it gives Corollary 2.17.*
6. *By substituting $\alpha=0$, $s=1$, $\delta=0$ and $c_t\ge 0$ in Theorem 2.3, then it gives Corollary 2.18.*
7. *By substituting $\delta=0$, $s=1$, $\alpha=0$ in Theorem 2.3, then it gives Corollary 2.19.*

Theorem 2.4. *Let $f(z)=\sum_{j=0}^{n}k_jz^j$ be the n^{th} degree polynomial with real coefficients, Let α be a real number, $\delta>0$, $0<r\le 1$ such that for some*

$$rb_n\le b_{n-1}\le\ldots\le b_4\le b_3\le b_2+\delta$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\le\frac{|b_n|-r(b_n+|b_n|)+b_2+|b_2|+2\delta}{|b_n|}$$

are simple, where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ $for\quad t=2,3,4,\ldots,n$

Corollary 2.20. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and α be a real number, $\delta>0$, $0<r\le 1$ such that for some*

$$0\le rb_n\le b_{n-1}\le\ldots\le b_4\le b_3\le b_2+\delta$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\le\frac{(1-2r)b_n+2b_2+2\delta}{|b_n|}$$

are simple, where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ $for\quad t=2,3,4,\ldots,n$

Corollary 2.21. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and α be a real number, such that for some*

$$b_n\le b_{n-1}\le\ldots\le b_4\le b_3\le b_2$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\le\frac{-b_n+b_2+|b_2|}{|b_n|}$$

are simple, where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ $for\quad t=2,3,4,\ldots,n$

Corollary 2.22. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and α be a real number, such that for some*

$$0\leq b_n\leq b_{n-1}\leq ...\leq b_4\leq b_3\leq b_2$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\leq\frac{-b_n+2b_2}{|b_n|}$$

are simple, where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

Corollary 2.23. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and $\delta>0$, $0<r\leq1$ such that for some*

$$rc_n\leq c_{n-1}\leq ...\leq c_4\leq c_3\leq c_2+\delta$$

then all zeros of $D_0f(z)$ which does not lie in

$$|z|\leq\frac{|c_n|-r(c_n+|c_n|)+c_2+|c_2|+2\delta}{|c_n|}$$

are simple, where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

Corollary 2.24. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, and $\delta>0$, $0<r\leq1$ such that for some*

$$0\leq rb_n\leq b_{n-1}\leq ...\leq b_4\leq b_3\leq b_2+\delta$$

then all zeros of $D_0f(z)$ which does not lie in

$$|z|\leq\frac{(1-2r)c_n+2c_2+2\delta}{|b_n|}$$

are simple, where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

Corollary 2.25. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, such that for some*

$$0\leq c_n\leq c_{n-1}\leq ...\leq c_4\leq c_3\leq c_2$$

then all zeros of $D_0f(z)$ which does not lie in

$$|z|\leq\frac{-c_n+2c_2}{|c_n|}$$

are simple, where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

Corollary 2.26. *If $f(z)=\sum_{j=0}^{n}k_jz^j$ is the n^{th} degree polynomial with real coefficients, such that for some*

$$c_n\leq c_{n-1}\leq ...\leq c_4\leq c_3\leq c_2$$

then all zeros of $D_0f(z)$ which does not lie in

$$|z|\leq\frac{-c_n+c_2+|c_2|}{|c_n|}$$

are simple, where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

Remark 2.4.

1. *By substituting $b_t\geq0$ in Theorem 2.4, then it gives Corollary 2.20.*
2. *By substituting $\delta=0$, $r=1$ in Theorem 2.4, then it gives Corollary 2.21.*
3. *By substituting $\delta=0$, $r=1$ and $b_t\geq0$ in Theorem 2.4, then it gives Corollary 2.22.*
4. *By substituting $\alpha=0$ in Theorem 2.4, then it gives Corollary 2.23.*
5. *By substituting $\alpha=0$ and $c_t\geq0$ in Theorem 2.4, then it gives Corollary 2.24.*
6. *By substituting $\alpha=0$, $r=1$, $\delta=0$ and $c_t\geq0$ in Theorem 2.4, then it gives Corollary 2.25.*
7. *By substituting $\delta=0$, $r=1$, $\alpha=0$ in Theorem 2.4, then it gives Corollary 2.26.*

3. Proof of the theorems

Proof of the Theorem 2.1.

