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Induced V_4-magic labeling of cycle related graphs

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Abstract
Let $V_4 = \{0,a,b,c\}$ be the Klein-4-group with identity element 0 and $G = (V,E)$ be a graph. Let $f : V \to V_4$ be a vertex labeling and $f^* : E \to V_4$ be the induced edge labeling of f, defined by $f^*(v_1v_2) = f(v_1) + f(v_2)$ for all $v_1v_2 \in E$. Then f^* again induces a vertex labeling say $f^{**} : V \to V_4$ defined by $f^{**}(v) = \sum_{vv_1 \in E} f^*(vv_1)$. A graph $G = (V,E)$ is said to be an Induced V_4-Magic Graph (IMG) if there exists a non zero labeling $f : V \to V_4$ such that $f \equiv f^{**}$. The function f, so obtained is called an Induced V_4-Magic labeling (IML) of G and a graph which has no such induced magic labeling is called a non-induced magic graph. In this paper, we discuss Induced V_4- magic labeling of some cycle related graphs.

Keywords
Klein-4-group, Induced V_4-magic graphs

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1. Introduction

This paper deals with only finite, un directed simple and connected graphs. We refer [1] for the phrasing and standard notations related to graph theory. The Klein 4-group is denoted by $V_4 = \{0,a,b,c\}$, which is a non cyclic abelian group of order 4 with every non identity element has order 2. Let $G = (V,E)$, be the graph with vertex set V and edge set E. Let $f : V \to V_4$ be a vertex labeling and $f^* : E \to V_4$ be the induced edge labeling of f, defined by $f^*(v_1v_2) = f(v_1) + f(v_2)$ for all $v_1v_2 \in E$. Then f^* again induces a vertex labeling say $f^{**} : V \to V_4$ defined by $f^{**}(v) = \sum_{vv_1 \in E} f^*(vv_1)$. A graph $G = (V,E)$ is said to be an induced V_4-Magic graph denoted by IMV_4G or simply IMG if there exists a non zero labeling $f : V \to V_4$ such that $f \equiv f^{**}$. The function f, so obtained is called a induced V_4-Magic labeling of G or simply induced Magic labeling of G and it is denoted by IMV_4L or simply IML. In this paper we discuss some cycle related Induced V_4 magic graphs that belongs to the following categories:

(i) $\Gamma(V_4) :=$ class of all induced V_4-magic graphs.

(ii) $\Gamma_{k,0}(V_4) :=$ class of all induced V_4-magic graphs with induced magic labeling f satisfies $f(V(G)) = \{k,0\}$ for some $k \in V_4$.

2. Main Results

Theorem 2.1. *(See[4]) If f is an induced magic labeling of a graph G and u be a pendant vertex adjacent to a vertex v in G,then $f(v) = 0$.*

Corollary 2.2. *(See[4]) If f is an induced magic labeling of a graph G and wuvz is a path in G with w and z are pendant vertices in G,then $f^*(uv) = 0$.*

Theorem 2.3. *(See[4]) Let f be any vertex labeling of a graph G and u, be a vertex in G with deg$(u) = m$. Then f is an induced magic labeling of G if and only if $(m-1)f(u) + \Sigma f(v) = 0$ where the summation is taken over all the vertices v which are adjacent to u.*

Corollary 2.4. *[Degree sum equation of a vertex]*
Let f be any vertex labeling of a graph G and u, be a vertex in G with deg$(u) = m$. Then f is an induced V_4magic labeling if and only if $f(u) + \Sigma f(v) = 0$ or $\Sigma f(v) = 0$ according as

$deg(u) = m$ is even or odd, where the summation is taken over all the vertices v which are adjacent to u.

