Malaya Journal of Matematik, Vol. 8, No. 2, 478-484, 2020
https://doi.org/ 10.26637/MJM0802/0026

On c-representability of Permutation Groups

Kavitha T.[1]* and Sini P.[2]

Abstract
In this paper we study c-representability of permutation groups. We prove that the Dihedral group D_n is a c-representable permutation group. We discuss the c-representability of some cyclic subgroups of the symmetric group S(X). Some properties of c- representable permutation groups are also discussed.

Keywords
Čech closure space, permutation groups, closure isomorphisms, group of closure isomorphisms.

AMS Subject Classification
54A05, 20B35.

[1,2] *Department of Mathematics, University of Calicut, Kerala-673635, India.*
*Corresponding author: [1] kavithatrnair@gmail.com; [2]sinimecheri@gmail.com
Article History: Received **3** November **2019**; Accepted **12** March **2020**

Contents

1. Introduction

The concept of Čech closure spaces was introduced by Edward Čech as a generalisation of topological spaces. Various concepts in Čech closure spaces were studied in [6, 12, 13, 15, 17, 19]. Adjacency in the lattice of closure operators were discussed in [14]. Boonpok C. investigated generalized closed sets in Čech closure spaces and determined some of their characterizations[3].

Ramachandran P. T. discussed the problem of representing permutation groups as the group of homeomorphisms of topological spaces[15, 18]. He proved that if $X = \{a_1, a_2, \dots, a_n\}$, $n \geq 3$, then the permutation group on X generated by the cycle $(a_1, a_2, \dots, a_n)$ cannot be represented as the group of homeomorphisms of (X,T) for any topology T on X [15]. The t-representability of normal subgroups of the symmetric group $S(X)$ was studied in [15, 18]. Then Sini P. and Ramachandran P. T. defined t-representability of permutation groups and studied t-representability of some subgroups of the symmetric group $S(X)$[21–23]. A permutation group K on a set X is said to be t-representable if there exists a topology T on X such that the group $H(X,T)$ of homeomorphisms of (X,T) is K [22]. In [22], it was proved that direct sum of t-representable finite permutation groups is t-representable on X. The t- representabity of transitive permutation groups, maximal subgroups of the symmetric group, dihedral groups and cyclic permutation groups etc. were studied in [20–24]. An analogous concept is introduced in Closure spaces in [13]. c-representability permutation groups is defined and it is proved that normal subgroups of $S(X)$ is c-representable if and only if $|X| \neq 3$[13].

In this paper we investigate some problems related to group of closure isomorphisms of Čech closure spaces. Here we continue the study of c-representablity of permutation groups.

2. Preliminaries

In this section, we discuss some basic concepts used in this paper. Set theoretical notions are adopted from [9]. Let $P(X)$ denotes the power set of X. A closure operator is defined as follows.

Definition 2.1. *[6] A Čech closure operator on a set X is a function $V : P(X) \to P(X)$ satisfying $V(\emptyset) = \emptyset$, $A \subseteq V(A)$, and $V(A \cup B) = V(A) \cup V(B)$ for every A, $B \in P(X)$. Simply we call V a closure operator on X and the pair (X,V) a closure space.*

A subset A of a closure space (X,V) is said to be closed if $V(A) = A$, and is said to be open if its complement is closed. A subset A of X is said to be dense if $V(A) = X$. The collection of all open sets in a closure space (X,V) is a topology on X, called the topology associated with V. A closure operator V

is said to be topological if and only if $V(V(A)) = V(A)$ for every $A \subseteq X$.

Let $I : P(X) \to P(X)$ be given by

$$I(A) = \begin{cases} \emptyset & ; \quad \text{if } A = \emptyset \\ X & ; \quad \text{otherwise.} \end{cases}$$

Then I is a closure operator on X. This closure operator is the topological closure operator associated with the indiscrete topology on X and is called the indiscrete closure operator. The closure operator D on X given by $D(A) = A$ for all $A \in P(X)$, is the topological closure operator associated with the discrete topology on X, called the discrete closure operator. A closure space (X,V) is said to be T_1 if $V(\{a\}) = \{a\}$ for each $a \in X$.

