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Solutions of ternary quadratic Diophantine equations $x^2+y^2\pm\lambda y=z^2$

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Abstract
The infinite integer solutions of the ternary quadratic Diophantine equations $x^2+y^2+\lambda y=z^2$ and $x^2+y^2-\lambda y=z^2$ are investigated in this study. It is shown that when $\lambda=2\beta,\beta\in Z_+$, $x^2+y^2\pm\lambda y=z^2$ has infinitely many pure integer solutions but the equations $x^2+y^2\pm\lambda y=z^2$ has infinitely many mixed integer solutions when $\lambda=2\beta+1,\beta\in Z_{\geqslant}$. A few interesting relations between solutions are also exhibited in this work.

Keywords
Diophantine Equation, Pell's Equation, Hyperbola.

AMS Subject Classification
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Contents

1. Introduction

Let $\lambda=2\beta$ and β be a positive integer. Let H_λ be the hyperbola of 3 dimension defined by the equations

$$x^2+y^2\pm\lambda y=z^2, x,y,z\in Z \tag{1.1}$$

The both equations have always the integer solution (0, 0, 0) and the second equation $x^2+y^2-\lambda y=z^2$ only has one more integer solution $(0,\lambda,0)$ which are called trivial solutions.

In [6], Pingzhi Yuana and Yongzhong Hu proved that the Diophantine equation $x^2-kxy+y^2+lx=0$, $l\in\{1,2,4\}$ has infinite number of positive integer solutions. This equation is one of many variety of (1.1). Moreover, the equation (1.1) becomes a Pythagorean equation when $\lambda=0$.

In this work we deal with this equation $x^2+y^2\pm\lambda y=z^2$, $x,y,z\in Z$ for $\lambda>0$ in order to find the infinite integer solutions based on Pell's equation. Using the generalized solution of Pell's Equation, we prove the following results.

Theorem 1.1. *The equation $x^2+y^2\pm\lambda y=z^2$ has infinitely many pure integer solutions when $\lambda=2\beta$, $\beta\in Z_+$.*

Theorem 1.2. *The equation $x^2+y^2\pm\lambda y=z^2$ has infinitely many mixed integer solutions when $\lambda=2\beta+1$, $\beta\in Z_{\geqslant}$.*

In order to prove the above theorems, the generalized solutions of the proposed equations $x^2+y^2\pm\lambda y=z^2$ are found with the help of Pell's equation.

The equation to be solved is

$$x^2+y^2\pm\lambda y=z^2$$

It is to be solved through the following two cases:

2. The case $x=y-a$, $a\neq 0$

Substitute $x=y-a$ in equation (1.1), we have

$$2y^2-(2a\mp\lambda)y+(a^2-z^2)=0$$

Treating this as quadratic in y, we obtain

$$y=\frac{(2a\mp\lambda)\pm\sqrt{(2a\mp\lambda)^2-8(a^2-z^2)}}{4}$$

After simplification, we get the Pell's Equation

$$(4y-(2a\mp\lambda))^2-8z^2=\lambda^2-4a(a\pm\lambda)$$

The above equation can be written as

$$Y^2-8z^2=N \tag{2.1}$$

where $Y=4y-(2a\mp\lambda)$, $N=\lambda^2-4a(a\pm\lambda)$ and $\lambda=2\beta$, $\beta\in Z_+$.

The general solution of equation (1.1)

$$y_n=\frac{1}{4}\left\{\frac{1}{2}\left[\left(3+\sqrt{8}\right)^n\left(Y_0+\sqrt{8}z_0\right)+\left(3-\sqrt{8}\right)^n\left(Y_0-\sqrt{8}z_0\right)\right]+(2a\mp\lambda)\right\} \tag{2.2}$$

$$z_n=\frac{1}{2\sqrt{8}}\left[\left(3+\sqrt{8}\right)^n\left(Y_0+\sqrt{8}z_0\right)-\left(3-\sqrt{8}\right)^n\left(Y_0-\sqrt{8}z_0\right)\right] \tag{2.3}$$

$n=0,1,2,\dots$
where $\left(Y_0+\sqrt{8}z_0\right)$ is the fundamental solution of (2.1) and $\left(3+\sqrt{8}\right)$ is the fundamental solutions of $Y^2-8z^2=1$.