Let $f(z)=k_nz^n+k_{n-1}z^{n-1}+...+k_1z+k_0$ be the n^{th} degree polynomial with real coefficients. by the definition of polar derivative, $D_\alpha f(z)=nf(z)+\alpha f'(z)-zf'(z)$
there fore,

$$\begin{aligned}D_\alpha f(z)&=n(k_nz^n+k_{n-1}z^{n-1}+...+k_1z+k_0)\\&\quad+\alpha(nk_nz^{n-1}+(n-1)k_{n-1}z^{n-2}+...+k_1)\\&\quad-z(nk_nz^{n-1}+(n-1)k_{n-1}z^{n-2}+...+k_1)\\&=[n\alpha k_n+(n-(n-1))k_{n-1}]z^{n-1}\\&\quad+[(n-1)\alpha k_{n-1}+(n-(n-2))k_{n-2}]z^{n-2}\\&\quad+...+[2\alpha k_2+(n-1))k_1]z+[\alpha k_1+nk_0]\end{aligned}$$

Now find $D'_\alpha f(z)$, we get
$D'_\alpha f(z)=b_nz^{n-2}+b_{n-1}z^{n-3}+...+b_4z^2+b_3z+b_2$
where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$.
Now Consider $g(z)=(1-z)D'_\alpha f(z)$, so that

$$\begin{aligned}g(z)&=(1-z)[b_nz^{n-2}+b_{n-1}z^{n-3}+b_{n-2}z^{n-4}+...+b_4z^2\\&\quad+b_3z+b_2],\end{aligned}$$

then

$$\begin{aligned}|g(z)|\geq|b_n||z|^{n-2}\Bigg[&|z+s-1|-\frac{1}{|b_n|}\{|sb_n-b_{n-1}|\\&+\frac{|b_{n-1}-b_{n-2}|}{|z|}+\frac{|b_{n-2}-b_{n-3}|}{|z|^2}+...+\frac{|b_4-b_3|}{|z|^{n-4}}\\&+\frac{|b_3-b_2|}{|z|^{n-3}}+\frac{|b_2|}{|z|^{n-2}}\}\Bigg]\end{aligned}$$

If $|z|>1$ then $\frac{1}{|z|}<1$

$$
\begin{aligned}
&\geq |b_n||z|^{n-2}\left[|z+s-1| - \frac{1}{|b_n|}\left\{|sb_n - b_{n-1}| + \ldots \right.\right.\\
&\quad \left.\left. + |b_3 - \delta - (b_2 - \delta)| + |b_2|\right\}\right]\\
&\geq |b_n||z|^{n-2}\left[|z+s-1| - \frac{1}{|b_n|}\left\{|sb_n - b_{n-1}| \right.\right.\\
&\quad \left.\left. + \ldots + |b_3 - (b_2 - \delta)| + |\delta| + |b_2|\right\}\right]\\
&\geq |b_n||z|^{n-2}\left[|z+s-1| - \frac{1}{|b_n|}\left\{(sb_n - b_{n-1}) + \ldots \right.\right.\\
&\quad \left.\left. + (b_3 - b_2 + \delta) + |\delta| + |b_2|\right\}\right]\\
&\geq |b_n||z|^{n-2}\left[|z+s-1| - \frac{1}{|b_n|}\left\{sb_n - b_2 + 2\delta + |b_2|\right\}\right]
\end{aligned}
$$

Hence $|g(z)| > 0$ if

$$|z+s-1| > \frac{1}{|b_n|}\left\{sb_n - b_2 + 2\delta + |b_2|\right\}.$$

This implies that all zeros of $g(z)$ whose modulus is greater than 1 are lie in

$$|z+s-1| \leq \frac{1}{|b_n|}\left\{sb_n - b_2 + 2\delta + |b_2|\right\}.$$

Since the zeros of $g(z)$ whose modulus is less than or equal to 1 are already lie in

$$|z+s-1| \leq \frac{1}{|b_n|}\left\{sb_n - b_2 + 2\delta + |b_2|\right\},$$

it follows that all the zeros of $g(z)$ lie in

$$|z+s-1| \leq \frac{1}{|b_n|}\left\{sb_n - b_2 + 2\delta + |b_2|\right\}.$$

Since all the zeros of $g(z)$ are also the zeros of $D'_\alpha f(z)$ lie in

$$|z+s-1| \leq \frac{1}{|b_n|}\left\{sb_n - b_2 + 2\delta + |b_2|\right\}.$$

Thus all the zeros of $D'_\alpha f(z)$ lie in

$$|z+s-1| \leq \frac{1}{|b_n|}\left\{sb_n - b_2 + 2\delta + |b_2|\right\}.$$

In other words all zeros of $D_\alpha f(z)$ which does not lie in

$$|z+s-1| \leq \frac{1}{|b_n|}\left\{sb_n - b_2 + 2\delta + |b_2|\right\}.$$

are simple, where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$
$for \quad t = 2,3,4,...,n$

Proof of the Theorem 2.2.

