Malaya Journal of Matematik, Vol. 8, No. 2, 469-472, 2020

https://doi.org/10.26637/MJM0802/0024

On solving linear Fredholm integro-differential equations via finite difference-Simpson's approach

Bashir Danladi Garba[1]* and Sirajo Lawan Bichi[2]

Abstract

In this paper, a combination of Finite difference-Simpson's approach were applied to solve Linear Fredholm integro-differential equations of second kind by discritising the unknown function, which leads in generating a system of linear algebraic equations. The numerical results obtained from the proposed method were compared with exact solutions of the tested problems which show that the method derived is effective and promising when compared with some existing method in the literature and error estimation of the scheme was derived.

Keywords

Error estimation, Finite difference equations, Fredholm integro-differential equation, Simpson's Method.

AMS Subject Classification

45A05, 45B05, 45J05, 45P05.

[1] *Department of Mathematics, Kano University of Science and technology Wudil, Nigeria.*

[2] *Department of Mathematics, Bayero University Kano, Nigeria.*

***Corresponding author**: [1] bashirdanladi@gmail.com; [2]slawan.mth@buk.edu.ng

Article History: Received **12** September **2019**; Accepted **26** March **2020**

Contents

1. Introduction

Mathematical modeling of real life problems often result in functional equations such as differential, integral and integro-differential equations. Many mathematical formulation of physical phenomena reduced to integro-differential equations like Fluid dynamics, Biological models and chemical kinetics [7, 14, 15].

In general, the exact solution of integro-differential equations is difficult to obtain. Therefore, that is why there have been a great attention by many researchers on how to obtain an approximate solution of integro-differential equations of different kind such as [2, 4, 7, 8, 13]. Nevertheless these methods require more effort and usually they are meant for special types of integro-differential equation problems. We consider a Linear integro-differential equation of the form:

$$u'(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, \quad u_0 = \alpha. \tag{1.1}$$

where a, b and λ are constants, $f(x)$ and $k(x,t)$ are known function with $k(x,t)$ is the kernel and u is the unknown function to be determined.

Darania and Ebadian [2] studied first order linear Fredholm Integro-differential equations, a Differential transform based on Taylor's series expansion has been successfully employed to obtained the approximate solution and provide applicable relation between the one and two dimensional transformation,numerical examples were given to show the effectiveness of the method. According to Danfu. H and Xufeng. [4] used a cos and sine (CAS) wavelet operational matrix of integration method was used to reduce IDEs into a system of linear equations. Numerical results showed that CAS wavelet method is better than a Differential Transform method.

Vahidi *et.al* [8] applied Adomian's decomposition method (ADM) and comparison was made wtih two different Numerical methods; CAS wavelet and Differential Transform, which showed that ADM gave a better approximation and is more efficient.

Tamamagar M. [13] obtained numerical solution of linear Fredholm integro-differential equations via Parametric

iteration method (PIM), some examples were considered and convergence analysis was studied.

Moreover, In this paper we propose a Finite difference-Simpson's approach to solve equation (1.1) by transforming the equation into a system of linear equations. Comparison were made with exact solution and two different methods presented in the literature. The remainder of the paper is organized as follows: In Section 2, we presented the derivation of the method. Error estimation of the scheme was proved in section 3, and numerical results were provided in section 4. Section 5 gives the conclusion.

2. Derivation of the Method

We defined N finite points of the domain $[a,b]$ of (1.1) as $a = t_0 < t_1 < ... < t_{N-1} < t_N = b$ using uniform step length $h = \dfrac{b-a}{N}$, such that $x_i = a + ih, \quad i = 1,2,...,N.$

To obtain numerical solution, we use Finite difference on the differential part and composite Simpson's on the integral part of (1.1). Now using Composite Simpson's with N subintervals and $t \in [a,b]$, the integral part of (1.1) is approximated by

$$\int_a^b k(x,t)u(t)dt \approx \frac{h}{3}[k(x,t_0)u(t_0) + 4k(x,t_1)u(t_1) + ... + 4k(x,t_{N-1})u(t_{N-1}) + k(x,t_N)u(t_N)].$$

