



# Solvability of two-point fractional boundary value problems at resonance

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## Abstract

A two-point boundary value problem of nonlinear fractional differential equations at resonance is considered in this work. An existence result is obtained with the use of the generalized Miranda theorem.

## Keywords

Fractional differential equations, Fractional Caputo derivative, boundary value problem, Resonance,  $R_\sigma$ -set, Miranda's theorem.

## AMS Subject Classification

34A08, 34B15.

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## 1. Introduction

The field of fractional calculus is concerned with the generalization of the integer order differentiation and integration to an arbitrary real or complex order. It has played a significant role in various branches of science such as, physics, chemistry, chemical physics, electrical networks, control of dynamic systems, science, engineering, biological science, optics and signal processing; see for example, [11, 15, 17]. The present work concerns a fractional differential boundary value problem, which can be transformed to the following equation  $Lx = Nx$ , where  $L$  is a linear operator and  $N$  is a given operator from a Banach space  $X$  to another Banach space  $Y$ . If the kernel of the linear part of the above equation contains only zero, the corresponding boundary value problem is called non-resonant, in this case  $L$  is invertible. Otherwise, if  $L$  is a non-invertible, i.e.  $\dim \ker L \geq 1$ , then the problem is said to be at resonance, an important class of resonant problems when  $L$  is a Fredholm operator with zero-index, the problem can be solved by using the coincidence degree theory. Recently, fractional boundary value problems at resonance have attracted

many scholar's attention, for instance see [1–4, 8–10, 12, 13] and the references therein.

Besides, for the recent advances in other techniques for solving nonlinear problems resonant and non resonant. The generalized Miranda theorem (see [18, 19]) can be applied to systems of ordinary differential equations, to both non-resonant and resonant cases. In [18] some examples of using this method for systems of differential equations of second order. In [21], the authors by means this method proved the existence of solutions for systems of differential equations of higher-order under non-resonant and resonant boundary conditions.

Using the generalized Miranda Theorem of the sublinear case, we are concerned with the existence results for two-point BVP of nonlinear fractional differential equation at resonance

$$\begin{cases} (\varphi(t) {}^C D_{0+}^\alpha u(t))' = f(t, u(t), u^{(N-1)}(t), {}^C D_{0+}^\alpha u(t)), t \in [0, 1] \\ u'(0) = u^{(N-1)}(0) = {}^C D_{0+}^\alpha u(0) = 0, \quad {}^C D_{0+}^\alpha u(1) = 0, \end{cases} \quad (1.1)$$

where  $N - 1 < \alpha \leq N$ ,  $N \geq 2$ ,  ${}^C D_{0+}^\alpha$  is the Caputo fractional derivatives,  $\varphi(t) \in C^1([0, 1], \mathbb{R})$ ,  $\min_{t \in [0, 1]} \varphi(t) > 0$ ,  $f: [0, 1] \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ , is a continuous function.

In this work, we suppose that the following conditions hold:

$$(H_1) \quad |f(t, u_0, u_1, \dots, u_N)| \leq \psi_0(t)|u_0| + \psi_1(t)|u_1| + \dots + \psi_N(t)|u_N| + \psi_{N+1}(t) \text{ where } \psi_0, \psi_1, \dots, \psi_{N+1} \in C([0, 1], \mathbb{R}_+).$$

(H<sub>2</sub>) There exists  $\eta > 0$  such that  $u_0 \cdot f(t, u_0, u_1, \dots, u_N) \geq 0$  for all  $(t, u_0, u_1, \dots, u_N) \in [0, 1] \times \mathbb{R}^{N+1}$  and  $|u_0| \geq \eta$ .

The remaining part of this article is organized as follows. In Section 2, we introduce some notations, definitions, lemmas and theorems which will be used later. In Section 3, we present and prove our main results by characterizing the  $R_\sigma$ -map and applying generalized Miranda Theorem.