Example 2.1. *If a = 1, then substitute $x=y-1$ in equation (1.1), we have*

$$2y^2-(2\mp\lambda)y+(1-z^2)=0$$

Treating this as quadratic in y, we obtain

$$y=\frac{(2\mp\lambda)\pm\sqrt{(2\mp\lambda)^2-8(1-z^2)}}{4}$$

After simplification, we get the Pell's equation

$$(4y-(2\mp\lambda))^2-8z^2=\lambda^2-4(1\pm\lambda)$$

The above equation can be written as

$$Y^2-8z^2=N_1 \tag{2.4}$$

where $Y=4y-(2\mp\lambda)$, $N_1=\lambda^2-4(1\pm\lambda)$ and $\lambda=2\beta$, $\beta\in Z_+$

The general solution of equation (1.1)

$$y_n=\frac{1}{4}\left[\frac{1}{2}\left[\left(3+\sqrt{8}\right)^n\left(Y_0+\sqrt{8}z_0\right)+\left(3-\sqrt{8}\right)^n\left(Y_0-\sqrt{8}z_0\right)\right]+(2\mp\lambda)\right] \tag{2.5}$$

$$z_n=\frac{1}{2\sqrt{8}}\left[\left(3+\sqrt{8}\right)^n\left(Y_0+\sqrt{8}z_0\right)-\left(3-\sqrt{8}\right)^n\left(Y_0-\sqrt{8}z_0\right)\right] \tag{2.6}$$

$n=0,1,2,\dots$
where $\left(Y_0+\sqrt{8}z_0\right)$ is the fundamental solution of (2.4) and $\left(3+\sqrt{8}\right)$ is the fundamental solutions of $Y^2-8z^2=1$.

Table 1

n	Solution of $\left(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)}\right)$		
	$\lambda=2, N=-8$	$\lambda=4, N=-4$	$\lambda=6, N=8$
0	(1, 2, 3)	(-1, 0, 1)	(-1, 0, 1)
1	(11, 12, 17)	(2, 3, 5)	(3, 4, 7)
2	(69, 70, 99)	(19, 20, 29)	(27, 28, 41)
3	(407, 408, 577)	(118, 119, 169)	(167, 168, 239)
4	(2377, 2378, 3363)	(695, 696, 985)	(983, 984, 1393)
5	(13859, 13860, 19601)	(4058, 4059, 5741)	(5739, 5740, 8119)

Table 2

n	Solution of $\left(x_{(n,d)},y_{(n,d)},z_{(n,d)}\right)$		
	$\lambda=8, N=28$	$\lambda=10, N=56$	$\lambda=12, N=92$
0	(-1, 0, 1)	(-1, 0, 1)	(-1, 0, 1)
1	(4, 5, 9)	(5, 6, 11)	(6, 7, 13)
2	(35, 36, 53)	(43, 44, 65)	(51, 52, 77)
3	(216, 217, 309)	(265, 266, 379)	(314, 315, 449)
4	(1271, 1272, 1801)	(1559, 1560, 2209)	(1847, 1848, 2617)
5	(7420, 7421, 10497)	(9101, 9102, 12875)	(10782, 10783, 15253)

For the sake of simplicity a few solution of $x^2+y^2+\lambda y=z^2$ for $N=-8,-4,8,28,\dots$ are presented in Tables 1 and 2 and a few solution of $x^2+y^2-\lambda y=z^2$ for $N=8,28,56,92,\dots$. are presented in Tables 3 and 4.

Further the solutions satisfy the following recurrence relation

(a) Recurrence relations for solution $\left(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)}\right)$ among the different values of λ

  (i) $x_{(n,\lambda)}-2x_{(n,\lambda+2)}+x_{(n,\lambda+4)}=0$, where $\lambda=2\beta$, $\beta>2$

  (ii) $y_{(n,\lambda)}-2y_{(n,\lambda+2)}+y_{(n,\lambda+4)}=0$, where $\lambda=2\beta$, $\beta>2$

  (iii) $z_{(n,\lambda)}-2z_{(n,\lambda+2)}+z_{(n,\lambda+4)}=0$, where $\lambda=2\beta$, $\beta>2$

  In particular $n=4,\lambda=6$ when $\beta=3$

  (i) $x_{(4,6)}-2x_{(4,8)}+x_{(4,10)}=0$

  (ii) $y_{(4,6)}-2y_{(4,8)}+y_{(4,10)}=0$

  (iii) $z_{(4,6)}-2z_{(4,8)}+z_{(4,10)}=0$

(b) Recurrence relations for solution $\left(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)}\right)$ among the different values of $\lambda=2,4,6$.

