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A parameter uniform numerical method for a singularly perturbed initial value problem with Robin initial condition

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Abstract

In this paper an initial value problem for a singularly perturbed first order differential equation with Robin initial condition is considered on the interval $(0,1]$. A numerical method composed of a classical finite difference scheme on a piecewise uniform Shishkin mesh is suggested. This method is proved to be first - order convergent in the maximum norm uniformly in the perturbation parameters. A numerical illustration is provided to support the theory.

Keywords

Singular perturbation problems, Robin initial condition, Finite difference schemes, Shishkin mesh, Parameter uniform convergence.

AMS Subject Classification

34K10, 34K20, 34K26, 34K28..

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Contents

1. The Continuous Problem

Consider the initial value problem with robin boundary condition on the finite interval $[0,1]$

$$\varepsilon u'(x)+a(x)u(x)=f(x) \text{ for all } x\in\Omega, \tag{1.1}$$

$$u(0)-\varepsilon u'(0)=l \tag{1.2}$$

where $\Omega=(0,1]$, the functions $a(x), f(x)\in C^2(\overline{\Omega})$ and assume that the singular perturbation parameter ε satisfies $0<\varepsilon\leq 1$. It is assumed furthermore that the coefficient function satisfies the condition

$$a(x)\geq\alpha>0, \text{ for all } x\in\overline{\Omega}. \tag{1.3}$$

The above problem can be rewritten in the operator form

$$Lu=f \text{ on } \Omega \tag{1.4}$$

with

$$\beta u(0)=l \tag{1.5}$$

where the operators L,β are defined by

$$L=\varepsilon D+a,\quad \beta=I-\varepsilon D$$

where I is the identity operator, $D=\dfrac{d}{dx}$ is the first order differential operator. The reduced problem corresponding to (1.1) - (1.2) is

$$u_0(x)=\frac{f(x)}{a(x)}. \tag{1.6}$$

2. Analytical Results

The operator L satisfies the following maximum principle.

Lemma 2.1. *Let ψ be any function in the domain of L such that $\beta\psi(0) \geq 0$. Then $L\psi(x) \geq 0$ on $(0,1]$ implies that $\psi(x) \geq 0$ on $[0,1]$.*

Proof. Let x^* be a point such that $\psi(x^*) = \min_x \psi(x)$ and assume that $\psi(x^*) < 0$. For $x^* = 0$, then

$$\begin{aligned}\beta\psi(0) &= \psi(0) - \varepsilon\psi'(0)\\ &< 0, \text{ which is a contradiction.}\end{aligned}$$

Therefore, $x^* \neq 0$.
Suppose $x^* \in (0,1]$, then

$$\begin{aligned}L\psi(x^*) &= \varepsilon\psi'(x^*) + a(x^*)\psi(x^*)\\ &\leq a(x^*)\psi(x^*)\\ &< 0, \text{ which is a contradiction.}\end{aligned}$$

Hence our assumption $\psi(x^*) < 0$ is wrong. It follows that $\psi(x^*) \geq 0$ and thus that $\psi(x) \geq 0$, for all $x \in \overline{\Omega}$, which proves the lemma. □

As an immediate consequence of the above lemma the stability result is established in the following

Lemma 2.2. *If ψ is any function in the domain of L such that for each $x \in [0,1]$, then*

$$|\psi(x)| \leq \max\{||\beta\psi(0)||, \frac{1}{\alpha}||L\psi||\}$$

Proof. The following two functions are defined:

$$\theta^{\pm}(x) = \max\left\{\| \beta\psi(0) \|, \frac{1}{\alpha} \| L\psi \|\right\} \pm \psi(x)$$
$$\theta^{\pm}(x) = M \pm \psi(x)$$

where $M = \max\{||\beta\psi(0)||, \frac{1}{\alpha}||L\psi||\}$. Then, it is not hard to verify that $\beta\theta^{\pm}(0) \geq 0$ and $L\theta^{\pm}(x) \geq 0$ on Ω. It follows from Lemma 2.1 that $\theta^{\pm}(x) \geq 0$ on $\overline{\Omega}$. Hence,