## 2. Preliminaries

First of all, we present the necessary definitions and lemmas from fractional calculus theory. These can be found in [11, 15, 17].

**Definition 2.1.** Let  $\alpha > 0$ , for a function  $u : [0, \infty) \rightarrow \mathbb{R}$ . The Riemann-Liouville fractional integral of order  $\alpha$  of  $u$  is defined by

$$I_{0+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

provided that the right-hand side is pointwise defined on  $(0, \infty)$ .

**Remark 2.2.** The notation  $I_{0+}^\alpha u(t) |_{t=0}$  means that the limit is taken at almost all points of the right-sided neighborhood  $(0, \varepsilon)$  ( $\varepsilon > 0$ ) of 0 as follows:

$$I_{0+}^\alpha u(t) |_{t=0} = \lim_{t \rightarrow 0+} I_{0+}^\alpha u(t).$$

Generally,  $I_{0+}^\alpha u(t) |_{t=0}$  is not necessarily to be zero. For instance, let  $\alpha \in (0, 1)$ ,  $u(t) = t^{-\alpha}$ . Then

$$I_{0+}^\alpha t^{-\alpha} |_{t=0} = \lim_{t \rightarrow 0+} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\alpha} ds = \Gamma(1-\alpha).$$

**Definition 2.3.** Let  $\alpha > 0$ . The Caputo fractional derivative of order  $\alpha$  of a function  $u : (0, \infty) \rightarrow \mathbb{R}$  is given by

$${}^C D_{0+}^\alpha u(t) = I_{0+}^{n-\alpha} u^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds,$$

where  $n = [\alpha] + 1$ ,  $[\alpha]$  denotes the integer part of real number  $\alpha$ , provided that the right-hand side is pointwise defined on  $(0, \infty)$ .

**Lemma 2.4.** Let  $\alpha > 0$ ,  $n = [\alpha] + 1$ , then

$$I_{0+}^\alpha {}^C D_{0+}^\alpha u(t) = u(t) - \sum_{k=0}^{n-1} c_k t^k, \quad c_k \in \mathbb{R}.$$

**Lemma 2.5.** Let  $\alpha > 0$ , and  $n = [\alpha] + 1$ . If  ${}^C D_{0+}^\alpha u(t) \in C[0, 1]$ , then  $u(t) \in C^{n-1}([0, 1])$ .

*Proof.* Let  $v(t) \in C[0, 1]$ , such that  ${}^C D_{0+}^\alpha u(t) = v(t)$ , then from Lemma 2.4, we have

$$u(t) = I_{0+}^\alpha v(t) + \sum_{k=0}^{n-1} c_k t^k, \quad c_k \in \mathbb{R}.$$

It is easy to check that  $u(t) \in C^{n-1}([0, 1])$ . □

Now, we shall set up notation and terminology.

Let  $X, Y$  be non empty metric spaces. We say that a space  $X$  is contractible, if there exist  $x_0 \in X$  and a homotopy  $h : X \times [0, 1] \rightarrow X$  such that  $h(x, 0) = x$  and  $h(x, 1) = x_0$  for every  $x \in X$ .

A compact space  $X$  is an  $R_\delta$ -set (we write  $X \in R_\delta$ ) if there is a decreasing sequence  $X_n$  of compact contractible spaces such that  $X = \bigcap_{n \geq 1} X_n$ .

A multivalued map  $\Phi : X \rightarrow Y$  is called upper semicontinuous (USC) provided for every open  $U \subset Y$  the set  $\Phi^{-1}(U)$  is open in  $X$ . We say  $\Phi$  is an  $R_\delta$ -map if it is USC for each  $x \in X$ ,  $\Phi(x) \in R_\delta$ .

The following theorem is very important concerning topological structure of the set of solutions for some nonlinear functional equation.