  (i) $x_{(n,2)}-2x_{(n,4)}-x_{(n,6)}-4=0$

  (ii) $y_{(n,2)}-2y_{(n,4)}-y_{(n,6)}-2=0$

  (iii) $z_{(n,2)}-2z_{(n,4)}-z_{(n,6)}=0$

(c) Recurrence relations for solution $\left(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)}\right)$ among the particular values of λ.

  (i) $x_{(n-1,\lambda)}-6x_{(n,\lambda)}+x_{(n+1,\lambda)}-(\lambda+2)=0$, where $n>0$

  (ii) $y_{(n-1,\lambda)}-6y_{(n,\lambda)}+y_{(n+1,\lambda)}-(\lambda-2)=0$, where $n>0$

(iii) $z_{(n-1,\lambda)}-6z_{(n,\lambda)}+z_{(n+1,\lambda)}=0$,
where $n>0$.

In particular $n=4,\lambda=6$ when $\beta=3$
$x_{(3,6)}-6x_{(4,6)}+x_{(5,6)}-(6+2)=0$
$y_{(3,6)}-6y_{(4,6)}+y_{(5,6)}+(6-2)=0$
$z_{(3,6)}-6z_{(4,6)}+z_{(5,6)}=0.$

(d) Recurrence relations for solution $\left(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)}\right)$ among the particular values of λ

$2\left(x_{(n,\lambda)}+x_{(n+1,\lambda)}+z_{(n,\lambda)}-z_{(n+1,\lambda)}+1\right)=-\lambda$
$2\left(y_{(n,\lambda)}+y_{(n+1,\lambda)}+z_{(n,\lambda)}-z_{(n+1,\lambda)}-1\right)=-\lambda.$

In particular $n=4,\lambda=6$ when $\beta=3$
$2\left(x_{(4,6)}+x_{(5,6)}+z_{(4,6)}-z_{(5,6)}+1\right)=-6$
$2\left(y_{(4,6)}+y_{(5,6)}+z_{(4,6)}-z_{(5,6)}-1\right)=-6$

Table 3

n	Solution of $\left(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)}\right)$		
	$\lambda=2, N=8$	$\lambda=4, N=28$	$\lambda=6, N=56$
0	(1, 2, 1)	(2, 3, 1)	(3, 4, 1)
1	(5, 6, 7)	(7, 8, 9)	(9, 10, 11)
2	(29, 30, 41)	(38, 39, 53)	(47, 48, 65)
3	(169, 170, 239)	(219, 220, 309)	(269, 270, 379)
4	(985, 986, 1393)	(1274, 1275, 1801)	(1563, 1564, 2209)
5	(5741, 5742, 8119)	(7423, 7424, 10497)	(9105, 9106, 12875)

Table 4

n	Solution of $\left(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)}\right)$		
	$\lambda=8, N=92$	$\lambda=10, N=136$	$\lambda=12, N=188$
0	(4, 5, 1)	(5, 6, 1)	(6, 7, 1)
1	(11, 12, 13)	(13, 14, 15)	(15, 16, 17)
2	(56, 57, 77)	(65, 66, 89)	(74, 75, 101)
3	(319, 320, 449)	(369, 370, 519)	(419, 420, 589)
4	(1852, 1853, 2617)	(2141, 2142, 3025)	(2430, 2431, 3433)
5	(10787, 10788, 15253)	(12469, 12470, 17631)	(14151, 14152, 20009)

Further the solutions satisfy the following recurrence relation

(a) Recurrence relations for solution $\left(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)}\right)$ among the different values of λ

(i) $x_{(n,\lambda)}-2x_{(n,\lambda+2)}+x_{(n,\lambda+4)}=0$,
where $\lambda=2\beta$, $\beta\in Z_+$

(ii) $y_{(n,\lambda)}-2y_{(n,\lambda+2)}+y_{(n,\lambda+4)}=0$,
where $\lambda=2\beta$, $\beta\in Z_+$