$$|\psi(x)| \leq \max\{||\beta\psi(0)||, \frac{1}{\alpha}||L\psi||\}.$$

□

Lemma 2.3. *Let u be the solution of (1.1), (1.2). Then, there exists a constant C such that*

$$\begin{aligned}|u(x)| &\leq C\{\| l \| + \| f \|\}\\ |u'(x)| &\leq C\varepsilon^{-1}\{\| l \| + \| f \|\}\\ |u''(x)| &\leq C\varepsilon^{-2}\{\| l \| + \| f \| + \| f' \|\}\end{aligned}$$

Proof. From Lemma 2.2, it is evident that,

$$|u(x)| \leq ||\beta\psi(0)|| + \frac{1}{\alpha}||L\psi||.$$

Thus,

$$|u(x)| \leq C\{\| l \| + \| f \|\}.$$

From (1.1), we get

$$u'(x) = \varepsilon^{-1}(f(x) - a(x)u(x)).$$
$$\text{Hence, } |u'(x)| \leq C\varepsilon^{-1}(||l|| + ||f||).$$

Differentiating once the equation (1.1), we get

$$\varepsilon u''(x) + a(x)u'(x) = f'(x) - a'(x)u(x).$$

Using the bounds of u' and u

$$\begin{aligned}|u''(x)| \leq \varepsilon^{-1}[&|f'(x)| + C\varepsilon^{-1}(||l|| + ||f||)\\ &+ C(||l|| + ||f||)]\end{aligned}$$

and hence,

$$|u''(x)| \leq C\varepsilon^{-2}[||f'|| + ||l|| + ||f||].$$

□

3. The Shishkin Decomposition

The Shishkin decomposition of the solution u of (1.1) is given by

$$u = v + w \tag{3.1}$$

where the smooth component v of the solution u satisfies

$$Lv = f \text{ on } \Omega \tag{3.2}$$

with

$$\beta v(0) = u_0(0) - \varepsilon u_0'(0) \tag{3.3}$$

and the singular component w is the solution of

$$Lw(x) = 0 \text{ on } \Omega \tag{3.4}$$

with

$$\beta w(0) = l - \beta v(0). \tag{3.5}$$

Lemma 3.1. *The smooth component v, satisfies for $x \in \overline{\Omega}$,*

$$|v(x)| \leq C(1+\varepsilon), \ |v'(x)| \leq C, \ |v''(x)| \leq C\varepsilon^{-1}.$$

Proof. The smooth component v is further decomposed into

$$v = v_0 + \varepsilon v_1 \tag{3.6}$$

where v_0 is the solution of the reduced problem

$$v_0(x) = \frac{f(x)}{a(x)}. \tag{3.7}$$

The component v_1 satisfies the following equation

$$\varepsilon v_1' + a v_1 = -v_0'(x) \tag{3.8}$$

with

$$v_1(0) - \varepsilon v_1'(0) = 0. \tag{3.9}$$

From the expressions (3.7) and using Lemma 2.2, it is found that for $k = 1,2$,

$$|v_0^{(k)}(x)| \le C. \tag{3.10}$$

From (3.8) and (3.10), the following bounds hold:

$$|v_1^{(k)}(x)| \le C\varepsilon^{-k}, \ \ k = 1,2 \tag{3.11}$$

Substitute (3.10) and (3.11) in (3.6), we get

$$|v^{(k)}(x)| \le C\varepsilon^{-k}, \ \ k = 1,2$$

as required.