**Theorem 2.6** ([7]). Let  $X$  be a space,  $(G; \|\cdot\|)$  a Banach space and  $g : X \rightarrow G$  a proper map, i.e.  $g$  is continuous and for every compact  $B \subset G$  the set  $g^{-1}(B)$  is compact. Assume further that for each  $\varepsilon > 0$  a proper map  $g_\varepsilon : X \rightarrow G$  is given and the following two conditions are satisfied:

1.  $\|g_\varepsilon(x) - g(x)\| < \varepsilon$ , for every  $x \in X$ ,
2. for any  $\varepsilon > 0$  and  $u \in G$  such that  $\|u\| \leq \varepsilon$ , the equation  $g_\varepsilon(x) = u$  has exactly one solution.

Then the set  $S = g^{-1}(0)$  is  $R_\delta$ .

The key tool in our approach is the following generalization of the Miranda theorem

**Theorem 2.7** ([18, 19]). Let  $A_i > 0$ ,  $i = 1, \dots, n$ , and  $F$  be an admissible map from  $\prod_{i=1}^n [-A_i, A_i]$  to  $\mathbb{R}^n$ , i.e. there exist a Banach space  $E$ ,  $\dim E \geq n$ , a linear, bounded and surjective map  $\phi : E \rightarrow \mathbb{R}^n$ , and an  $R_\delta$ -map  $\Phi$  from  $\prod_{i=1}^n [-A_i, A_i]$  to  $E$  such that  $F = \phi \circ \Phi$ . If for any  $i = 1, \dots, n$ , and very  $y \in F(a)$ , where  $|a_i| = A_i$ , we have

$$a_i \cdot y_i \geq 0, \tag{2.1}$$

or

$$a_i \cdot y_i \leq 0, \tag{2.2}$$

then there exists  $a \in \prod_{i=1}^n [-A_i, A_i]$  such that  $0 \in F(a)$ .

## 3. Main Results

We use the Banach space  $G = C([0, 1], \mathbb{R})$  with the norm  $\|v\|_\infty = \max_{t \in [0, 1]} |v(t)|$ .

Now, we consider the following initial value problem

$$\begin{cases} (\varphi(t) {}^C D_{0+}^\alpha u(t))' = f(t, u(t), u^{(N-1)}(t), {}^C D_{0+}^\alpha u(t)), t \in [0, 1], \\ u(0) = a, \quad u'(0) = u^{(N-1)}(0) = {}^C D_{0+}^\alpha u(0) = 0, \end{cases} \tag{3.1}$$



where  $a \in \mathbb{R}$  is fixed. Let  $v(t) = \varphi(t) {}^C D_{0+}^\alpha u(t)$ , then from lemma 2.4 the IVP (3.1) is equivalent to

$$\begin{cases} v'(t) = f\left(t, a + I_{0+}^\alpha\left(\frac{v}{\varphi}\right)(t), I_{0+}^{\alpha-1}\left(\frac{v}{\varphi}\right)(t), \dots, \right. \\ \left. I_{0+}^{\alpha-(N-1)}\left(\frac{v}{\varphi}\right)(t), \left(\frac{v}{\varphi}\right)(t)\right), t \in [0, 1] \\ v(0) = 0. \end{cases} \quad (3.2)$$

We can write down IVP (3.2) in the following form

$$v(t) = \int_0^t f\left(s, a + I_{0+}^\alpha\left(\frac{v}{\varphi}\right)(s), I_{0+}^{\alpha-1}\left(\frac{v}{\varphi}\right)(s), \dots, \right. \\ \left. I_{0+}^{\alpha-(N-1)}\left(\frac{v}{\varphi}\right)(s), \left(\frac{v}{\varphi}\right)(s)\right) ds. \quad (3.3)$$