Let $f(z) = k_n z^n + k_{n-1}z^{n-1} + \ldots + k_1 z + k_0$ be the n^{th} degree polynomial with real coefficients. by the definition of polar derivative, $D_\alpha f(z) = nf(z) + \alpha f'(z) - zf'(z)$
therefore,

$$
\begin{aligned}
D_\alpha f(z) &= (k_n z^n + k_{n-1}z^{n-1} + \ldots + k_1 z + k_0)\\
&\quad + \alpha(nk_n z^{n-1} + (n-1)k_{n-1}z^{n-2} + \ldots + k_1)\\
&\quad - z(nk_n z^{n-1} + (n-1)k_{n-1}z^{n-2} + \ldots + k_1)\\
&= [n\alpha k_n + (n-(n-1))k_{n-1}]z^{n-1}\\
&\quad + [(n-1)\alpha k_{n-1} + (n-(n-2))k_{n-2}]z^{n-2}\\
&\quad + \ldots + [2\alpha k_2 + (n-1))k_1]z + [\alpha k_1 + nk_0]
\end{aligned}
$$

Now find $D'_\alpha f(z)$, we get
$D'_\alpha f(z) = b_n z^{n-2} + b_{n-1}z^{n-3} + \ldots + b_4 z^2 + b_3 z + b_2$
where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$
$for \quad t = 2,3,4,...,n$
Now Consider $g(z) = (1-z)D'_\alpha f(z)$, so that
$g(z) = (1-z)[b_n z^{n-2} + b_{n-1}z^{n-3} + \ldots + b_4 z^2 + b_3 z + b_2]$, then

$$
\begin{aligned}
|g(z)| \geq &|b_n||z|^{n-2}\left[|z| - \frac{1}{|b_n|}\left\{|b_n - b_{n-1}| + \frac{|b_{n-1} - b_{n-2}|}{|z|} + \ldots \right.\right.\\
&\left.\left. + \frac{|b_4 - b_3|}{|z|^{n-4}} + \frac{|b_3 - b_2|}{|z|^{n-3}} + \frac{|b_2|}{|z|^{n-2}}\right\}\right]
\end{aligned}
$$

If $|z|>1$ then $\frac{1}{|z|}<1$

$$
\begin{aligned}
|g(z)| &\geq |b_n||z|^{n-2}\left[|z| - \frac{1}{|b_n|}\left\{|b_n - rb_n + rb_n - b_{n-1}| \right.\right.\\
&\quad \left.\left. + \ldots + |b_3 - \delta - (b_2 - \delta)| + |b_2|\right\}\right]\\
&\geq |b_n||z|^{n-2}\left[|z| - \frac{1}{|b_n|}\left\{|b_n - rb_n| + |rb_n - b_{n-1}| \right.\right.\\
&\quad \left.\left. + \ldots + |b_3 - (b_2 - \delta)| + |\delta| + |b_2|\right\}\right]\\
&\geq |b_n||z|^{n-2}\left[|z| - \frac{1}{|b_n|}\left\{(1-r)|b_n| + (rb_n - b_{n-1}) \right.\right.\\
&\quad \left.\left. + \ldots + (b_3 - b_2 + \delta) + |\delta| + |b_2|\right\}\right]\\
&\geq |b_n||z|^{n-2}\left[|z| - \frac{1}{|b_n|}\left\{(1-r)|b_n| + rb_n - b_2 + 2\delta + |b_2|\right\}\right]\\
&\geq |b_n||z|^{n-2}\left[|z| - \frac{|b_n| + r(b_n - |b_n|) - b_2 + 2\delta + |b_2|}{|b_n|}\right].
\end{aligned}
$$

Hence $|g(z)| > 0$ if

$$|z| > \frac{1}{|b_n|}\left\{|b_n| + r(b_n - |b_n|) - b_2 + 2\delta + |b_2|\right\}.$$

This implies that all zeros of $g(z)$ whose modulus is greater than 1 are lie in

$$|z| \leq \frac{1}{|b_n|}\left\{|b_n| + r(b_n - |b_n|) - b_2 + 2\delta + |b_2|\right\}.$$