By discritizing along x and taking $u_i' = u'(x_i)$, $k(x_i,t_i) = k_{ij}$, we have

$$u_i' = f_i + \frac{h}{3}[k_{i0}u_0 + 4k_{i1}u_1 + 2k_{i2}u_2 ... + 2k_{iN2}u_{N-2} + 4k_{iN-1}u_{N-1} + k_{iN}u_N], \quad (2.1)$$

using Central difference we can approximate the derivative part of (2.1) as

$$u_i' = \frac{u_{i+1} - u_{i-1}}{2h}, \quad i = 1,2,...,N-1.$$

and at the end point N we use second Backward Finite difference

$$u_i' = \frac{3u_N - 4u_{N-1} + u_{N-2}}{2h}, \quad i = N,$$

and by replacing u_i' in (2.1) we have

for $i = 1,2,...,N-1$.

$$\frac{u_{i+1} - u_{i-1}}{2h} = f_i + \frac{h}{3}[k_{i0}u_0 + 4k_{i1}u_1 + 2k_{i2}u_2 + ... + 4k_{iN-1}u_{N-1} + k_{iN}u_N], \quad (2.2)$$

and for $i = N$

$$\frac{3u_N - 4u_{N-1} + u_{N-2}}{2h} = f_N + \frac{h}{3}[k_{N0}u_0 + 4k_{N1}u_1 + 2k_{N2}u_2 + ... + 4k_{NN-1}u_{N-1} + k_{NN}u_N], \quad (2.3)$$

using the above equations (2.2) and (2.3), we can generate a systems of equations for $u_1, u_2, ..., u_N$ which can be represent in a matrix form

$$\mathbf{MU=W}$$

$$\mathbf{M} = \begin{pmatrix} A_{11} & B_{12}+1 & A_{13} & ... & A_{1N-1} & C_{1N} \\ A_{21}-1 & B_{22} & A_{23}+1 & ... & A_{1N-1} & C_{2N} \\ A_{31}-1 & B_{32}-1 & A_{33} & ... & A_{3N-1} & C_{3N} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ A_{N-21} & B_{N-22} & A_{N-23} & ... & A_{N-2N-1+1} & C_{N-2N} \\ A_{N-11} & B_{N-12} & A_{N-13} & ... & A_{N-1N-1} & C_{N-1N+1} \\ A_{N1} & B_{N2} & A_{N3} & ... & A_{NN-1}-4 & C_{NN+3} \end{pmatrix},$$

$$\mathbf{U} = \begin{pmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_N \end{pmatrix},$$

$$\mathbf{W} = \begin{pmatrix} 2hf_1 + (\frac{2}{3}h^2k_{10})u_0 \\ 2hf_2 + \frac{2}{3}h^2k_{20}u_0 \\ . \\ . \\ 2hf_{N-1} + \frac{2}{3}h^2k_{N-10}u_0 \\ 2hf_N + \frac{2}{3}h^2k_{N0}u_0 \end{pmatrix},$$

and we define $A_{ij} = -\frac{8}{3}h^2k_{ij}$, $B_{ij} = -\frac{4}{3}h^2k_{ij}$, $C_{ij} = -\frac{2}{3}h^2k_{ij}$.

3. Error Estimation

Theorem 3.1. *Suppose that $\mu_1, \mu_2, \mu_3 \in (a,b)$ such that the errors e_1 of central difference, e_2 of second backward difference approximation and e_3 of Simpson's rule respectively are given by $\frac{h^2}{6}u^{(3)}(\mu_1)$, $\frac{h^2}{4}u^{(4)}(\mu_3)$ and $\frac{(b-a)}{180}h^4u^{(4)}(\mu_2)$. Then the error estimation of approximate solution of linear Fredholhm integro-differential equation (1.1) obtained by the scheme (2.2) and (2.3) is given by*

$$e \leq \frac{5(b-a)^2}{12N^2}G \quad (3.1)$$

where $G = max\{u^3(\mu_1), u^4(\mu_2), u^4(\mu_3)\}$ and N is the number of subinterval.

Proof. From the problems of LFIDES (2.2) and (2.3), the exact solution after discritising for $i = 1,2,...,N-1$.

$$\frac{u_{i+1} - u_{i-1}}{2h} + \frac{h^2}{6}u^{(3)}(\mu_1) = f_i + \frac{h}{3}[k_{i0}u_0 + ... + k_{iN}u_N] + \frac{b-a}{180}h^4u^{(4)}(\mu_2), \quad (3.2)$$

and for $i = N$

$$\frac{3u_N - 4u_{N-1} + u_{N-2}}{2h} + \frac{h^2}{4}u^{(4)}(\mu_3)$$
$$= f_N + \frac{h}{3}[k_{N0}u_0 + \ldots + k_{NN}u_N] \quad (3.3)$$
$$+ \frac{(b-a)}{180}h^4 u^{(4)}(\mu_2), \quad (3.4)$$