Proof. From the above theorem we have f is an induced V_4magic labeling of G if and only if $(m-1)f(u)+\Sigma f(v) = 0$, where v is adjacent to u, then the result follows directly from the fact that $f(u) \in V_4$. □

3. Cycle related graphs

Theorem 3.1. *(See[4]) $C_n \in \Gamma(V_4)$ if and only if $n \equiv 0(mod\ 3)$.*

Corollary 3.2. *(See[4]) $C_n \in \Gamma_{a,0}(V_4)$ if and only if $n \equiv 0(mod\ 3)$.*

Definition 3.3. *(See [3]) The sum of the graphs C_n and K_1 is called a Wheel graph and it is denoted by W_n, that is $W_n = C_n + K_1$.*

Theorem 3.4. *The Wheel graph $W_n \in \Gamma(V_4)$ if and only if n is even.*

Proof. Let $V(W_n) = \{w, v_1, v_2, v_3, \cdots, v_n\}$, where w is the central vertex. Suppose n is even,then $n = 4k$ or $4k+2$ for some positive integer k.

Case 1: $n = 4k$

In this case, define $f : V(W_n) \to V_4$ as :

$$f(v) = \begin{cases} 0 & \text{if} \quad v = w \\ a & \text{if} \quad v = v_1, v_3, v_5, \cdots v_{4k-1} \\ b & \text{if} \quad v = v_2, v_4, v_6, \cdots v_{4k} \end{cases}$$

Case 2: $n = 4k+2$

In this case, define $f : V(W_n) \to V_4$ as :

$$f(v) = \begin{cases} 0 & \text{if} \quad v = w \\ a & \text{if} \quad v = v_i \end{cases}$$

then, in both case we can verify that f is an induced V_4 magic labeling of W_n. Conversely suppose n is an odd number, if f is an induced V_4 magic labeling of W_n then by the degree sum equation of vertices in W_n, f must satisfy the following system of equations.

$$\begin{aligned} f(v_2)+f(v_n)+f(w) &= 0 \\ f(v_1)+f(v_3)+f(w) &= 0 \\ f(v_2)+f(v_4)+f(w) &= 0 \\ f(v_3)+f(v_5)+f(w) &= 0 \\ &\vdots \\ f(v_1)+f(v_{n-1})+f(w) &= 0 \\ f(v_1)+f(v_2)+f(v_3)+\cdots+f(v_n) &= 0 \end{aligned}$$

From the system of equations we have, $f(v_1) = f(v_2) = f(v_3) = \cdots = f(v_n)$, thus from the last equation we have $nf(v_i) = 0$, for $i = 1, 2, 3, \cdots, n$. Since n is odd this happens only when $f(v_i) = 0$ for $i = 1, 2, 3, \cdots, n$. Using this in the first equation of the above system of equations we have $f(w) = 0$ also. Thus in this case $f \equiv 0$, hence f is not an Induced V_4 magic labeling. □

Corollary 3.5. *$W_n \in \Gamma_{a,0}(V_4)$ if and only if n is even.*

Proof. Let $V(W_n) = \{w, v_1, v_2, v_3, \cdots, v_n\}$, where w is the central vertex. Suppose n is even number. Define $f : V(W_n) \to V_4$ as :

$$f(v) = \begin{cases} 0 & \text{if} \quad v = w \\ a & \text{if} \quad v = v_i \end{cases}$$

Then we can verify that f is an induced V_4 magic labeling of W_n. Converse part follows from the above theorem. □

Definition 3.6. *(See [3]) The helm H_n is a graph obtained from a wheel W_n by attaching a pendent edge at each vertex of the n-cycle.*

Theorem 3.7. *The Helm graph $H_n \in \Gamma(V_4)$ if and only if n is odd.*

Proof. Let $V(H_n) = \{w, v_1, v_2, v_3, \cdots, v_n, w_1, w_2, w_3, \cdots, w_n\}$, where w be the central vertex and $w_1, w_2, w_3, \cdots, w_n$be the pendent vertices adjacent to $v_1, v_2, v_3, \cdots, v_n$. Suppose n is odd,then define $f : V(W_n) \to V_4$ as :

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_1, v_2, v_3, \cdots v_n \\ a & \text{if} \quad v = w, w_1, w_2, w_3, \cdots, w_n \end{cases}$$

Then clearly f is an induced V_4 magic labeling of H_n. Conversely suppose n is an even number, if f is an induced V_4 magic labeling of H_n then by the degree sum equation of vertices in H_n, f must satisfy the following system of equations.