(iii) $z_{(n,\lambda)}-2z_{(n,\lambda+2)}+z_{(n,\lambda+4)}=0$,
where $\lambda=2\beta$, $\beta\in Z_+$

In particular $n=4,\lambda=6$ when $\beta=3$

(i) $x_{(4,6)}-2x_{(4,8)}+x_{(4,10)}=0$

(ii) $y_{(4,6)}-2y_{(4,8)}+y_{(4,10)}=0$

(iii) $z_{(4,6)}-2z_{(4,8)}+z_{(4,10)}=0$

(b) Recurrence relations for solution $\left(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)}\right)$ among the particular values of λ

(i) $x_{(n-1,\lambda)}-6x_{(n,\lambda)}+x_{(n+1,\lambda)}+\lambda-2=0$,
where $n>0$

(ii) $y_{(n-1,\lambda)}-6y_{(n,\lambda)}+y_{(n+1,\lambda)}+\lambda+2=0$,
where $n>0$

(iii) $z_{(n-1,\lambda)}-6z_{(n,\lambda)}+z_{(n+1,\lambda)}=0$ where $n>0$,

In particular $n=4,\lambda=6$ when $\beta=3$
$x_{(3,6)}-6x_{(4,6)}+x_{(5,6)}+6-2=0$
$y_{(3,6)}-6y_{(4,6)}+y_{(5,6)}+6+2=0$
$z_{(3,6)}-6z_{(4,6)}+z_{(5,6)}=0.$

(c) Recurrence relations for solution $\left(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)}\right)$ among the particular values of λ
$2\left(x_{(n,\lambda)}+x_{(n+1,\lambda)}+z_{(n,\lambda)}-z_{(n+1,\lambda)}+1\right)=\lambda$
$2\left(y_{(n,\lambda)}+y_{(n+1,\lambda)}+z_{(n,\lambda)}-z_{(n+1,\lambda)}-1\right)=\lambda.$

In particular $n=4,\lambda=6$ when $\beta=3$
$2\left(x_{(4,6)}+x_{(5,6)}+z_{(4,6)}-z_{(5,6)}+1\right)=6$
$2\left(y_{(4,6)}+y_{(5,6)}+z_{(4,6)}-z_{(5,6)}-1\right)=6$
For the sake of clarity the geometrical representation of $x^2+y^2+\lambda y=z^2$ for $\lambda=2$ has been shown in Fig. 1:

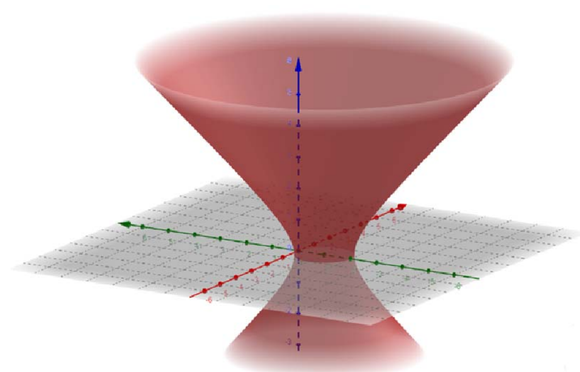

Figure 1. Geometrical representation of $x^2+y^2+2y=z^2$.

Proposition 2.1. *If x, y, and z are relatively prime integers such that $x^2+y^2\pm\lambda y=z^2$, $\lambda=2\beta$, and $x<y$, then x is odd, y is even, and z is odd when β is odd.*

Proposition 2.2.

1. *If x, y, and z are relatively prime integers such that $x^2+y^2+\lambda y=z^2$, $\lambda=2\beta$, and $x<y$, then x is even, y and z are odd when β is even and n is odd.*

2. *If x, y, and z are relatively prime integers such that $x^2+y^2+\lambda y=z^2$, $\lambda=2\beta$, and $x<y$, then x and z are odd and y is even when β is even and n is zero and even.*

Proposition 2.3.

1. *If x, y, and z are relatively prime integers such that* $x^2+y^2-\lambda y=z^2$, $\lambda=2\beta$, *and* $x<y$, *then x and z are odd, y is even when* β *is even and n is odd.*

2. *If x, y, and z are relatively prime integers such that* $x^2+y^2-\lambda y=z^2$, $\lambda=2\beta$, *and* $x<y$, *then x is even and y and z are odd when* β *is even and n is zero and even.*

Proof. The assertions of the above propositions are easily checked from Tables 1 1 to 4.