Since the zeros of $g(z)$ whose modulus is less than or equal to 1 are already lie in

$$|z| \leq \frac{1}{|b_n|}\left\{|b_n| + r(b_n - |b_n|) - b_2 + 2\delta + |b_2|\right\},$$

it follows that all the zeros of $g(z)$ lie in

$$|z| \leq \frac{1}{|b_n|}\left\{|b_n| + r(b_n - |b_n|) - b_2 + 2\delta + |b_2|\right\}.$$

Since all the zeros of $g(z)$ are also the zero of $D'_\alpha f(z)$ lie in

$$|z| \leq \frac{1}{|b_n|}\left\{|b_n| + r(b_n - |b_n|) - b_2 + 2\delta + |b_2|\right\}.$$

Thus all the zeros of $D'_\alpha f(z)$ lie in

$$|z| \leq \frac{1}{|b_n|}\left\{|b_n| + r(b_n - |b_n|) - b_2 + 2\delta + |b_2|\right\}.$$

In other words all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{1}{|b_n|}\left\{|b_n| + r(b_n - |b_n|) - b_2 + 2\delta + |b_2|\right\}.$$

are simple, where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$
$for \quad t = 2,3,4,.....,n$

Proof of the Theorem 2.3.

Let $f(z) = k_n z^n + k_{n-1}z^{n-1} + ... + k_1 z + k_0$ be the n^{th} degree polynomial with real coefficients. by the definition of polar derivative, $D_\alpha f(z) = nf(z) + \alpha f'(z) - zf'(z)$
there fore,

$$\begin{aligned} D_\alpha f(z) &= n(k_n z^n + k_{n-1}z^{n-1} + ... + k_1 z + k_0) \\ &\quad + \alpha(nk_n z^{n-1} + (n-1)k_{n-1}z^{n-2} + ... + k_1) \\ &\quad - z(nk_n z^{n-1} + (n-1)k_{n-1}z^{n-2} + ... + k_1) \\ &= [n\alpha k_n + (n-(n-1))k_{n-1}]z^{n-1} \\ &\quad + [(n-1)\alpha k_{n-1} + (n-(n-2))k_{n-2}]z^{n-2} \\ &\quad + ... + [2\alpha k_2 + (n-1))k_1]z + [\alpha k_1 + nk_0] \end{aligned}$$

Now find $D'_\alpha f(z)$, we get
$D'_\alpha f(z) = b_n z^{n-2} + b_{n-1}z^{n-3} + ... + b_4 z^2 + b_3 z + b_2$
where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$
$for \quad t = 2,3,4,...,n$
Now Consider $g(z) = (1-z)D'_\alpha f(z)$, so that
$g(z) = (1-z)[b_n z^{n-2} + b_{n-1}z^{n-3} + ... + b_4 z^2 + b_3 z + b_2]$, then

$$\begin{aligned} |g(z)| \geq |b_n||z|^{n-2}\Bigg[&|z+s-1| - \frac{1}{|b_n|}\Big\{|sb_n - b_{n-1}| \\ &+ \frac{|b_{n-1} - b_{n-2}|}{|z|} + \frac{|b_{n-2} - b_{n-3}|}{|z|^2} + ... \\ &+ \frac{|b_4 - b_3|}{|z|^{n-4}} + \frac{|b_3 - b_2|}{|z|^{n-3}} + \frac{|b_2|}{|z|^{n-2}}\Big\}\Bigg] \end{aligned}$$

If $|z|>1$ then $\frac{1}{|z|}<1$

$$\begin{aligned} |g(z)| &\geq |b_n||z|^{n-2}\Big[|z+s-1| - \frac{1}{|b_n|}\Big\{|sb_n - b_{n-1}| \\ &\quad + ... + |b_3 + \delta - (b_2 + \delta)| + |b_2|\Big\}\Big] \\ &\geq |b_n||z|^{n-2}\Big[|z+s-1| - \frac{1}{|b_n|}\Big\{|sb_n - b_{n-1}| \\ &\quad + ... + |b_3 - (b_2 + \delta)| + |\delta| + |b_2|\Big\}\Big] \\ &\geq |b_n||z|^{n-2}\Big[|z+s-1| - \frac{1}{|b_n|}\Big\{(b_{n-1} - sb_n) + \\ &\quad ... + (b_2 + \delta - b_3) + |\delta| + |b_2|\Big\}\Big] \\ &\geq |b_n||z|^{n-2}\Big[|z+s-1| - \frac{1}{|b_n|}\Big\{2\delta + b_2 + |b_2| - sb_n\Big\}\Big]. \end{aligned}$$