$$\begin{aligned} f(v_i) &= 0 \text{ for } i = 1, 2, 3, \cdots, n \\ f(w)+f(w_i) &= 0 \text{ for } i = 1, 2, 3, \cdots, n \\ f(w) &= 0 \end{aligned}$$

Thus $f(v_i) = f(w_i) = f(w) = 0$, that is $f \equiv 0$. Hence f is not an Induced V_4 magic labeling. □

From the proof of above theorem we have the following corollary

Corollary 3.8. *If n an odd number then $H_n \in \Gamma_{a,0}(V_4)$.*

Definition 3.9. *(See [3]) The Web graph $W(2,n)$ is a graph obtained by joining the pendent points of a helm to form a cycle and then adding a single pendent edge to each vertex of this outer cycle.*

Theorem 3.10. *The Web graph $W(2,n) \in \Gamma(V_4)$ if and only if $n \equiv 0(mod\ 3)$.*

Proof. Let $\{w, u_i, v_i, w_i / i = 1, 2, 3, \cdots, n\}$, be the vertex set of $W(2,n)$, where w is the central vertex, $u_1, u_2, u_3, \cdots, u_n$ are the vertices of inner cycle, $v_1, v_2, v_3, \cdots, v_n$ are the vertices of outer cycle and $w_1, w_2, w_3, \cdots, w_n$ are the pendent vertices

adjacent to $v_1, v_2, v_3, \cdots, v_n$ of $W(2,n)$.
Suppose $n \equiv 0(mod\ 3)$, then define $f : V(W(2,n)) \to V_4$ as :

$$f(v) = \begin{cases} 0 & \text{if} \quad v = w, v_1, v_2, v_3, \cdots, v_n \\ a & \text{if} \quad v = u_i, w_i, \text{ for } i \equiv (1\ mod\ 3) \\ b & \text{if} \quad v = u_i, w_i, \text{ for } i \equiv (2\ mod\ 3) \\ c & \text{if} \quad v = u_i, w_i, \text{ for } i \equiv (0\ mod\ 3) \end{cases}$$

Then clearly f is an induced V_4 magic labeling of $W(2,n)$. Conversely suppose that $n \not\equiv 0(mod\ 3)$, then $n = 3k+1$ or $3k+2$ for some positive integer k. If possible suppose f is an induced V_4 magic labeling of $W(2,n)$ then by the degree sum equation of vertices in $W(2,n)$, f must satisfy the following system of equations.

$$\begin{aligned} f(v_i) &= 0 \text{ for } i = 1,2,3,\cdots,n \\ f(u_1)+f(u_2)+f(u_n)+f(w) &= 0 \\ f(u_1)+f(u_2)+f(u_3)+f(w) &= 0 \\ f(u_2)+f(u_3)+f(u_4)+f(w) &= 0 \\ &\vdots \\ f(u_1)+f(u_{n-1})+f(u_n)+f(w) &= 0 \\ f(u_i)+f(w_i) &= 0 \text{ for } i = 1,2,3,\cdots,n \\ (n-1)f(w)+\sum_{i=1}^{n} f(u_i) &= 0. \end{aligned}$$

Since $n = 3k+1$ or $3k+2$, from the above system of equations we have, $f(v_i) = 0, f(u_i) = f(w_i), f(u_1) = f(u_2) = f(u_3) = \cdots = f(u_n)$ and $(n-1)f(w)+nf(u_i) = 0$ for $i = 1,2,3,\cdots,n$.

Case 1: $n = 3k+1$

Subcase 1: k is even
Note that k is even implies $n = 3k+1$ is odd. Therefore the equation $(n-1)f(w)+nf(u_i) = 0$ for $i = 1,2,3,\cdots,n$ reduces to $f(u_i) = 0$ for $i = 1,2,3,\cdots,n$. Hence in this case $f(u_i) = f(v_i) = f(w_i) = f(w) = 0$.

Subcase 2: k is odd
Note that k is odd implies $n = 3k+1$ is even. Thus the equation $(n-1)f(w)+nf(u_i) = 0$ for $i = 1,2,3,\cdots,n$ reduces to $f(w) = 0$. Thus from the system of equations we have, $f(u_i) = 0$. Hence in this case also, $f(u_i) = f(v_i) = f(w_i) = f(w) = 0$.