For the sake of simplicity a few solution of $x^2+y^2+\lambda y=z^2, \lambda=2\beta+1, \beta\in Z_{\geqslant}$ for $N=-7,-7,1,\ldots$ are presented in Table 5 which are mixed integers in nature. Similarly the equation $x^2+y^2-\lambda y=z^2$, $\lambda=2\beta+1$, $\beta\in Z_{\geqslant}$ also having mixed integer solutions for $N=1,17,41,\ldots$. □

Table 5

n	Solution of $(x_{(n,\lambda)}, y_{(n,\lambda)}, z_{(n,\lambda)})$ $\lambda=1, N=-7$	$\lambda=3, N=-7$	$\lambda=5, N=1$
0	(-0.5, 0.5, 1)	(-1, 0, 1)	(-1, 0, 1)
1	(2, 3, 4)	(1.5, 2.5, 4)	(2.5, 3.5, 6)
2	(15.5, 16.5, 23)	(15, 16, 23)	(23, 24, 35)
3	(94, 95, 134)	(93.5, 94.5, 134)	(142.5, 143.5, 204)
4	(551.5, 552.5, 781)	(551, 552, 781)	(839, 840, 1189)
5	(3218, 3219, 4552)	(3217.5, 3218.5, 4552)	(4898.5, 4899.5, 6930)

The present work is discussing about only the pure integer solutions of the proposed equations of this study. So that mixed integer solutions are not discussed in detail.

In the same way, the mixed integer solutions are arrived for $\lambda=2\beta+1$ in the following case which also has not been discussed in this work as per its objective.

3. The Case $x=y+a, a\neq 0$

Substitute $x=y+a$ in equation (1.1), we have

$$2y^2+(2a\pm\lambda)y+(a^2-z^2)=0$$

Treating this as quadratic in y, we obtain

$$y=\frac{-(2a\pm\lambda)\pm\sqrt{(2a\pm\lambda)^2-8(a^2-z^2)}}{4}$$

After simplification, we get the Pell's equation

$$(4y+(2a\pm\lambda))^2-8z^2=\lambda^2-4a(a\mp\lambda)$$

The above equation can be written as

$$Y^2-8z^2=N' \tag{3.1}$$

where $Y=4y+(2a\pm\lambda)$, $N'=\lambda^2-4a(\lambda\mp a)$ and $\lambda=2\beta, \beta\in Z_+$.

The general solution of equation (1.1)

$$y_n=\frac{1}{4}\left[\frac{1}{2}\left[\left(3+\sqrt{8}\right)^n\left(Y_0+\sqrt{8}z_0\right)+\left(3-\sqrt{8}\right)^n\left(Y_0-\sqrt{8}z_0\right)\right]-(2a\pm\lambda)\right] \tag{3.2}$$

$$z_n=\frac{1}{2\sqrt{8}}\left[\left(3+\sqrt{8}\right)^n\left(Y_0+\sqrt{8}z_0\right)-\left(3-\sqrt{8}\right)^n\left(Y_0-\sqrt{8}z_0\right)\right] \tag{3.3}$$

$n=0,1,2,\ldots$
where $\left(Y_0+\sqrt{8}z_0\right)$ is the fundamental solution of (3.1) and $\left(3+\sqrt{8}\right)$ is the fundamental solutions of $Y^2-8z^2=1$.

Example 3.1. *If* $a=1$ *then substitute* $x=y+1$ *in equation (1.1), we have*

$$2y^2+(2\pm\lambda)y+(1-z^2)=0$$

Treating this as quadratic in y, we obtain

$$y=\frac{-(2\pm\lambda)\pm\sqrt{(2\pm\lambda)^2-8(1-z^2)}}{4}$$

After simplification, we get the Pell's equation

$$(4y+(2\pm\lambda))^2-8z^2=\lambda^2-4(1\mp\lambda)$$

The above equation can be written as

$$Y^2-8z^2=N_1' \tag{3.4}$$

where $Y=4y+(2\pm\lambda)$, $N_1'=\lambda^2-4(1\pm\lambda)$ *and* $\lambda=2\beta$, $\beta\in Z_+$