Hence $|g(z)| > 0$ if

$$|z+s-1| > \frac{1}{|b_n|}\left\{2\delta + b_2 + |b_2| - sb_n\right\}.$$

This implies that all zeros of $g(z)$ whose modulus is greater than 1 are lie in

$$|z+s-1| \leq \frac{1}{|b_n|}\left\{2\delta + b_2 + |b_2| - sb_n\right\}.$$

Since the zeros of $g(z)$ whose modulus is less than or equal to 1 are already lie in

$$|z+s-1| \leq \frac{1}{|b_n|}\left\{2\delta + b_2 + |b_2| - sb_n\right\},$$

it follows that all the zeros of $g(z)$ lie in

$$|z+s-1| \leq \frac{1}{|b_n|}\left\{2\delta + b_2 + |b_2| - kb_n\right\}.$$

Since all the zeros of $g(z)$ are also the zeros of $D'_\alpha f(z)$ lie in

$$|z+s-1| \leq \frac{1}{|b_n|}\left\{2\delta + b_2 + |b_2| - sb_n\right\}.$$

Thus all the zeros of $D'_\alpha f(z)$ lie in

$$|z+s-1| \leq \frac{1}{|b_n|}\left\{2\delta + b_2 + |b_2| - sb_n\right\}.$$

In other words all zeros of $D_\alpha f(z)$ which does not lie in

$$|z+s-1| \leq \frac{1}{|b_n|}\left\{2\delta + b_2 + |b_2| - sb_n\right\}.$$

are simple. where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$
$for \quad t = 2,3,4,.....,n$

Proof of the Theorem 2.4.

Let $f(z) = k_n z^n + k_{n-1}z^{n-1} + ... + k_1 z + k_0$ be the n^{th} degree polynomial with real coefficients. by the definition of polar

derivative, $D_\alpha f(z) = nf(z) + \alpha f^{'}(z) - zf^{'}(z)$
there fore,

$$\begin{aligned}D_\alpha f(z) = & n(k_n z^n + k_{n-1}z^{n-1} + ... + k_1 z + k_0) \\ & + \alpha(nk_n z^{n-1} + (n-1)k_{n-1}z^{n-2} + ... + k_1) \\ & - z(nk_n z^{n-1} + (n-1)k_{n-1}z^{n-2} + ... + k_1) \\ = & [n\alpha k_n + (n-(n-1))k_{n-1}]z^{n-1} \\ & + [(n-1)\alpha k_{n-1} + (n-(n-2))k_{n-2}]z^{n-2} \\ & + ... + [2\alpha k_2 + (n-1))k_1]z + [\alpha k_1 + nk_0]\end{aligned}$$

Now find $D^{'}_\alpha f(z)$, we get
$D^{'}_\alpha f(z) = b_n z^{n-2} + b_{n-1}z^{n-3} + ... + b_4 z^2 + b_3 z + b_2$
where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$
$for \quad t = 2,3,4,...,n$
Now Consider $g(z) = (1-z)D^{'}_\alpha f(z)$, so that
$g(z) = (1-z)[b_n z^{n-2} + b_{n-1}z^{n-3} + ... + b_4 z^2 + b_3 z + b_2]$, then

$$\begin{aligned}|g(z)| \geq & |b_n||z|^{n-2}\Bigg[|z| - \frac{1}{|b_n|}\{|b_n - b_{n-1}| \\ & + \frac{|b_{n-1} - b_{n-2}|}{|z|} + \frac{|b_{n-2} - b_{n-3}|}{|z|^2} + ... \\ & + \frac{|b_4 - b_3|}{|z|^{n-4}} + \frac{|b_3 - b_2|}{|z|^{n-3}} + \frac{|b_2|}{|z|^{n-2}}\}\Bigg]\end{aligned}$$