Case 2: $n = 3k+2$

Subcase 1: k is even
Note that k is even implies $n = 3k+2$ is even. Thus the equation $(n-1)f(w)+nf(u_i) = 0$ for $i = 1,2,3,\cdots,n$ reduces to $f(w) = 0$. Thus from the system of equations we have, $f(u_i) = 0$. Hence in this case $f(u_i) = f(v_i) = f(w_i) = f(w) = 0$.

Subcase 2: k is odd
Note that k is odd implies $n = 3k+2$ is odd. Therefore the equation $(n-1)f(w)+nf(u_i) = 0$ for $i = 1,2,3,\cdots,n$ reduces to $f(u_i) = 0$ for $i = 1,2,3,\cdots,n$. Hence in this case also $f(u_i) = f(v_i) = f(w_i) = f(w) = 0$.

Hence in both case we have $f \equiv 0$, that is f is not an Induced V_4 magic labeling. □

Definition 3.11. *(See [3]) A closed helm CH_n is a graph obtained from a helm by joining each pendent vertex to form a cycle.*

Theorem 3.12. *The closed helm $CH_n \in \Gamma(V_4)$ for n is odd.*

Proof. Let $V(CH_n) = \{w, v_1, v_2, v_3, \cdots, v_n, w_1, w_2, w_3, \cdots, w_n\}$, where w be the central vertex and $w_1, w_2, w_3, \cdots, w_n$ be the pendant vertices adjacent to $v_1, v_2, v_3, \cdots, v_n$ in corresponding H_n. Suppose n is odd,then define $f : V(CH_n) \to V_4$ as :

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_1, v_2, v_3, \cdots v_n \\ a & \text{if} \quad v = w, w_1, w_2, w_3, \cdots, w_n \end{cases}$$

□

Then f is an IML of CH_n. Hence the proof.

Corollary 3.13. *$CH_n \in \Gamma_{a,0}(V_4)$ for n is odd.*

Proof. Follows directly from the proof of above theorem. □

Definition 3.14. *(See [3]) A Flower graph Fl^n is a graph obtained from a helm by joining each pendent vertex to the central vertex of the helm.*

Theorem 3.15. *The Flower graph $Fl^n \in \Gamma(V_4)$ for all n.*

Proof. Let $V(Fl^n) = \{w, u_1, u_2, u_3, \cdots, u_n, v_1, v_2, v_3, \cdots, v_n,\}$, where w is the central vertex, $u_1, u_2, u_3, \cdots, u_n$ are the vertices of corresponding cycle and $v_1, v_2, v_3, \cdots, v_n$ are the vertices adjacent to the central vertex w.

Case 1: n is odd
In this case, define $f : V(Fl^n) \to V_4$ as :

$$f(v) = \begin{cases} a & \text{if} \quad v = w \\ b & \text{if} \quad v = u_i \\ c & \text{if} \quad v = v_i \end{cases}$$

Case 2: n is even
In this case, define $f : V(Fl^n) \to V_4$ as :

$$f(v) = \begin{cases} 0 & \text{if} \quad v = w \\ a & \text{if} \quad v = u_i, v_i \text{ for } i \text{ is odd} \\ b & \text{if} \quad v = u_i, v_i \text{ for } i \text{ is even} \end{cases}$$

Then, in both case we can verify that f is an induced V_4 magic labeling of Fl^n. □

Corollary 3.16. *$Fl^n \in \Gamma_{a,0}(V_4)$ for all n.*

Proof. Let $V(Fl^n)=\{w,u_1,u_2,u_3,\cdots,u_n,v_1,v_2,v_3,\cdots,v_n,\}$, where w is the central vertex, $u_1,u_2,u_3,\cdots,u_n$ are the vertices of corresponding cycle and $v_1,v_2,v_3,\cdots,v_n$ are the vertices adjacent to the central vertex w. Define $f:V(Fl^n)\to V_4$ as :

$$f(v)=\begin{cases} 0 & \text{if} \quad v=w \\ a & \text{if} \quad v=u_i,v_i \end{cases}$$