By  $(H_1)$ , we get

$$\begin{aligned} |v(t)| &\leq \int_0^t \left( \psi_0(s) \left| a + I_{0+}^\alpha\left(\frac{v}{\varphi}\right)(s) \right| + \psi_1(s) \left| I_{0+}^{\alpha-1}\left(\frac{v}{\varphi}\right)(s) \right| + \right. \\ &\quad \dots + \psi_{N-1}(s) \left| I_{0+}^{\alpha-(N-1)}\left(\frac{v}{\varphi}\right)(s) \right| \\ &\quad \left. + \psi_N(s) \left| \left(\frac{v}{\varphi}\right)(s) \right| + \psi_{N+1}(s) \right) ds \\ &\leq \max_{r \in [0,1]} \frac{1}{\varphi(r)} \int_0^t \left( \frac{\psi_0(s)}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} |v(\tau)| d\tau + \right. \\ &\quad \frac{\psi_1(s)}{\Gamma(\alpha-1)} \int_0^s (s-\tau)^{\alpha-2} |v(\tau)| d\tau + \dots \\ &\quad \left. + \frac{\psi_{N-1}(s)}{\Gamma(\alpha-N+1)} \int_0^s (s-\tau)^{\alpha-N} |v(\tau)| d\tau \right. \\ &\quad \left. + \psi_N(s) |v(s)| \right) ds + \int_0^t (|\psi_0(s) + \psi_{N+1}(s)|) ds \\ &\leq \max_{r \in [0,1]} \frac{1}{\varphi(r)} \int_0^t \left( \sum_{i=0}^{N-1} \frac{\psi_i(s) s^{\alpha-i}}{\Gamma(\alpha-i+1)} \right. \\ &\quad \left. + \psi_N(s) \right) \max_{\tau \in [0,s]} |v(\tau)| ds + \int_0^1 (|\psi_0(s) + \psi_{N+1}(s)|) ds. \end{aligned}$$

Set  $w(t) = \max_{s \in [0,t]} |v(s)|$ . We obtain

$$w(t) \leq \max_{r \in [0,1]} \frac{1}{\varphi(r)} \int_0^t \left( \sum_{i=0}^{N-1} \frac{\psi_i(s) s^{\alpha-i}}{\Gamma(\alpha-i+1)} \right. \\ \left. + \psi_N(s) \right) w(s) ds + C_a$$

where

$$C_a = \int_0^1 (|\psi_0(s) + \psi_{N+1}(s)|) ds$$

Now, due to Gronwall's Lemma [16, p. 17], we obtain

$$w(t) \leq C_a \exp \max_{r \in [0,1]} \frac{1}{\varphi(r)} \int_0^t \left( \sum_{i=0}^{N-1} \frac{\psi_i(s) s^{\alpha-i}}{\Gamma(\alpha-i+1)} + \psi_N(s) \right) ds$$

Thus

$$|v(t)| \leq C_a \exp \max_{r \in [0,1]} \frac{1}{\varphi(r)} \int_0^t \left( \sum_{i=0}^{N-1} \frac{\psi_i(s) s^{\alpha-i}}{\Gamma(\alpha-i+1)} + \psi_N(s) \right) ds \quad (3.4)$$

Hence, by the Theorem on a priori bounds [16, p. 146], for each fixed  $a \in \mathbb{R}$  there exists at least one global solution  $v$  to the IVP (3.2), i.e.  $v \in G$ . Moreover, by assumption  $(H_1)$  and (3.4), we have

$$|v(t)| \leq C_a \exp C < \infty \quad (3.5)$$

where

$$C = \max_{r \in [0,1]} \frac{1}{\varphi(r)} \int_0^1 \left( \sum_{i=0}^{N-1} \frac{\psi_i(s) s^{\alpha-i}}{\Gamma(\alpha-i+1)} + \psi_N(s) \right) ds.$$

Thus all global solutions are bounded for  $t \in [0, 1]$ .