where $\mu_1, \mu_2, \mu_3 \in (a,b)$. Subtract (2.2), (2.3) from (3.2), (3.3), we obtained the error term as follows:

$$e = \left|\frac{h^2}{6}u^{(3)}(\mu_1) + \frac{h^2}{4}u^{(4)}(\mu_3) - 2\frac{(b-a)}{180}h^4 u^{(4)}(\mu_2)\right|,$$
$$= \left|\frac{h^2}{6}u^{(3)}(\mu_1) + \frac{h^2}{4}u^{(4)}(\mu_3) - \frac{(b-a)}{90}h^4 u^{(4)}(\mu_2)\right|,$$
$$\leq \left|\frac{h^2}{6}u^{(3)}(\mu_1) + \frac{h^2}{4}u^{(4)}(\mu_3)\right|.$$

Let $G_1 = u^{(3)}(\mu_1)$ and $G_2 = u^{(4)}(\mu_3)$, we have

$$e \leq \left|\frac{h^2}{6}G_1 + \frac{h^2}{4}G_2\right|,$$

If we take $G = max\{G_1, G_2\}$, then we have

$$e \leq \left|\frac{h^2}{6}G + \frac{h^2}{4}G\right|,$$
$$\leq \left|\frac{5h^2}{12}G\right|, \quad (3.5)$$

substituting $h = \frac{b-a}{N}$ in (3.5) we get

$$e \leq \left|\frac{5(b-a)^2}{12N^2}G\right|, \quad (3.6)$$

which is the error estimation. Hence the proposed scheme is of second order convergence. □

4. Numerical Examples

In this section, we presented some problems of linear Fredholm integro-differential equations and applied the proposed method of finite difference-Simpson's approach to obtain the numerical solution, $N = 10$ was used in all the examples. The results obtained are compared in terms of absolute errors. A comparison between our method and two different methods of [2] and [13] are presented.

Example 4.1. *Consider LFIDE equation:*

$$u'(x) = xe^x + e^x - x + \int_0^1 xu(t)dt, \quad u(0) = 0, \quad (4.1)$$

Table 1. The exact and approximate solution of Example 4.1

x_i	Exact solution	FDSM	Abs. error
0.1	0.1105170918	0.1088364750	1.6806E-3
0.2	0.2442805516	0.2430225346	1.2580E-3
0.3	0.4049576424	0.4017429978	3.2146E-3
0.4	0.5967298792	0.5936406224	3.0893E-3
0.5	0.8243606355	0.8189936476	5.3670E-3
0.6	1.093271280	1.087681668	5.5896E-3
0.7	1.409626895	1.401381265	8.2456E-3
0.8	1.780432742	1.771552116	8.8806E-3
0.9	2.213642800	2.201655459	1.1987E-2
1	2.718281828	2.705165692	1.3116E-2

Table 2. Comparison of the absolute errors of differential transform ($DTransf$), parametric iteration method (PIM) and finite difference simpson method ($FDSM$) of Example 4.1

x_i	$DTransf$ [2]	PIM [13]	Abs. error of FDSM
0.1	1.00118319E-2	1.10517092E-1	1.6806E-3
0.2	2.78651355E-2	2.44280552E-1	1.2580E-3
0.3	5.08730892E-2	4.04957642E-1	3.0893E-3
0.4	7.55356316E-2	5.96729879E-1	3.0893E-3
0.5	9.71888952E-2	8.24360635E-1	5.3670E-3
0.6	1.09551714E-1	1.09327129E-0	5.5896E-3
0.7	1.04133232E-1	1.40962689E-0	8.2456E-3
0.8	1.94512700E-2	1.78043274E-0	8.8806E-3
0.9	1.0034260E-2	2.21364280E-0	1.1987E-2
1.0	1.55147712E-1	-	1.3116E-2

with exact solution $u(x) = xe^x$. We obtain the numerical result, exact solution and absolute error at different values of x which is represented in Table 1.
Table 1 shows the exact solution of problem in Example 4.1 and the approximate solution obtained by our method. The absolute error obtained indicated that our method can give good approximation to Linear Fredholm Integro-Differential Equations of second kind.
In Table 2 the absolute errors obtained from $DTransf$ and PIM are compared with absolute error obtained from our $FDSM$ method and the numerical results show that our method is more accurate.