Then one can easily verify that f is an IML of Fl^n. Hence the proof. □

Definition 3.17. *(See [3]) A Gear graph is a graph G_n obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the n-cycle.*

Theorem 3.18. *The Gear graph $G_n\in\Gamma(V_4)$ if and only if n is even.*

Proof. Let $V(G_n)=\{w,u_1,u_2,u_3,\cdots,u_n,v_1,v_2,v_3,\cdots,v_n,\}$, where w is the central vertex, $u_1,u_2,u_3,\cdots,u_n$ are the vertices of the corresponding Wheel graph W_n and $v_1,v_2,v_3,\cdots,v_n$ are the remaining vertices with $u_iv_i,v_iu_{i+1}\in E(G_n)$ where i is taken modulo n.

Suppose n is even, then define $f:V(G_n)\to V_4$ as :

$$f(v)=\begin{cases} 0 & \text{if} \quad v=w \\ a & \text{if} \quad v=u_i \text{ for } i \text{ is odd} \\ b & \text{if} \quad v=u_i \text{ for } i \text{ is even} \\ c & \text{if} \quad v=v_i \end{cases}$$

Then we can easily verify that this f is an induced V_4 magic labeling of G_n. Conversely suppose that n is an odd number.Then by the degree sum equation of vertices in G_n we have: if f is an induced V_4 magic labeling of G_n then f must satisfy the following system of equations.

$$\begin{aligned} f(v_1)+f(v_2)+f(w) &= 0 \\ f(v_2)+f(v_3)+f(w) &= 0 \\ &\vdots \\ f(v_{n-1})+f(v_n)+f(w) &= 0 \\ f(v_n)+f(v_1)+f(w) &= 0 \end{aligned}$$

$$\begin{aligned} f(u_1)+f(u_2)+f(u_3)+\cdots+f(u_n) &= 0 \\ f(v_1)+f(u_1)+f(u_2) &= 0 \\ f(v_2)+f(u_2)+f(u_3) &= 0 \\ f(v_3)+f(u_3)+f(u_4) &= 0 \\ &\vdots \\ f(v_n)+f(u_n)+f(u_1) &= 0 \end{aligned}$$

Since n is odd the equations corresponding to the vertices $u_1,u_2,u_3,\cdot,u_n$ (that is the first n equations)implies that $f(v_1)=f(v_2)=f(v_3)=\cdots=f(v_n)$, substituting these in the above system of equations we get $f(w)=0$. Now using the fact n is odd and $f(v_1)=f(v_2)=f(v_3)=\cdots=f(v_n)$, the last n equations in the above system of equations implies that $f(u_1)=f(u_2)=f(u_3)=\cdots=f(u_n)$. Substituting these in the equation $f(u_1)+f(u_2)+f(u_3)+\cdots+f(u_n)=0$ we get $f(u_i)=0$, for $i=1,2,3,\cdots,n$ which implies $f(v_i)=0$, $i=1,2,3,\cdots,n$. Hence $f\equiv 0$,that is f is not an induced V_4 magic labeling. □

Definition 3.19. *(See [3]) A Fan graph is denoted by F_n and is defined as P_n+K_1, where P_n is the path graph with n vertices.*

Theorem 3.20. *The Fan graph $F_n\in\Gamma(V_4)$ for n is even.*

Proof. Suppose n is an even number. we have $F_n=P_n+K_1$. Let $V(F_n)=\{w,v_1,v_2,v_3,\cdots,v_n,\}$, where w is the vertex of K_1 and $v_1,v_2,v_3,\cdots,v_n$, be the vertices of P_n. Then define $f:V(F_n)\to V_4$ as follows :

$$f(v)=\begin{cases} 0 & \text{if} \quad v=w \\ a & \text{if} \quad v=v_i, \text{ for } i=1,2,3,\cdots,n. \end{cases}$$

Then we can easily verify that this f is an induced V_4 magic labeling of F_n. Hence the proof follows. □