The general solution of equation (1.1)

$$y_n=\frac{1}{4}\left[\frac{1}{2}\left[\left(3+\sqrt{8}\right)^n\left(Y_0+\sqrt{8}z_0\right)+\left(3-\sqrt{8}\right)^n\left(Y_0-\sqrt{8}z_0\right)\right]-(2\pm\lambda)\right] \tag{3.5}$$

$$z_n=\frac{1}{2\sqrt{8}}\left[\left(3+\sqrt{8}\right)^n\left(Y_0+\sqrt{8}z_0\right)-\left(3-\sqrt{8}\right)^n\left(Y_0-\sqrt{8}z_0\right)\right] \tag{3.6}$$

$n=0,1,2,\ldots$
where $\left(Y_0+\sqrt{8}z_0\right)$ *is the fundamental solution of* (3.4) *and* $\left(3+\sqrt{8}\right)$ *is the fundamental solutions of* $Y^2-8z^2=1$.

For the sake of simplicity a few solution of (1.1) *for* $N=8,28,56,92,\ldots$ *are presented in Tables 6 and 7 and* $N=-8,-4,8,28,\ldots$ *are presented in Tables 8 and 9.*

Further the solutions satisfy the following recurrence relation

Table 6

n	Solution of $(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)})$		
	$\lambda=2, N=8$	$\lambda=4, N=28$	$\lambda=6, N=56$
0	(1, 0, 1)	(1, 0, 1)	(1, 0, 1)
1	(5, 4, 7)	(6, 5, 9)	(7, 6, 11)
2	(29, 28, 41)	(37, 36, 53)	(45, 44, 65)
3	(169, 168, 239)	(218, 217, 309)	(267, 266, 379)
4	(985, 984, 1393)	(1273, 1272, 1801)	(1561, 1560, 2209)
5	(5741, 5740, 8119)	(7422, 7421, 10497)	(9103, 9102, 12875)

Table 7

n	Solution of $(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)})$		
	$\lambda=8, N=92$	$\lambda=10, N=136$	$\lambda=12, N=188$
0	(1, 0, 1)	(1, 0, 1)	(1, 0, 1)
1	(8, 7, 13)	(9, 8, 15)	(10, 9, 17)
2	(53, 52, 77)	(61, 60, 89)	(69, 68. 101)
3	(316, 315, 449)	(365, 364, 519)	(414, 413, 589)
4	(1849, 1848, 2617)	(2137, 2136, 3025)	(2425, 2423, 3433)
5	(10784, 10783, 15253)	(12465, 12464, 17631)	(14146, 14145, 20009)

(a) Recurrence relations for solution $(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)})$ among the different values of λ

(i) $x_{(n,\lambda)}-2x_{(n,\lambda+2)}+x_{(n,\lambda+4)}=0$, where $\lambda=2\beta$, $\beta\in Z_+$

(ii) $y_{(n,\lambda)}-2y_{(n,\lambda+2)}+y_{(n,\lambda+4)}=0$, where $\lambda=2\beta$, $\beta\in Z_+$

(iii) $z_{(n,\lambda)}-2z_{(n,\lambda+2)}+z_{(n,\lambda+4)}=0$, where $\lambda=2\beta$, $\beta\in Z_+$

In particular $n=4,\lambda=6$ when $\beta=3$

(i) $x_{(4,6)}-2x_{(4,8)}+x_{(4,10)}=0$

(ii) $y_{(4,6)}-2y_{(4,8)}+y_{(4,10)}=0$

(iii) $z_{(4,6)}-2z_{(4,8)}+z_{(4,10)}=0$

(b) Recurrence relations for solution $(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)})$ among the particular values of λ

(i) $x_{(n-1,\lambda)}-6x_{(n,\lambda)}+x_{(n+1,\lambda)}-(\lambda-2)=0$, where $n>0$

(ii) $y_{(n-1,\lambda)}-6y_{(n,\lambda)}+y_{(n+1,\lambda)}-(\lambda+2)=0$, where $n>0$

(iii) $z_{(n-1,\lambda)}-6z_{(n,\lambda)}+z_{(n+1,\lambda)}=0$, where $n>0$.