Now, let us consider the nonlinear operator  $T : \mathbb{R} \times G \rightarrow G$ ,  $(a, v) \mapsto T_a(v)$  defined as

$$T_a(v)(t) = \int_0^t f\left(s, a + I_{0+}^\alpha\left(\frac{v}{\varphi}\right)(s), I_{0+}^{\alpha-1}\left(\frac{v}{\varphi}\right)(s), \dots, \right. \\ \left. I_{0+}^{\alpha-(N-1)}\left(\frac{v}{\varphi}\right)(s), \left(\frac{v}{\varphi}\right)(s)\right) ds. \quad (3.6)$$

It is easy to see that  $T$  is well defined and continuous. Using the Arzelà-Ascoli theorem under assumption  $(H_1)$ , we get the following results.

**Lemma 3.1.** *Let assumption  $(H_1)$  hold. Then the operator  $T$  is completely continuous.*

Notice that the solution of the IVP (3.2) are fixed points of the operator  $T$  defined by (3.6).

Let  $FixT_a(\cdot)$  denote the set of fixed points of operator  $T_a$ , where  $a \in \mathbb{R}$  is given.

$$FixT_a(\cdot) = \{v \in G : T_a(v) = v\},$$

Now, let us define a map  $\Phi : \mathbb{R} \rightarrow G$  by

$$\Phi(a) = FixT_a(\cdot). \quad (3.7)$$

and define a map  $\phi : G \rightarrow \mathbb{R}$  by

$$\phi(v) = v(1). \quad (3.8)$$

then  $\phi$  is continuous, linear and surjective.

Now, let a multifunction  $F : \mathbb{R} \rightarrow \mathbb{R}$  be such that  $F = \phi \circ \Phi$ .

$$F(a) = \{v(1) \mid v \in FixT_a(\cdot)\}. \quad (3.9)$$

**Lemma 3.2.** *Let assumption  $(H_1)$  hold. Then the set-valued map  $\Phi$  is USC with compact values.*

*Proof.* The set valued map  $\Phi$  is upper semi-continuous with compact values if given a sequence  $(a_n)$  in  $\mathbb{R}$ ,  $a_n \rightarrow a_0$  and  $(v_n) \in \Phi(a_n)$ ,  $(v_n)$  has converging sub-sequence to some  $v_0 \in \Phi(a_0)$ . Taking any sequence  $(a_n)$ ,  $a_n \rightarrow a_0$  and  $v_n \in \Phi(a_n)$ , we get

$$v_n = T_{a_n}(v_n). \quad (3.10)$$



Since  $(a_n)$  is bounded, by (3.5), we get that the fixed points of  $T_a(\cdot)$  are equibounded for any  $a$ . Hence the sequences  $(v_n)$  is bounded in  $G$ . From the Lemma 3.1 the operator  $T$  is completely continuous. Then, by (3.10),  $(v_n)$  is relatively compact in  $G$ . Hence  $(v_n)$ , has a sub-sequence  $(v_{n_k})$  convergent to some  $v_0 \in G$ . Moreover,

$$v_{n_k} = T_{a_{n_k}}(v_{n_k}).$$

On the other hand,  $a_{n_k} \rightarrow a_0$ . From the continuity of  $T$  we find when  $k$  tends to infinity

$$v_0 = Ta_0(v_0).$$

So,  $v_0 \in \Phi(a_0)$ . This achieves the proof.  $\square$

**Lemma 3.3.** *Let assumption  $(H_1)$  hold. The set of all solutions of the IVP (3.2) is an  $R_\delta$ -set.*

*Proof.* Let  $B_a = \{v \in G \mid \|v\|_\infty \leq C_a \exp C\}$ . Define an operator  $g : B_a \rightarrow G$  as  $g(v) = v - T_a(v)$ . By lemma 3.1,  $T_a : B_a \rightarrow G$  is compact,  $g$  is compact vector field associated with  $T_a(\cdot)$ . We shall prove that all the assumptions of Theorem 2.6 are satisfied.