Bounds on the singular component w of u and its derivatives are contained in □

Lemma 3.2. *There exists a constant C, such that, for each* $x \in [0,1]$

$$|w(x)| \le Ce^{-\frac{\alpha x}{\varepsilon}}$$
$$|w'(x)| \le C\varepsilon^{-1}e^{-\frac{\alpha x}{\varepsilon}}$$
$$|w''(x)| \le C\varepsilon^{-2}e^{-\frac{\alpha x}{\varepsilon}}$$

Proof. To derive the bound of w, define the two functions

$$\psi^{\pm}(x) = Ce^{-\alpha x/\varepsilon} \pm w(x).$$

For a proper choice of C, $\beta\psi^{\pm}(0) \ge 0$ and for $x \in \Omega$

$$L\psi^{\pm}(x) = -C\varepsilon\frac{\alpha}{\varepsilon}e^{-\alpha x/\varepsilon} + Cae^{-\alpha x/\varepsilon} \ge 0.$$

By Lemma 2.2, $\psi^{\pm} \ge 0$ on $\overline{\Omega}$ and it follows that

$$|w(x)| \le Ce^{-\alpha x/\varepsilon}.$$

From (3.4) and differentiating (3.4) once, and using Lemma 2.2, it is not hard to see that

$$|w'(x)| \le C\varepsilon^{-1}e^{-\frac{\alpha x}{\varepsilon}}$$

and

$$|w''(x)| \le C\varepsilon^{-2}e^{-\frac{\alpha x}{\varepsilon}}$$

as required. □

4. The Shishkin mesh

A piecewise uniform Shishkin mesh $\Omega^N = \{x_j\}_1^N$ with N mesh-intervals is now constructed on $\overline{\Omega} = [0,1]$ as follows. A simpler construction requiring just one parameter τ suffices. The interval $[0,1]$ is subdivided into 2 sub-intervals $[0,\tau) \cup (\tau,1]$. The parameter τ which determine the point separating the uniform mesh, is defined by

$$\tau = \min\left\{\frac{1}{2}, \frac{\varepsilon}{\alpha}\ln N\right\} \tag{4.1}$$

Clearly,

$$0 < \tau \le \frac{1}{2}$$

Then, on each of the sub-intervals $[0,\tau)$ and $(\tau,1]$, a uniform mesh of $\frac{N}{2}$ mesh points is placed.

5. The Discrete Problem

The initial value problem (1.1) is discredited using the backward Euler scheme applied on the piecewise uniform fitted mesh $\overline{\Omega}^N = \{x_j\}_0^N$. The discrete problem is

$$L^N U(x_j) = \varepsilon D^- U(x_j) + a(x_j)U(x_j) = f(x_j), \quad j = 1(1)N \tag{5.1}$$

$$U(x_0) - \varepsilon D^+ U(x_0) = l. \tag{5.2}$$

The problem (5.1) can also be written in the operator form

$L^N U = f$ on Ω^N with

$\beta^N U(0) = l$

where $L^N = \varepsilon D^- + a$ with

$\beta^N = I - \varepsilon D^+$

and D^+, D^- are the difference operators

$$D^- U(x_j) = \frac{U(x_j) - U(x_{j-1})}{x_j - x_{j-1}}, \quad D^+ U(x_j) = \frac{U(x_{j+1}) - U(x_j)}{x_{j+1} - x_j}.$$

Lemma 5.1. *Let Ω^N be any mesh on $\overline{\Omega}$. Assume that the mesh function Φ_i satisfies $\beta\Phi_0 \ge 0$. Then $L^N\Phi_i \ge 0$, for all $1 \le i \le N$, implies that $\Phi_i \ge 0$, for all $0 \le i \le N$.*

Proof. Let k be such that $\Phi_k = \min_{0 \le i \le N} \Phi_i$ and assume that $\Phi_k < 0$. Suppose that $k = 0$, then

$$\beta^N\Phi_0 = \Phi_0 - \varepsilon D^+\Phi_0$$
$$< 0, \ \text{which is a contradiction}$$