In particular $n=4,\lambda=6$ when $\beta=3$
$x_{(3,6)}-6x_{(4,6)}+x_{(5,6)}-(6-2)=0$
$y_{(3,6)}-6y_{(4,6)}+y_{(5,6)}-(6+2)=0$
$z_{(3,6)}-6z_{(4,6)}+z_{(5,6)}=0$

(c) Recurrence relations for solution $(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)})$ among the particular values of λ

(i) $x_{(n,\lambda)}+x_{(n+1,\lambda)}+z_{(n,\lambda)}-z_{(n+1,\lambda)}-(\beta+1)=-\lambda$, where $\lambda=2\beta$, $\beta\in Z_+$

(ii) $y_{(n,\lambda)}+y_{(n+1,\lambda)}+z_{(n,\lambda)}-z_{(n+1,\lambda)}-(\beta-1)=-\lambda$, where $\lambda=2\beta$, $\beta\in Z_+$.

In particular $n=4,\lambda=6$ when $\beta=3$
$x_{(4,6)}+x_{(5,6)}+z_{(4,6)}-z_{(5,6)}-4=-6$
$y_{(4,6)}+y_{(5,6)}+z_{(4,6)}-z_{(5,6)}-2=-6.$

Table 8

n	Solution of $(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)})$		
	$\lambda=2, N=-8$	$\lambda=4, N=-4$	$\lambda=6, N=8$
0	(3, 2, 3)	(2, 1, 1)	(3, 2, 1)
1	(13, 12, 17)	(5, 4, 5)	(7, 6, 7)
2	(71, 70, 99)	(22, 21, 29)	(31, 30. 41)
3	(409, 408, 577)	(121, 120, 169)	(171, 170, 239)
4	(2379, 2378, 3363)	(698, 697, 985)	(987, 986, 1393)
5	(13861, 13860, 19601)	(4061, 4060, 5741)	(5743, 5742, 8119)

Table 9

n	Solution of $(x_{(n,\lambda)},y_{(n,\lambda)},z_{(n,\lambda)})$		
	$\lambda=8, N=28$	$\lambda=10, N=56$	$\lambda=12, N=92$
0	(4, 3, 1)	(5, 4, 1)	(6, 5, 1)
1	(9, 8, 9)	(11, 10, 11)	(13, 12, 13)
2	(40, 39, 53)	(49, 48, 65)	(58, 57, 77)
3	(221, 220, 309)	(271, 270, 379)	(321, 320, 449)
4	(1276, 1275, 1801)	(1565, 1564, 2209)	(1854, 1853, 2617)
5	(7425, 7424, 10497)	(9107, 9106, 12875)	(10789, 10788, 15253)

Further the solutions satisfy the following recurrence relation (see Tables 8 and 9).

(a) Recurrence relations for solution $\left(x_{(n,\lambda)}, y_{(n,\lambda)}, z_{(n,\lambda)}\right)$ among the different values of λ

(i) $x_{(n,\lambda)} - 2x_{(n,\lambda+2)} + x_{(n,\lambda+4)} = 0$, where $\lambda = 2\beta$, $\beta > 2$

(ii) $y_{(n,\lambda)} - 2y_{(n,\lambda+2)} + y_{(n,\lambda+4)} = 0$, where $\lambda = 2\beta$, $\beta > 2$

(iii) $z_{(n,\lambda)} - 2z_{(n,\lambda+2)} + z_{(n,\lambda+4)} = 0$, where $\lambda = 2\beta$, $\beta > 2$

In particular $n = 4, \lambda = 6$ when $\beta = 3$

(i) $x_{(4,6)} - 2x_{(4,8)} + x_{(4,10)} = 0$

(ii) $y_{(4,6)} - 2y_{(4,8)} + y_{(4,10)} = 0$

(iii) $z_{(4,6)} - 2z_{(4,8)} + z_{(4,10)} = 0$

(b) Recurrence relations for solution $\left(x_{(n,\lambda)}, y_{(n,\lambda)}, z_{(n,\lambda)}\right)$ among the different values of $\lambda = 2, 4, 6$.

(i) $x_{(n,2)} - 2x_{(n,4)} - x_{(n,6)} + 4 = 0$

(ii) $y_{(n,2)} - 2y_{(n,4)} - y_{(n,6)} + 2 = 0$

(iii) $z_{(n,2)} - 2z_{(n,4)} - z_{(n,6)} = 0$

(c) Recurrence relations for solution $\left(x_{(n,\lambda)}, y_{(n,\lambda)}, z_{(n,\lambda)}\right)$ among the particular values of λ.
$x_{(n-1,\lambda)} - 6x_{(n,\lambda)} + x_{(n+1,\lambda)} + \lambda + 2 = 0$, where $n > 0$
$y_{(n-1,\lambda)} - 6y_{(n,\lambda)} + y_{(n+1,\lambda)} + \lambda - 2 = 0$, where $n > 0$
$z_{(n-1,\lambda)} - 6z_{(n,\lambda)} + z_{(n+1,\lambda)} = 0$ where $n > 0$.