Definition 2.2. *[7] A permutation of a set X is a function $\phi : X \to X$ that is both one-one and onto.*

The function composition $\circ$ is a binary operation on the collection of all permutations of a set A. This operation is called permutation multiplication. The set of all permutations of a set X forms a group under permutation multiplication, denoted by the symmetric group $S(X)$[7]. We write S_n to denote the group $S(X)$ when n is a positive integer and $X = \{1,2,\dots,n\}$[7]. A permutation group is a subgroup of the symmetric group $S(X)$. A cycle of length 2 is a transposition. Any cycle is a product of transpositions and any permutation of a finite set of at least two elements is a product of transpositions.

Definition 2.3. *[15] Let $(X,\ V)$ and $(Y,\ V')$ be two closure spaces. A closure isomorphism from $(X,\ V)$ to $(Y,\ V')$ is a bijection $f : X \longrightarrow Y$ such that $f(V(A)) = V'(f(A))$ for all $A \in P(X)$.*

If $(X,\ V)$ is a closure space, then the set of all closure isomorphisms from $(X,\ V)$ onto itself is a group under function composition and is called the group of closure isomorphisms of $(X,\ V)$, denoted by $CI(X,V)$. Note that $CI(X,V)$ is a subgroup of the symmetric group $S(X)$.

Definition 2.4. *[13] A subgroup H of $S(X)$ is said to be c-representable on X if there exists a closure operator V on X such that $CI(X,\ V) = H$.*

3. Main Results

We determined c-representability of normal subgroups of $S(X)$ in[13]. In this section we study the c-representability of dihedral group and permutation groups generated by product of cycles. We use the following results in [13].

Theorem 3.1. *[13] If a permutation group H is t-representable on a set X, then it is c-representable on X.*

Theorem 3.2. *[13] Let X be a finite set $\{a_1,a_2,\dots,a_n\}$ and H be the group of permutations of X generated by the cycle $f = (a_1,a_2,\dots,a_n)$. Then H is c-representable on X.*

Note that any permutation generated by an infinite cycle on an infinite set is t-representable hence it is c-representable[16]. The following Theorem says that in order to determine the c-representability of a permutation group H on a set X, we have to consider only the c-representability of H on the set of all points which are moved by the permutations of H.

We need the following definition.

Definition 3.3. *[5] Let G_1 and G_2 be two permutation groups on X_1 and X_2 respectively. The direct product $G_1 \times G_2$ acts on the disjoint union $X_1 \cup X_2$ by the rule*

$$(g_1,g_2)(x) = \begin{cases} g_1(x) & \text{if } x \in X_1 \\ g_2(x) & \text{if } x \in X_2. \end{cases}$$

Theorem 3.4. *Let X be any set and $Y \subseteq X$. If H is a c-representable permutation group on Y, then the permutation group $\{I_{X\setminus Y}\} \times H$ is c-representable on X, where $I_{X\setminus Y}$ denotes the identity permutation on $X \setminus Y$.*

Proof. Since H is c-representable on Y, there exists a closure operator V_1 on Y such that $CI(X,V_1) = H$. Let $Z = X \setminus Y$. If $Z = \emptyset$, there is nothing to prove. Suppose $Z \neq \emptyset$. By the well ordering theorem, well order the set Z by the order relation $<$. We can use the ordinals to index the members of Z. Let x_0 be the first element of Z and x_1 be the first element of $Z \setminus \{x_0\}$. In general x_α denotes the first element of $Z \setminus \{x \in Z : x < x_\alpha\}$ provided $\{x \in Z : x < x_\alpha\}$ is non-empty. Now we define a closure operator V_2 on Z as follows.
$V_2(A) = \bigcup\limits_{x_\alpha \in A} V_2(x_\alpha)$ for $A \subseteq Z$ where $V_2(x_\alpha) = Z \setminus \{x \in Z : x < x_\alpha\}$. Then V_2 is a closure operator on Z. Consider X as $X = Y \cup Z$. Let $A \subseteq X$. Then $A = A_1 \cup A_2$ where $A_1 = A \cap Y$ and $A_2 = A \cap Z$. Define $V : P(X) \to P(X)$ as follows:

$$V(A) = \begin{cases} \emptyset & ; \quad \text{if } A = \emptyset \\ V_1(A_1) & ; \quad \text{if } A_2 = \emptyset \\ Y \cup V_2(A_2) & ; \quad \text{if } A_2 \neq \emptyset. \end{cases}$$

We have to prove that V is a closure operator on X.