Therefore, $k \ne 0$.
Suppose $\Phi_k \in \Omega^N$, then

$$L^N\Phi_k = \varepsilon D^-\Phi_k + a\Phi_k.$$

Then, clearly, $\Phi_k - \Phi_{k-1} \leq 0$ and so

$$L^N\Phi_k = \varepsilon\,\frac{\Phi_k - \Phi_{k-1}}{x_k - x_{k-1}} + \Phi_k < 0$$

which is false. Therefore $\Phi_k \geq 0$ and hence $\Phi_i \geq 0$, for all i, $0 \leq i \leq N$, as required. □

Lemma 5.2. *Let Ω^N be any mesh on $\overline{\Omega}$. Then, for any mesh function Φ, the following estimate holds for all $i, 0 \leq i \leq N$,*

$$|\Phi_i| \leq |\beta^N\Phi_0| + \max_{0\leq j\leq N} |L^N\Phi_j|.$$

Proof. Consider the two mesh functions

$$\Psi_i^{\pm} = |\Phi_0| + \max_{0\leq j\leq N} |L^N\Phi_j| \pm \Phi_i.$$

It is not hard to see that $\beta^N\Psi_0^{\pm} \geq 0$ and that $L^N\Psi_i^{\pm} \geq 0$. Applying the discrete maximum principle (Lemma 5.1) then gives $\Psi_i^{\pm} \geq 0$, and so

$$|\Phi_i| \leq |\Phi_0| + \max_{0\leq j\leq N} |L^N\Phi_j|$$

as required. □

6. The Local Truncation Error

From the discrete stability result, it is seen that in order to bound the error $U-u$, it suffices to bound $L^N(U-u)$. Notice that, for $x_j \in \Omega^N$,

$$\begin{aligned} L^N(u(x_j) - U(x_j)) &= L^N u(x_j) - L^N U(x_j) \\ &= L^N u(x_j) - f(x_j) \\ &= L^N u(x_j) - Lu(x_j) \\ &= \varepsilon\left(D^- - \frac{d}{dx}\right)u(x_j). \end{aligned}$$

Using integration by parts to reduce the order of differentiation in the integral, it is not hard to verify that

$$\left(D^- - \frac{d}{dx}\right)u(x_j) = \frac{1}{x_j - x_{j-1}}\int_{x_{j-1}}^{x_j} (x_{j-1} - s)u''(s)ds. \quad (6.1)$$

It follows that

$$\begin{aligned} \left|\left(D^- - \frac{d}{dx}\right)u(x_j)\right| &\leq \frac{|u|_2}{x_j - x_{j-1}}\int_{x_{j-1}}^{x_j}(s - x_{j-1})ds \\ &\leq \frac{1}{2}|u|_2(x_j - x_{j-1}). \end{aligned}$$

Therefore,

$$\left|L^N(u_\varepsilon(x_j) - U_\varepsilon(x_j))\right| \leq \left|\frac{\varepsilon}{2}|u|_2(x_j - x_{j-1})\right|. \quad (6.2)$$

Analogous to the continuous case, the discrete solution U can be decomposed into V and W which are defined to be solutions of the following discrete problems

$$(L^N V)(x_j) = \varepsilon D^- V(x_j) + a(x_j)V(x_j) = f(x_j) \text{ on } \Omega^N$$

$$V(0) - \varepsilon D^+ V(0) = v(0) - \varepsilon v'(0)$$

and

$$(L^N W)(x_j) = \varepsilon D^- W(x_j) + a(x_j)W(x_j) = 0 \text{ on } \Omega^N$$

$$W(0) - \varepsilon D^+ W(0) = w(0) - \varepsilon w'(0).$$

The error at each point $x_j \in \overline{\Omega}^N$ is denoted by $U(x_j) - u(x_j)$. Then the local truncation error $L^N(U-u)(x_j)$ has the decomposition

$$L^N(U-u)(x_j) = L^N(V-v)(x_j) + L^N(W-w)(x_j).$$

It is to be noted that for any smooth function ϕ, the following two distinct estimates of the local truncation of its first derivative hold.

$$|(D^- - D)\phi(x_j)| \leq 2\max_{s\in I_j}|\phi'(s)| \quad (6.3)$$

and

$$|(D^- - D)\phi(x_j)| \leq \frac{h_j}{2}\max_{s\in I_j}|\phi''(s)| \quad (6.4)$$

where $I_j = x_j - x_{j-1}$.
The error in the smooth and singular components are bounded in the following section.