We define the sequence  $g_n : B_a \rightarrow G$  as follows:

$$g_n(v) = v - T_a^n(v). \tag{3.11}$$

Where,  $(T_a^n(\cdot))$  is a sequence of continuous and compact mappings defined as follows :

$$T_a^n(v)(t) = T_a(v)(\theta_n(t)), \quad v \in B_a, \quad n \in \mathbb{N}^*. \tag{3.12}$$

and  $\theta_n$  is auxiliary mappings defined by

$$\theta_n(t) = \begin{cases} 0, & t \in [0, \frac{1}{n}], \\ t - \frac{1}{n}, & t \in [\frac{1}{n}, 1]. \end{cases}$$

It satisfies

$$|\theta_n(t) - t| \leq \frac{1}{n}, \quad \text{for all } t \in [0, 1], n \in \mathbb{N}^*. \tag{3.13}$$

We see that  $g_n$  is continuous and proper mappings. We find from the compactness of  $(T_a^n(\cdot))$ , (3.11) and (3.13) that  $T_a^n(v) \rightarrow T_a(v)$  uniformly in  $B_a$ .

Now, we shall prove that  $g_n$  is one to one map. Assume that for some  $v, w \in B_a$ , we have  $g_n(v) = g_n(w)$ . This means that

$$v - w = T_a^n(v) - T_a^n(w).$$

If  $t \in [0, \frac{1}{n}]$ , then we find

$$\begin{aligned} v(t) - w(t) &= T_a(v)(\theta_n(t)) - T_a(w)(\theta_n(t)) \\ &= T_a(v)(0) - T_a(w)(0) = 0. \end{aligned}$$

So, we get

$$v(t) = w(t) \quad \text{for } t \in [0, \frac{1}{n}].$$

If  $t \in [\frac{j}{n}, \frac{j+1}{n}]$ ,  $j = 1, \dots, n-1$ , then

$$\theta_n(t) \in [\frac{j-1}{n}, \frac{j}{n}], \quad \theta_n(\theta_n(t)) \in [\frac{j-2}{n}, \frac{j-1}{n}].$$

And so on we get

$$\underbrace{\theta_n \circ \theta_n \cdots \circ \theta_n(t)}_{j \text{ times}} \in [0, \frac{1}{n}],$$

and that implies

$$\underbrace{\theta_n \circ \theta_n \cdots \circ \theta_n(t)}_{j+1 \text{ times}} = 0.$$

So, by the property of operator  $T_a(\cdot)$  mentioned above and (3.13), we find

$$\begin{aligned} T_a(v)(t) &= \lim_{n \rightarrow \infty} T_a^n(v)(\theta_n(t)) = \lim_{n \rightarrow \infty} T_a^n(v)(\theta_n(\theta_n(t))) = \dots \\ &= \lim_{n \rightarrow \infty} T_a^n(v)(\underbrace{\theta_n \circ \theta_n \cdots \circ \theta_n(t)}_{j \text{ times}}) \\ &= \lim_{n \rightarrow \infty} T_a(v)(\underbrace{\theta_n \circ \theta_n \cdots \circ \theta_n(t)}_{j+1 \text{ times}}) = 0. \end{aligned}$$

So,  $T_a(v)(t) = T_a(w)(t) = 0$  for  $t \in [\frac{j}{n}, \frac{j+1}{n}]$ , thus we have  $v(t) = w(t)$  for  $t \in [\frac{j}{n}, \frac{j+1}{n}]$ ,  $j = 1, \dots, n-1$ . So,  $v(t) = w(t)$  for  $t \in [0, 1]$ . Hence,  $g_n$  is one to one map. This means that the assumptions of Theorem 2.6 hold and  $g^{-1}(0) = \text{Fix}T_a(\cdot)$  is an  $R_\delta$  set. The proof is complete.  $\square$

**Lemma 3.4.** *Let assumption  $(H_1)$  hold. Then  $\Phi$  is an  $R_\delta$ -map.*

*Proof.* By lemma 3.2 and 3.3 the set valued map  $\Phi$  is USC and for any  $a \in \mathbb{R}$ , the set  $\text{Fix}T_a(\cdot)$  is an  $R_\delta$ -set ( i.e  $\Phi(a) \in R_\delta$ ). This means that,  $\Phi$  is a  $R_\delta$ -map.  $\square$