Let $A \subseteq X$. If $A = \emptyset$, then there is nothing to prove. Now suppose that $A \neq \emptyset$. We have $A = A_1 \cup A_2$. If $A_2 = \emptyset$, then $V(A) = V_1(A)$ and hence $A \subseteq V(A)$. If $A_2 \neq \emptyset$, $V(A) = Y \cup V_2(A_2)$. Then clearly $A \subseteq V(A)$.

Let $A,B \subseteq X$. $A = A_1 \cup A_2$, $B = B_1 \cup B_2$, where $A_1,B_1 \subseteq Y$ and $A_2,B_2 \subseteq Z$.

Case (i): $A_2 = \emptyset$, $B_2 = \emptyset$
In this case $A,\ B \subseteq Y$ and hence $V(A) = V_1(A_1)$ and $V(B) = V_1(B_1)$. Then

$$\begin{aligned} V(A \cup B) &= V_1(A_1 \cup B_1) \\ &= V_1(A_1) \cup V_1(B_1) \\ &= V(A) \cup V(B). \end{aligned}$$

Case (ii): $A_2 \neq \emptyset$, $B_2 = \emptyset$ Then $V(A) = Y \cup V_2(A_2)$, $V(B) = V_1(B_1)$.

Now

$$\begin{aligned} V(A) \cup V(B) &= Y \cup V_2(A_2) \cup V_1(B_1) \\ &= Y \cup V_2(A_2), \text{ since } V_1(B_1) \subseteq Y. \end{aligned}$$

and

$$\begin{aligned} V(A \cup B) &= V[(A_1 \cup B_1) \cup (A_2 \cup B_2)] \\ &= Y \cup V_2(A_2 \cup B_2) \\ &= Y \cup V_2(A_2) \end{aligned}$$

Hence $V(A \cup B) = V(A) \cup V(B)$.
Case (iii): $A_2 = \emptyset$, $B_2 \neq \emptyset$.
Similar to Case (ii).
Case (iv): $A_2 \neq \emptyset$, $B_2 \neq \emptyset$
Here $V(A) = Y \cup V_2(A_2)$ and $V(B) = Y \cup V_2(B_2)$. Then

$$\begin{aligned} V(A \cup B) &= Y \cup V_2(A_2 \cup B_2) \\ &= Y \cup V_2(A_2) \cup V_2(B_2) \\ &= [Y \cup V_2(A_2)] \cup [Y \cup V_2(B_2)] \\ &= V(A) \cup V(B). \end{aligned}$$

Thus V is a closure operator on X.

Next we claim that $CI(X,V) = \{I_Z\} \times Y$.
Let $f = (I_Z, h) \in \{I_Z\} \times H$ and $A \subseteq X$. Then we have to show that $V(f(A)) = f(V(A))$.
Now

$$\begin{aligned} V(f(A)) &= V(f(A_1 \cup A_2)) \\ &= V((I_Z,h)(A_1 \cup h(A_2))) \\ &= V(A_1 \cup h(A_2)). \end{aligned}$$

Since $A = A_1 \cup A_2$, we consider the following cases.
Case (i): $A_2 = \emptyset$
Then

$$\begin{aligned} V(f(A)) &= V(h(A_1)) \\ &= V_1(h(A_1)). \end{aligned}$$

Now

$$\begin{aligned} f(V(A)) &= f(V_1(A_1)) \\ &= h(V_1(A_1)) \\ &= V_1(h(A_1)). \end{aligned}$$

Hence $V(f(A)) = f(V(A))$.
Case (ii): $A_2 \neq \emptyset$
Then $V(f(A)) = V(h(A_1) \cup A_2) = Y \cup V_2(A_2)$.