7. Error estimate

The proof of the theorem on the error estimate is broken into two parts. First, a theorem concerning the error in the smooth component is established. Then the error in the singular component is established.

The following theorem gives the estimate of the error in the smooth component V.

Theorem 7.1. *Let v denote the smooth component of the solution of (1.1), (1.2) and V denote the smooth component of the solution of the problem (5.1), (5.2). Then*

$$|L^N(V-v)(x_j)| \leq CN^{-1}.$$

Proof. From the expression (6.4),

$$\begin{aligned} |\beta^N(V-v)(0)| &\leq C(x_1 - x_0)\max_{s\in[x_0,x_1]}|v''(s)| \quad (7.1)\\ &\leq CN^{-1}. \end{aligned}$$

From the differential and difference equations

$$\begin{aligned} L^N(V-v) &= L^N V - L^N v \\ &= Lv - L^N v \\ &= \varepsilon\left(D^- - \frac{d}{dx}\right)v. \end{aligned}$$

By the local truncation error, we have

$$\left|L^N(V-v)(x_j)\right| \le C\varepsilon(x_j - x_{j-1})|v|_2$$

It is to be noted that $x_j - x_{j-1} \le 2N^{-1}$ holds for all choices of the piecewise uniform mesh the estimate for v obtained above then yields

$$\begin{aligned}|L^N(V-v)(x_j)| &\le C2N^{-1} \\ &\le CN^{-1}.\end{aligned} \tag{7.2}$$

□

The following theorem gives the estimate of the error in the singular component W.

Theorem 7.2. *Let w be the singular component of the solution of* (1.1), (1.2) *and W be the singular component of the solution of the problem* (5.1), (5.2). *Then*

$$|L^N(W-w)(x_j)| \le CN^{-1}\ln N.$$

Proof. The solution argument depends on whether the transition parameter $\tau = \dfrac{1}{2}$ or $\tau = \dfrac{\varepsilon}{\alpha}\ln N$.

Case (i) $\tau = \dfrac{1}{2}$

When $\tau = \dfrac{1}{2}$, the mesh is uniform and it satisfies $\dfrac{\varepsilon}{\alpha}\ln N \ge \dfrac{1}{2}$. From the expression (6.4)

$$\begin{aligned}|\beta^N(W-w)(0)| &\le C\varepsilon(x_1 - x_0)\max_{s\in[x_0,x_1]}|w''(s)| \\ &\le CN^{-1}\ln N.\end{aligned} \tag{7.3}$$

The solution argument used above then yields

$$\left|L^N(W-w)(x_j)\right| \le C\varepsilon(x_j - x_{j-1})|w|_2.$$

Since $x_j - x_{j-1} \le 2N^{-1}$, the estimate for $|w|_2$ obtained which gives

$$\begin{aligned}\left|L^N(W-w)(x_j)\right| &\le \frac{\varepsilon}{2}(x_j - x_{j-1})|w|_2 \\ &\le \varepsilon N^{-1}C\varepsilon^{-2}e^{-\frac{\alpha x}{\varepsilon}} \\ &\le C\varepsilon^{-1}N^{-1}.\end{aligned}$$

Therefore,

$$\left|L^N(W-w)(x_j)\right| \le CN^{-1}\ln N, \text{ since } \varepsilon^{-1} \le \frac{2\ln N}{\alpha}. \tag{7.4}$$

Case (ii): $\tau = \dfrac{\varepsilon}{\alpha}\ln N$

In the second case the mesh is peicewise uniform, with the mesh spacing $\dfrac{2\tau}{N}$ in the subinterval $[0,\tau]$ and $\dfrac{2(1-\tau)}{N}$ in the subinterval $[\tau,1]$. A different argument is used to bound $|W_\varepsilon - w_\varepsilon|$ in each of these subintervals.