**Corollary 3.5.** *Under assumption  $(H_1)$ ,  $F$  defined in (3.9) is an admissible map in the sense of Theorem 2.7 (of the case  $n = 1$ ).*

**Theorem 3.6.** *Under assumptions  $(H_1)$  and  $(H_2)$ , then the BVP (1.1) has at least one solution.*

*Proof.* Let  $v \in \text{Fix}T_a(\cdot)$  be a bounded global solution of IVP (3.2). Observe that  $u(t) = a + I_{0+}^\alpha \left( \frac{v}{\varphi} \right)(t)$  is a solution of the BVP (1.1) if there exists  $a \in \mathbb{R}$  such that  $0 \in F(a)$ . It remains to show the condition (2.1) or (2.2) of Theorem 2.7 .

Let  $a = \eta + 1$ , where  $\eta$  is as in  $(H_2)$ . First, we will show that  $v(t) \geq 0$ ,  $t \in [0, 1]$ . Notice that  $v(0) = 0$ . Assume that for some  $t \in [0, 1]$  we have  $v(t) < 0$ . Then there exists  $t_0 = \inf \{t \in [0, 1] : v(t) < 0\}$  such that  $v(t_0) = 0$  and  $v(t) > 0$  for  $t < t_0$ . Consequently, since  $(\frac{v}{\varphi})(\cdot)$  is continuous function, there exists  $t_1 > t_0$  such that

$$\int_{t_0}^t (t-s)^{\alpha-1} \left| \left( \frac{v}{\varphi} \right)(s) \right| ds < \Gamma(\alpha), \quad t \in [t_0, t_1]$$



so

$$a - \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \left| \left( \frac{v}{\varphi} \right) (s) \right| ds \geq \eta, \quad t \in [t_0, t_1],$$

thus,

$$a + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \left( \frac{v}{\varphi} \right) (s) ds \geq \eta, \quad t \in [t_0, t_1],$$

we get

$$u(t) = a + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \left( \frac{v}{\varphi} \right) (s) ds \geq \eta, \quad t \in [t_0, t_1].$$

Now, by assumption  $(H_2)$ , we find

$$u(t) \cdot f(t, u(t), u^{(N-1)}(t), {}^C D_{0^+}^\alpha u(t)) \geq 0, \quad t \in [t_0, t_1].$$

Consequently

$$u(t) \cdot v'(t) \geq 0, \quad t \in [t_0, t_1].$$

Thus,  $v'(t) \geq 0$  for  $t \in [t_0, t_1]$ . This implies that  $v(t)$  is non-decreasing on  $[t_0, t_1]$ . Since  $v(0) = 0$ , we have  $v(t) \geq 0$  for  $t \in [t_0, t_1]$ , we get a contradiction. Hence  $v(t) \geq 0$ , for all  $t \in [0, 1]$ . So,  $a \cdot v(1) \geq 0$ .

Let  $a = -\eta - 1$ , in the same previous way we find  $v(t) \leq 0$ , for all  $t \in [0, 1]$ . Thus,  $a \cdot v(1) \geq 0$ .

Consequently, for each  $v(1) \in F(a)$  with  $|a| = \eta + 1$ , one has

$$a \cdot v(1) \geq 0. \tag{3.14}$$

Therefore, the condition (2.1) of Theorem 2.7 is satisfied for  $A = \eta + 1$ . Hence, there exists  $a \in [-A, A]$  such that,  $0 \in F(a)$ , i.e.  ${}^C D_{0^+}^\alpha u(1) = 0$ . The proof is finished.  $\square$