$$\begin{aligned} \text{Now } f(V(A)) &= f(V(A_1 \cup A_2)) \\ &= f(Y \cup V_2(A_2)) \\ &= h(Y) \cup V_2(A_2) \\ &= Y \cup V_2(A_2). \end{aligned}$$

Thus $f(V(A)) = V(f(A))$, for every $A \subseteq X$. It follows that f is a closure isomorphism on (X,V). Hence

$$\{I_{X \setminus Y}\} \times H \subseteq CI(X,V). \tag{3.1}$$

Now let $f \in CI(X,V)$. We have $V(X \setminus \{x_0\}) = X \setminus \{x_0\}$. Hence $\{x_0\}$ is open in X. Then $f(\{x_0\})$ is open in X. Since the only one point set open in X is $\{x_0\}$, $f(x_0) = x_0$. Also $V(X \setminus \{x_0, x_1\}) = X \setminus \{x_0, x_1\}$. That is $\{x_0, x_1\}$ is open in X. Therefore $f(\{x_0, x_1\})$ is open in X. Since the only two point set which is open in X is $\{x_0, x_1\}$, we have $f(\{x_1\})) = x_1$. Let x_α be any element of Z such that $f(x) = x$ for every $x < x_\alpha$. If x_α has no immediate successor, then x_α is the last element of Z. Since $V(Y) = V_1(Y) = Y$, we have Z is open in X and hence $f(Z)$ is open in X. Thus $f(Z) = (Z \setminus \{x_\alpha\}) \cup \{f(x_\alpha)\}$ which implies that $f(x_\alpha) = x_\alpha$.

If x_α has an immediate successor x_β, then $V(X \setminus \{x \in Z : x < x_\beta\}) = X \setminus \{x \in Z : x < x_\beta\}$. This implies that $U = \{x \in X \setminus Y : x < x_\beta\}$ is an open set. Then $f(U) = \{x \in Z : x < x_\beta\} \cup \{f(x_\alpha)\}$. By the definition of V, $f(U) = U$ and hence $f(x_\alpha) = x_\alpha$. Thus we get $f|_Z = I_Z$.

Since f is a closure isomorphism, $f(V(A)) = V(f(A))$ for every $A \subseteq X$. If $A \subseteq Y$, then $f(V(A)) = f(V_1(A)) = f|_Y(V_1(A))$. Since f is a bijection on X and $f|_Z = I_Z$, we have $f(A) \subseteq Y$ and hence $V(f(A)) = V_1(f(A)) = V_1(f|_Y(A))$. Therefore $f|_Y(V_1(A)) = V_1(f|_Y(A))$. Thus we have $f|_Y \in H$. That is $f = (I_Z, h)$, where $h = f|_Y \in H$. Since $Z = X \setminus Y$ it follows that

$$CI(X,V) \subseteq \{I_{X \setminus Y}\} \times H. \tag{3.2}$$

From 3.1 and 3.2, $CI(X,V) = \{I_{X \setminus Y}\} \times H$ □

Remark 3.5. *By Theorem 3.4, in order to determine the c-representability of a non- trivial permutation group H on a set X, we have to consider only the c-representability of H on the set of all points which are moved by the permutations of H.*

Theorem 3.6. *Let X be a set and f be a cycle on X. Then the permutation group generated by f is c-representable on X.*

Proof. Theorem is clear from Theorem 3.2 and Theorem 3.4. □

In [22] it is proved that the Dihedral group D_n is not t-representable for $n \geq 5$. Here we investigate the c-representability of the Dihedral group D_n.

Definition 3.7. *[8] For $n \geq 3$, the Dihedral group D_n is defined as the rigid motions of the plane preserving a regular n-gon with the operations being composition. The order of the Dihedral group D_n is 2n.*

Theorem 3.8. *The Dihedral group D_n is c-representable.*

Proof. Let $X = \{a_1, a_2, \ldots, a_n\}$. Define the closure operator $V : P(X) \to P(X)$ as $V(a_k) = \{a_k, a_{k\oplus 1}, a_{k\oplus(n-1)}\}$, $V(A) = \bigcup_{a_k \in A} V(\{a_k\})$ for each $A \subseteq X$. Recall that the generators of the Dihedral group D_n on $X = \{a_1, a_2, \ldots, a_n\}$ are the cycle $f = (a_1, a_2, \ldots, a_n)$ and