In the subinterval $[\tau,1]$, with no boundary layer, both W_ε and w_ε are small, and because $|W_\varepsilon - w_\varepsilon| \le |W_\varepsilon| + |w_\varepsilon|$, it suffices to bound w_ε and W_ε separately.

Sub - case (i): For the subinterval $[0,\tau]$.

Since the mesh is peicewise uniform, with the mesh spacing $\dfrac{2\tau}{N}$ in the subinterval $[0,\tau]$. It is to be noted that

$$\begin{aligned}|\beta^N(W-w)(0)| &= W(0) - \varepsilon D^+W(0) - w(0) + \varepsilon D^+ w(0) \\ &= \varepsilon\left[D^+ - \frac{d}{dx}\right]w(x_0) \\ &\le \varepsilon C(x_1 - x_0)|w|_2 \\ &\le \varepsilon C\frac{2\tau}{N}C\varepsilon^{-2}e^{-\frac{\alpha x}{\varepsilon}} \\ &\le C\varepsilon^{-1}N^{-1}\frac{\varepsilon}{\alpha}\ln N \\ &\le CN^{-1}\ln N.\end{aligned}$$

The classical argument leads, to the following estimate of the local truncation error

$$\begin{aligned}|L^N(W-w)(x_j)| &\le \frac{\varepsilon}{2}(x_j - x_{j-1})|w|_2 \\ &\le \frac{\varepsilon}{2}\frac{2\tau}{N}C\varepsilon^{-2}e^{-\frac{\alpha x}{\varepsilon}} \\ &\le CN^{-1}\ln N, \quad \text{since } \tau = \frac{\varepsilon}{\alpha}\ln N.\end{aligned}$$

The above estimates show that, in the interval $[0,\tau]$

$$|(W-w)(x_j)| \le CN^{-1}\ln N. \tag{7.5}$$

Sub - case (ii): For the subinterval $[\tau,1]$

From Lemma 2.2 it is not hard to see that

$$|w(x_j)| \le CN^{-1} \text{ for } j \ge \frac{N}{2}$$

Consider the barrier function

$$\psi^\pm(x_j) = CB^N(x_j) \pm W(x_j), \quad 0 \le j \le N.$$

where

$$B^N(x_j) = \prod_{i=1}^{j}\left(1 + \frac{\alpha h_i}{\varepsilon}\right)^{-1}$$

By applying Lemma 5.2, it can be show that

$$\begin{aligned}L^N\psi^\pm(x_j) &= \varepsilon D^-\psi(x_j) + a(x_j)\psi(x_j) \\ &= \varepsilon D^-B^N(x_j) + a(x_j)B^N(x_j) \pm L^N W(x_j) \\ &= \varepsilon D^-B^N(x_j) + a(x_j)B^N(x_j) \\ &= -\alpha B^N(x_{j-1}) + a(x_j)B^N(x_j) \\ &\ge -\alpha B^N(x_{j-1}) + a(x_j)B^N(x_{j-1}) \\ &\ge 0.\end{aligned}$$

It is not hard to find that

$$\begin{aligned}\beta^N \psi^{\pm}(0) &= C\beta^N(B^N(x_0)) \pm B^N(W(x_0)) \\ &= B^N(x_0) - \varepsilon\left[\frac{B^N(x_1) - B^N(x_0)}{x_1 - x_0}\right] \\ &\geq 0.\end{aligned}$$

Hence, $L^N\psi^{\pm} \geq 0$ and $\beta^N\psi^{\pm} \geq 0$, by applying Lemma 5.1 we have $\psi^{\pm}(x_j) \geq 0,\ 0 \leq j \leq N$.