## References

- [1] Z. Bai, *On solutions of some fractional m-point boundary value problems at resonance*, Electron. J. Qual. Theory Differ. Equ. 2010, No. 37, 1-15.
- [2] Z. Bai, *Solvability for a class of fractional m-point boundary value problem at resonance*, Comput. Math. Appl. 62(2011), 1292-1302.
- [3] Z. Bai, Y. Zhang, *The existence of solutions for a fractional multi-point boundary value problem*. Comput.Math.Appl. 60(8),2364-2372(2010) 24.
- [4] Z. Bai, Y. Zhang, *Solvability of fractional three-point boundary value problems with nonlinear growth*, Appl. Math. Comput., 218 (2011), 1719-1725.
- [5] M. Benchohra, S. Hamani, S. K. Ntouyas, *Boundary value problems for differential equations with fractional order and nonlocal conditions*. Nonlinear Anal. 71, 2391-2396 (2009).
- [6] L. Gorniewicz, *Homological methods in fixed point theory of multi-valued mappings*, in *Topological Fixed Point Theory of Multivalued Mappings*, Kluwer, Dordrecht, 1999.
- [7] L. Gorniewicz, *Topological approach to differential inclusions*, in *Topological Fixed Point Theory of Multivalued Mappings*, Kluwer, Dordrecht, 1999.
- [8] Z. Hu, W. Liu, T. Chen, *Two-point boundary value problems for fractional differential equations at resonance*, Bulletin of the Malaysian Mathematical Sciences Society 2, vol. 36,no.3.pp.747-755,2013.
- [9] L. Hu, S. Zhang, A. Shi, *Existence of Solutions for Two-Point Boundary Value Problem of Fractional Differential Equations at Resonance*, Hindawi Publishing Corporation International Journal of Differential Equations Volume 2014, Article ID 632434, 7 pages.
- [10] W. Jiang, *The existence of solutions to boundary value problems of fractional differential equations at resonance*. Nonlinear Anal. TMA 74 (2011) 1987-1994.
- [11] A.A. Kilbas, Hari M. Srivastava, J. Juan Trujillo, *Theory and Applications of Fractional Differential Equations*, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.
- [12] N. Kosmatov, *Multi-point boundary value problems on an unbounded domain at resonance*, Nonlinear Anal. 68 (2010), 2158-2171.
- [13] N. Kosmatov, *A boundary value problem of fractional order at resonance*, Electron. J. Differ. Equ. 2010, No. 135, 10 pp.
- [14] S. Liang, J. Zhang, *Existence and uniqueness of positive solutions to m-point boundary value problem for nonlinear fractional differential equation*. J Appl Math Comput (2012) 38:225-241.
- [15] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication, John Wiley Sons, New York, NY, USA, 1993.
- [16] L.C. Piccinnini, G. Stampacchia, G. Vidossich, *Ordinary differential equations in  $\mathbb{R}^n$  problems and methods*, Applied Mathematical Sciences, Vol.39, Springer-Verlag, New York, 1984.
- [17] I. Podlubny, *Geometric and physical interpretation of fractional integration and fractional differentiation*, Fract. Calc. Appl. Anal. 5 (2002) 367-386.
- [18] K. Szymańska-Dębowska, *On a generalisation of the Miranda Theorem and its application to boundary value problems*. J Differential Equations 258 (2015) 2686-2700.
- [19] K. Szymańska-Dębowska, *Solutions to nonlocal Neumann boundary value problems*. Electron.J.Qual.Theory Differ.Equ.2018, No. 28, 1-14; <https://doi.org/10.14232/ejqtde.2018.1.28>.
- [20] N. Xu, W. Liu, and L. Xiao, *The existence of solutions for nonlinear fractional multipoint boundary value problems at resonance*, Boundary Value Problems, vol. 2012, article 65, 10 pages, 2012.
- [21] A. Yang, J. Henderson and D.J. Wang, *The study of higher-order resonant and non-resonant boundary value problems*, Electronic Journal of Qualitative Theory of Differential Equations 2016.
- [22] X. Zhang, L. Wang, Q. Sun, *Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter*, Appl. Math. Comput., 226 (2014), 708-718.

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