$g = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots a_k \ldots & a_{n-1} & a_n \\ a_1 & a_n & a_{n-1} & \ldots a_{n+2-k} \ldots & a_3 & a_2 \end{pmatrix}$. We have

$$\begin{aligned} f(V(a_1)) &= p(\{a_n, a_1, a_2\}) \\ &= \{a_1, a_2, a_3\}. \end{aligned}$$

Also

$$V(f(\{a_1\})) = V(\{a_2\}) = \{a_1, a_2, a_3\}.$$

That is $f(V(a_1)) = V(f(\{a_1\}))$.
Similarly

$$\begin{aligned} f(V\{a_k\}) &= f(\{a_k, a_{k\oplus 1}, a_{k\oplus(n-1)}\}) \\ &= \{a_{k\oplus 1}, a_{k\oplus 2}, a_{k\oplus n}\} \end{aligned}$$

and

$$\begin{aligned} V(f(\{a_k\})) &= V(\{a_{k\oplus 1}\}) \\ &= \{a_{k\oplus 1}, a_{k\oplus 2}, a_{k\oplus n}\}. \end{aligned}$$

Thus
$f(V\{a_k\}) = V(f(\{a_k\}))$ for $k = 1, 2, \ldots, n$. Thus f is a closure isomorphism of (X, V). Next we prove that g is a closure isomorphism. We have

$$\begin{aligned} g(V(\{a_1\})) &= g(\{a_1, a_2, a_n\}) \\ &= \{a_1, a_n, a_2\} \end{aligned}$$

and

$$\begin{aligned} V(g(\{a_1\})) &= V(\{a_1\}) \\ &= \{a_1, a_2, a_n\}. \end{aligned}$$

That is
$g(V(\{a_1\})) = V(g(\{a_1\}))$.
Now

$$\begin{aligned} g(V(\{a_k\})) &= g(\{a_k, a_{k\oplus 1}, a_{k\oplus(n-1)}\}) \\ &= \{a_{n+2-k}, a_{n+2-(k\oplus 1)}, a_{n+2-(k\oplus(n-1))}\} \end{aligned}$$

and

$$\begin{aligned} V(g(\{a_k\})) &= V(\{a_{n+2-k}) \\ &= \{a_{n+2-k}, a_{n+2-(k\oplus 1)}, a_{n+2-(k\oplus(n-1))}\}) \end{aligned}$$

.
Hence $g \in CI(X,V)$. Then every element of D_n is a closure isomorphism. That is

$$D_n \subseteq CI(X,V) \tag{3.3}$$

Now suppose that $h \in CI(X,V)$. Then $h(V(\{a_1\}) = V(h(\{a_1\}))$.
Suppose $h(a_1) = a_k$. Then

$$\begin{aligned} h(V(\{a_1\}) &= h(\{a_1, a_2, a_n\}) \\ &= \{a_k, h(a_2), h(a_k)\}. \end{aligned}$$

And

$$\begin{aligned} V(h(\{a_1\})) &= V(\{a_k\}) \\ &= \{a_k, a_{k\oplus 1}, a_{k\oplus(n-1)}\}. \end{aligned}$$

Then $h(a_2)$ is either $a_{k\oplus 1}$ or $a_{k\oplus(n-1)}$, and $h(a_n)$ is either $a_{k\oplus 1}$ or $a_{k\oplus(n-1)}$
Case (i): $h(a_2) = a_{k\oplus 1}$ and $h(a_n) = a_{k\oplus(n-1)}$.
Since h is a closure isomorphism, $V(h(a_2)) = h(V(\{a_2\}))$.
But
$V(\{a_{k\oplus 1}\}) = \{a_{k\oplus 1}, a_k, a_{k\oplus 2}\}$ and $h(V(\{a_2\})) = h(\{a_1, a_2, a_3\})$.
This implies that $h(a_3) = a_{k\oplus 2}$ and $h(a_{n-1}) = a_{k\oplus n-2}$. That is
$h = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_n \\ a_k & a_{k\oplus 1} & a_{k\oplus 2} & \ldots & a_{k\oplus(n-2)} & a_{k\oplus(n-1)} \end{pmatrix} = f^{k-1}$.
Hence $h \in D_n$.