Hence,

$$|W(x_j)| \leq C\prod_{i=1}^{j}\left(1+\frac{\alpha h_i}{\varepsilon}\right)^{-1}$$

Then,

$$\begin{aligned}&|W(x_j)| \\ &\leq C\left(1+\frac{\alpha h_i}{\varepsilon}\right)^{-\frac{N}{2}},\ j \geq \frac{N}{2} \\ &\leq C\left(1+\frac{2\ln N}{N}\right)^{-\frac{N}{2}} (\text{since}\, h_1 = \frac{2\tau}{N}, \tau = \frac{\varepsilon}{\alpha}\ln N) \\ &\leq CN^{-1}\end{aligned}$$

as required. □

Theorem 7.3. *Let u be the solution of the continuous problem* (1.1)*,* (1.2) *and U be the solution of the discrete problem* (5.1)*,* (5.2)*. Then*

$$||U-u|| \leq CN^{-1}$$

Proof. From Lemma 5.2, it is clear that, in order to prove the above theorem it suffices to to prove that $||(L^N(U-u))|| \leq CN^{-1}$. But, $||(L^N(U-u))|| \leq ||(L^N(V-v))|| + ||(L^N(W-w))||$. Hence using theorems 7.1 and 7.2, the above result is derived. □

8. Numerical Illustration

The numerical method proposed above is illustrated through an example presented in this section.

Example 8.1. *Consider the initial value problem*

$$\varepsilon u'(x) + (1+x)u(x) = 5, \quad \forall\, x \in (0,1] \ \textit{with}$$

$$u(0) - \varepsilon u'(0) = 2.$$

The numerical solution obtained by applying the fitted mesh method (5.1) and (5.2) to the Example is shown in Figure 1. The order of convergence and the error constant are calculated and are presented in Table 1.

Table 1. Values of D_{ε}^N, D^N, p^N, p^* and $C_{p^*}^N$ generated for the example

η	Number of mesh points N				
	32	64	128	256	512
0.100E+01	0.193E+00	0.145E+00	0.781E-01	0.405E-01	0.206E-01
0.125E+00	0.159E+00	0.109E+00	0.694E-01	0.420E-01	0.245E-01
0.156E-01	0.156E+00	0.107E+00	0.680E-01	0.411E-01	0.240E-01
0.195E-02	0.156E+00	0.107E+00	0.678E-01	0.410E-01	0.240E-01
0.244E-03	0.156E+00	0.107E+00	0.678E-01	0.410E-01	0.239E-01
D^N	0.193E+00	0.145E+00	0.781E-01	0.420E-01	0.245E-01
p^N	0.411E+00	0.895E+00	0.895E+00	0.776E+00	
C_p^N	0.324E+01	0.324E+01	0.232E+01	0.166E+01	0.129E+01
The order of $\vec{\varepsilon}$ -uniform convergence $p^* = 0.411E+00$					
Computed $\vec{\varepsilon}$ -uniform error constant, $C_{p^*}^N = 0.324E+01$					

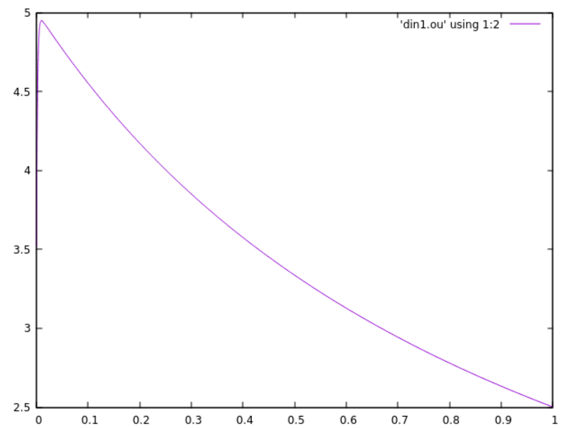

Figure 1

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