If $|z|>1$ then $\frac{1}{|z|}<1$

$$\begin{aligned}\geq & |b_n||z|^{n-2}\Bigg[|z| - \frac{1}{|b_n|}\{|b_n - rb_n + rb_n - b_{n-1}| \\ & + |b_{n-1} - b_{n-2}| + ... + |b_4 - b_3| \\ & + |b_3 + \delta - (b_2 + \delta)| + |b_2|\}\Bigg] \\ \geq & |b_n||z|^{n-2}\Bigg[|z| - \frac{1}{|b_n|}\{|b_n - rb_n| + |rb_n - b_{n-1}| \\ & + |b_{n-1} - b_{n-2}| + ... + |b_4 - b_3| \\ & + |b_3 - (b_2 + \delta)| + |\delta| + |b_2|\}\Bigg] \\ \geq & |b_n||z|^{n-2}\Bigg[|z| - \frac{1}{|b_n|}\{(1-r)|b_n| + (b_{n-1} - rb_n) + \\ & ... + (b_2 + \delta - b_3) + |\delta| + |b_2|\}\Bigg] \\ \geq & |b_n||z|^{n-2}\left[|z| - \frac{(1-r)|b_n| - rb_n + b_2 + 2\delta + |b_2|}{|b_n|}\right] \\ \geq & |b_n||z|^{n-2}\left[|z| - \frac{|b_n| - r(b_n + |b_n|) + b_2 + 2\delta + |b_2|}{|b_n|}\right]\end{aligned}$$

Hence $|g(z)| > 0$ if

$$|z| > \frac{1}{|b_n|}\{|b_n| - r(b_n + |b_n|) + b_2 + 2\delta + |b_2|\}.$$

This implies that all zeros of $g(z)$ whose modulus is greater than 1 are lie in

$$|z| \leq \frac{1}{|b_n|}\{|b_n| - r(b_n + |b_n|) + b_2 + 2\delta + |b_2|\}.$$

Since the zeros of $g(z)$ whose modulus is less than or equal to 1 are already lie in

$$|z| \leq \frac{1}{|b_n|}\{|b_n| - r(b_n + |b_n|) + b_2 + 2\delta + |b_2|\},$$

it follows that all the zeros of $g(z)$ lie in

$$|z| \leq \frac{1}{|b_n|}\{|b_n| - r(b_n + |b_n|) + b_2 + 2\delta + |b_2|\}.$$

Since all the zeros of $g(z)$ are also the zeros of $D^{'}_\alpha f(z)$ lie in

$$|z| \leq \frac{1}{|b_n|}\{|b_n| - r(b_n + |b_n|) + b_2 + 2\delta + |b_2|\}.$$

Thus all the zeros of $D^{'}_\alpha f(z)$ lie in

$$|z| \leq \frac{1}{|b_n|}\{|b_n| - r(b_n + |b_n|) + b_2 + 2\delta + |b_2|\}.$$

In other words all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{1}{|b_n|}\{|b_n| - r(b_n + |b_n|) + b_2 + 2\delta + |b_2|\}.$$

are simple, where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$
$for \quad t = 2,3,4,...,n$

References

[1] Abdul Aziz, Q.G.Mahammad, On the zeros of certain class of polynomials and related analytic functions, *J. Math. Anal. Appl.*, 75(1980), 495–502.

[2] Abdul Aziz, Q.G.Mahammad, Zero free regions for polynomials and some generalizations of Enestrom Kakeya theorem, *Cana. Math. Bull.*, 27(3)91984), 265–272.

[3] Bairagi, Vinay kumar, Saha, T.K. Mishra, On the location of zeros of certain polynomials ,publication de L'institut mathematics, *Nuvelle serie, tome*, 99(113)(2016), 287–294.

[4] G.Eneström, Remarquee sur un théorème relatif aux racines de l'equation $a_n + ... + a_0 = 0$ oü tous les coefficient sont et positifs, *Tôhoku Math. J*, 18 (1920), 34–36.

[5] C. Gangadhar, P.Ramulu and G.L Reddy, Zero free regions of polar derivatives of polynomial with restricted coefficients, *IJPEM*, 4(30(2016), 67–74.

[6] M.H.Gulzar, B.A.Zargar, R.Akhter, On the zeros of the polar derivative of polynomial, *Communication Nonlinear Analysis*, 6(1)(2019), 32–39.

[7] S.KAKEYA, On the limits of the roots of an alegebraic equation with positive coefficient, *Tôhoku Math. J*, 2(1912-1913), 140–142.

[8] P. Ramulu and G.L Reddy, On the zeros of polar derivatives, *International Journal of Recent Research in Mathematics Computer Science and Information Technology*, 2(1)(2015), 143-145.

[9] G.L. Reddy, P.Ramulu and C.Gangadhar, On the zeros of polar derivatives of polynomial, *Journal of Research in Applied Mathematics*, 2(4)(2015), 07–10.

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