Definition 3.21. *(See [3]) A Flag graph is denoted by Fl_n and is obtained by joining one vertex of C_n to an extra vertex called the root.*

Theorem 3.22. *The Flag graph $Fl_n\in\Gamma(V_4)$ if and only if $n\equiv 0(mod\ 3)$.*

Proof. Let $V(Fl_n)=\{w,v_1,v_2,v_3,\cdots,v_n,\}$, where v_i for $i=1,2,3,\cdots,n$, is the vertex of corresponding cycle graph C_n and w is the root vertex adjacent to the vertex v_1.
Suppose $n\equiv 0(mod\ 3)$, then define $f:V(Fl_n)\to V_4$ as :

$$f(v)=\begin{cases} 0 & \text{if} \quad v=v_i, i\equiv 1(mod\ 3) \\ a & \text{if} \quad v=v_i, i\equiv 0,2(mod\ 3) \\ 0 & \text{if} \quad v=w \end{cases}$$

Then we can easily verify that this f is an induced V_4 magic labeling of Fl_n.
Conversely suppose that $n\not\equiv 0(mod\ 3)$. If possible suppose f is an induced V_4 magic labeling of Fl_n, then by the degree sum equation of vertices w and v_i in Fl_n, f must satisfy the following system of equations.

$$\begin{aligned} f(v_1) &= 0 \\ f(v_2)+f(v_n)+f(w) &= 0 \\ f(v_2)+f(v_3) &= 0 \\ f(v_2)+f(v_3)+f(v_4) &= 0 \\ f(v_3)+f(v_4)+f(v_5) &= 0 \\ &\vdots \\ f(v_{n-2})+f(v_{n-1})+f(v_n) &= 0 \\ f(v_{n-1})+f(v_n) &= 0 \end{aligned}$$

Note that $n \not\equiv 0(mod\ 3)$ implies that $n = 3k+1$ or $n = 3k+2$ for some integer k, also from the above system of equations we have $f(v_1) = f(v_{n-2}) = 0$ and $f(v_2) = f(v_3)$. Using these facts we can prove that $f(v_1) = f(v_2) = f(v_3) = \cdots = f(v_n) = 0$ and $f(w) = 0$. Hence f is not an induced V_4 magic labeling.
Hence the proof follows. □

Definition 3.23. *(See [3]) A Sunflower graph is denoted by SF_n and is obtained by taking a wheel with the central vertex v_0 and the n-cycle $v_1, v_2, v_3, \cdots, v_n$ and additional vertices $w_1, w_2, w_3, \cdots, w_n$ where w_i is joined by edges to v_i, v_{i+1} where $i+1$ is taken modulo n.*

Theorem 3.24. *The Sun flower graph $SF_n \in \Gamma(V_4)$, for n is even.*

Proof. Suppose the given Sun flower graph is obtained by taking a wheel graph with the central vertex v_0, the $n-$cycle $v_1, v_2, v_3, \cdots, v_n$ and additional vertices $w_1, w_2, w_3, \cdots, w_n$ where w_i is joined by edges to v_i, v_{i+1}, where $i+1$ is taken modulo n.
Suppose n is even, then define $f : V(SF_n) \to V_4$ as :

$$f(v) = \begin{cases} 0 & \text{if } v = v_0, w_i \quad \text{for } i = 1,2,3,\cdots,n \\ a & \text{if } v = v_i \quad \text{for } i = 1,2,3,\cdots,n \end{cases}$$

Then we can easily prove that f is an induced V_4 magic labeling of SF_n. □

From the proof of above theorem we have the following corollary.