In particular $n = 4, \lambda = 6$ when $\beta = 3$
$x_{(3,6)} - 6x_{(4,6)} + x_{(5,6)} + 6 + 2 = 0$
$y_{(3,6)} - 6y_{(4,6)} + y_{(5,6)} + 6 - 2 = 0$
$z_{(3,6)} - 6z_{(4,6)} + z_{(5,6)} = 0$

(d) Recurrence relations for solution $\left(x_{(n,\lambda)}, y_{(n,\lambda)}, z_{(n,\lambda)}\right)$ among the particular values of λ

$x_{(n,\lambda)} + x_{(n+1,\lambda)} + z_{(n,\lambda)} - z_{(n+1,\lambda)} + \beta - 1 = \lambda$, where $\lambda = 2\beta$, $\beta \in Z_+$
$y_{(n,\lambda)} + y_{(n+1,\lambda)} + z_{(n,\lambda)} - z_{(n+1,\lambda)} + \beta + 1 = \lambda$, where $\lambda = 2\beta$, $\beta \in Z_+$.

In particular $n = 4, \lambda = 6$ when $\beta = 3$
$x_{(4,6)} + x_{(5,6)} + z_{(4,6)} - z_{(5,6)} + 2 = 6$
$y_{(4,6)} + y_{(5,6)} + z_{(4,6)} - z_{(5,6)} + 4 = 6$
For the sake of clarity the geometrical representation of $x^2 + y^2 - \lambda y = z^2$ for $\lambda = 2$ has been shown in Fig. 2.

Proposition 3.1. *If x, y, and z are relatively prime integers such that $x^2 + y^2 \pm \lambda y = z^2$, $\lambda = 2\beta$, and $x > y$, then x is odd, y is even, and z is odd when β is odd.*

Proposition 3.2.

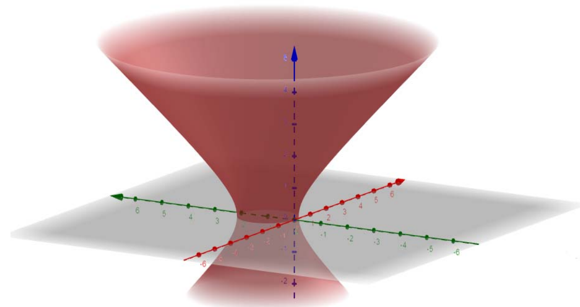

Figure 2. Geometrical representation of $x^2 + y^2 - 2y = z^2$.

1. *If x, y, and z are relatively prime integers such that $x^2 + y^2 + \lambda y = z^2$, $\lambda = 2\beta$, and $x > y$, then x and z are odd, y is even when β is even and n is zero and even.*

2. *If x, y, and z are relatively prime integers such that $x^2 + y^2 + \lambda y = z^2$, $\lambda = 2\beta$, and $x > y$, then x is even and y and z are odd when β is even and n is odd.*

Proposition 3.3.

1. *If x, y, and z are relatively prime integers such that $x^2 + y^2 - \lambda y = z^2$, $\lambda = 2\beta$, and $x > y$, then x is even, y and z are odd when β is even and n is zero and even.*

2. *If x, y, and z are relatively prime integers such that $x^2 + y^2 - \lambda y = z^2, \lambda = 2\beta$, and $x > y$, then x and z are odd and y is even when β is even and n is odd.*

Proof. The assertions of the above propositions are easily checked from Tables 6 to 9. ☐

Remark: In the above two cases, if $a = 0$ the proposed equations $x^2 + y^2 \pm \lambda y = z^2$ become of the form $2y^2 \pm \lambda y = z^2$. It is observed that, these equations are the equations of two variables. Moreover, geometrically the equations are of two dimensional hyperbolic forms. As per the objective of this work, we have planned to discuss these equations $2y^2 \pm \lambda y = z^2$ and its generalized form in our future work.

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