Example 4.2. *consider LFIDE equation:*

$$u'(x) = 1 + \frac{1 - e^{x+1}}{x+1} + \int_0^1 e^{xt}u(t)dt, \quad u(0) = 1, \quad (4.2)$$

with exact solution $u(x) = e^x$. We used maple to compute the numerical solution at different values of x as shown in Table 3
Table3 shows the exact solution of problem in Example 4.2 and the approximate solution obtained by our method and from the absolute error obtained it indicated that our method can give good approximation of Linear Fredholm Integro-Differential Equations of second kind.

5. Conclusion

In this paper, a Finite difference-Simpson's approach was used to solve linear Fredholm integro differential equations

Table 3. The exact and approximate solution of Example 4.2

x_i	Exact solution	FDSM	Abs. error of FDSM
0.1	1.105170918	1.081849538	2.33E-2
0.2	1.221402758	1.178615322	4.28E-2
0.3	1.349858808	1.297756098	5.21E-2
0.4	1.491824698	1.434534859	5.73E-2
0.5	1.648721271	1.596984604	5.17E-2
0.6	1.822118800	1.780935610	4.12E-2
0.7	2.013752707	1.995000966	1.87E-2
0.8	2.225540928	2.235618721	1.01E-2
0.9	2.459603111	2.512049535	5.24E-2
1	2.718281828	2.821429404	1.03E-1

(LFIDES). Error estimation of the scheme was derived, which show that the scheme is of second order convergence. However, the numerical results were presented in terms of absolute error and comparison was made with methods of *DTransf* in [2] and *PIM* in [13], the errors obtained showed that the derived method is good tool for approximating Linear integro-differential equation.

References

[1] Aruchunan. E and Sulaiman, Numerical Solution of First-Order Linear Fredholm Integro-Differential Equations using Conjugate Gradient Method, *Curtin Sarawak* 1*st Int. Symposium on Geology,* (2009), 11–13.

[2] Darania. P and Ali Ebadian, A method for the numerical solution of the integro-differential equations, *Appl. Math. and Comp.,* 188(2007), 657–668.

[3] Wazwaz. A, *Linear and Nonlinear Integral Equations Methods and Applications*, Springer Heidelberg Dordrecht London New York.

[4] Danfu. H, Xufeng. S, Numerical solution of integro differential equations by using CAS wavelet operational matrix of integration, *Appl. math. and Comp.*, 194(2007), 460–466.

[5] Kurt. N, Sezer. M, Polynomial Solution of high order linear Fredholm Integro differential equations with constant coeffients, *J. of Franklin institute,* 345(2008), 839–850.

[6] Rahman. M, *Integral Equations and their Applications*, Ashurst Lodge, Ashurst, Southampton, SO40 7AA, UK.

[7] Saadati. R, B. Raftari. B, Adibi. H, Vaezpour. S, Shakeri. S, A comparison between Variational iteration method and Trapezoidal rule for solving LIDEs, *World Appl. Sci. J.* 4(3)(2008), 321–325 .

[8] Vahidi. A, Babolian. E, Cordshooli. Gh, and Azimzadeh. Z, Numerical solution of FIDEs by Adomian decompostion method, *Int. J. of Math. Analy.*, 3(36)(2009), 1769–1773.

[9] Pandey. P.K, Numerical Solution of Linear Fredholm Integro-Differential equations by Non-standard Finite Difference Method, *Appl. and Appl. Math.* 10(2)(2015), 1019–1026.

[10] Manafianheris. J, Solving Integro-differential Equations Using the Modified Laplace Adomian Decomposition Method, *J. of Math. Ext.,* 6(1)(2012), 41–55.

[11] Mohammad. S, Hosseini, Shahmorad. S, Numerical piecewise approximate solution of Fredholm integro-differential equations by the Tau method, *Appl. Math. Model.,* 29(2005), 12–20.

[12] Raftari. B, Numerical Solutions of the Linear Volterra Integro-differential Equations: Homotopy Perturbation Method and Finite Difference Method, *World Appl. Sci. J.* 9 (Spec. Issue of Appl. Math (2010).

[13] Tamamagar M, The Numerical Solution of linear Fredholm integro-differential equations via Parametric iteration method, *Appl. and Comput. Math.*, 3(2014), 4–10.

[14] Jerri A. J, *Introduction to Integral Equations with Applications*, John Wiley and Sons, New York, (1999).

[15] Alwaneh A, Al-Khaled K, Al-Towaiq M, Reliable Algorithms for Solving Integro-Differential Equations with Applications, *Int. J. of Comput. Math.*, 87(2010), 1538–1554.

★★★★★★★★★
ISSN(P):2319 – 3786
Malaya Journal of Matematik
ISSN(O):2321 – 5666
★★★★★★★★★