Corollary 3.25. *If n is even then the Sun flower graph $SF_n \in \Gamma_{a,0}(V_4)$.*

Definition 3.26. *(See [3]) Jelly fish graph $J(m,n)$ is obtained from a $4-$cycle $v_1v_2v_3v_4v_1$ by joining v_1 and v_3 with an edge and appending the central vertex of $K_{1,m}$ to v_2 and appending the central vertex of $K_{1,n}$ to v_4.*

Theorem 3.27. *The Jelly fish $J(m,n) \in \Gamma(V_4)$ for all m and n.*

Proof. Consider the Jelly fish graph with $V(J(m,n)) = \{v_k/k = 1,2,3,4\} \cup \{u_i/i = 1,2,3,\cdots,m\} \cup \{w_j/j = 1,2,3,\cdots,n\}$ where v_i's are the vertices of C_4 and u_i, w_j are the vertices of corresponding $K_{1,m}$ and $K_{1,n}$ respectively. Now consider the following cases:

Case 1 : both m and n are odd.
Define $f : V(J(m,n)) \to V_4$ as:

$$f(v) = \begin{cases} 0 & \text{if} \quad u_1, w_1, v = v_k, \text{ for } k = 1,2,3,4 \\ a & \text{if} \quad v = u_i, \text{ for } i = 2,3,4,\cdots,m \\ a & \text{if} \quad v = w_j, \text{ for } j = 2,3,4,\cdots,n \end{cases}$$

Case 2 : m and n are even.
Define $f : V(J(m,n)) \to V_4$ as:

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_k, \text{ for } k = 1,2,3,4 \\ a & \text{if} \quad v = u_i, \text{ for } i = 1,2,3,\cdots,m \\ a & \text{if} \quad v = w_j, \text{ for } j = 1,2,3,\cdots,n \end{cases}$$

Case 3 : m odd and n even.
Define $f : V(J(m,n)) \to V_4$ as:

$$f(v) = \begin{cases} 0 & \text{if} \quad v = u_1, v_k, \text{ for } k = 1,2,3,4 \\ a & \text{if} \quad v = u_i, \text{ for } i = 2,3,4,\cdots,m \\ a & \text{if} \quad v = w_j, \text{ for } j = 1,2,3,\cdots,n \end{cases}$$

Case 4 : m even and n odd.
Define $f : V(J(m,n)) \to V_4$ as:

$$f(v) = \begin{cases} 0 & \text{if} \quad v = w_1, v_k, \text{ for } k = 1,2,3,4 \\ a & \text{if} \quad v = u_i, \text{ for } i = 1,2,3,\cdots,m \\ a & \text{if} \quad v = w_j, \text{ for } j = 2,3,4,\cdots,n \end{cases}$$

In all the above cases, we can prove that f is an induced magic labeling of $J(m,n)$.
Hence the proof. □

Corollary 3.28. *The Jelly fish $J(m,n) \in \Gamma_{a,0}(V_4)$ for all m and n.*

Definition 3.29. *(See [3]) The Sun graph on $m = 2n$ vertices, denoted by Sun_n, is the graph obtained by attaching a pendant vertex to each vertex of a $n-$cycle.*

Theorem 3.30. *The Sun graph $Sun_n \notin \Gamma(V_4)$ for all n.*

Proof. Consider a Sun graph Sun_n with $\{v_1, v_2, v_3, \cdots, v_n\}$ as vertex set of the corresponding C_n and $w_i, 1 \leq i \leq n$, be the pendant vertices attached to each $v_i, 1 \leq i \leq n$. If possible suppose $f : V(Sun_n) \to V_4$ is an IML of Sun_n. Then the degree sum equation of w_i, we have $f(v_i) = 0$. using this in the degree sum equation of v_i we get $f(w_i) = 0$. Thus $f \equiv 0$, which is a contradiction.
Hence the proof. □

References

[1] Chartrand G, Zhang P, *Introduction to Graph Theory,* McGraw-Hill, Boston; 2005.

[2] Frank Harary *Graph Theory*, Narosa Publishing House.

[3] J.A.Gallian, A dynamic survey of labeling, *The Electronics Journal of Combinatorics*,**17**(2014), 12–20.

[4] K.B. Libeeshkumar and V. Anil Kumar, Induced Magic Labeling of Some Graphs, *Malaya Journal of Matematik*, 8(1)(2020), 59–61.

[5] Lee SM, Saba F, Salehi E and Sun H, On the V_4-Magic Graphs, *Congressus Numerantium*,**156**(2002), 59–67.

[6] R Balakrishnan and K Ranganathan, *A Text Book of Graph Theory,* Springer-Verlag, NewYork, 2012.

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