Case (ii): $h(a_2) = a_{k\oplus(n-1)}$ and $h(a_n) = a_{k\oplus 1}$.
In this case $h(a_3) = a_{k\oplus(n-2)}$ and $h(a_{n-1}) = a_{k\oplus 2}$.
Hence $h = (a_1, a_k)(a_2, a_{k\oplus(n-1)}) \cdots (a_n, a_{k\oplus 1})(a_{n-1}, a_{k\oplus 2})$.
Then $h = f^{n-k} g \in D_n$. Hence

$$CI(X,V) \subseteq D_n \tag{3.4}$$

From equations (3.3) and (3.4), $D_n = CI(X,V)$. This completes the proof. □

Example 3.9. *Let $X = \{1,2,3,4\}$. Consider the dihedral group $D_4 = \{I, (1,2,3,4), (1,3)(2,4), (1,4,3,2), (1,2)(3,4), (1,4)(2,3), (1,3), (2,4)\}$. Define $V : P(X) \to P(X)$ as in Theorem 3.8. That is $V(\{1\}) = \{1,2,4\}$, $V(\{2\}) = \{1,2,3\}$, $V(\{3\}) = \{2,3,4\}$, $V(\{4\}) = \{3,4,1\}$ and $V(A) = \underset{a\in A}{\cup} V(\{a\})$. Then the group of closure isomorphisms of (X,V) is equal to D_4. Also note that the above mentioned closure operator is not a topological closure operator, since $V(V(\{1\})) = V(\{1,2,4\}) = \{1,2,3,4\} \neq V(\{1\})$.*

Now we can have a topological closure operator whose group of closure isomorphisms is D_4. Consider $V'(\{1\}) = \{1,3\} = V'(\{3\})$ and $V'(\{2\}) = \{2,4\} = V'(\{4\})$ and $V'(A) = \underset{a\in A}{\cup} V'(\{a\})$. Then the group of closure isomorphisms of (X,V') is equal to D_4.

Thus in the case of D_4, we can define a topological closure operator V such that $CI(X,V) = D_4$. But for $n > 4$, there exists no topological closure operator V such that $CI(X,V) = D_4$.

Now we investigate the c-representability of some cyclic subgroups of $S(X)$. Sini P. studied the t-representability of cyclic permutation groups[20, 23].

Theorem 3.10. *[23] If f is a permutation on X which is an arbitrary product of more than two disjoint cycles having equal length n, then the group generated by f is $t-$representable on X.*

By Theorem 3.10 and 3.1 we have the group generated by f where f is a permutation on X which is an arbitrary product of more than two disjoint cycles having equal length n is c-representable on X.

Let f be a permutation on X which is a product of two disjoint cycles having equal length n where $n \geq 3$. Then the cyclic group generated by f is not t-representable on X[23]. Here we show that the cyclic group generated by f is c-representable on X.

Theorem 3.11. *Let X be a set and f be a permutation which is a product of two disjoint finite cycles having equal lengths. Then the cyclic group generated by f is c-representable on X.*

Proof. Let $f = (a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n)$ be a permutation on X and H be the cyclic group generated by f. If $n < 3$, then the cyclic group generated by f is t-representable and hence c-representable on X. Assume that $n \geq 3$. Suppose that $Y = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$. By Theorem 3.4, it is enough to prove that H is c-representable on Y.
Let $X_1 = \{a_1, a_2, \ldots, a_n\}$ and $X_2 = \{b_1, b_2, \ldots, b_n\}$. Define $V : P(Y) \to P(Y)$ as $V(\emptyset) = \emptyset$, $V(\{a_j\} = \{a_j, a_{j\oplus 1}, b_j\})$ and $V(\{b_j\}) = \{b_j, b_{j\oplus 1}\}$, $j = 1, 2, \ldots, n$ and $V(A) = \underset{a\in A}{\cup} V(\{a\})$, $A \subseteq Y$. Then

$$\begin{aligned} f(V(a_i)) &= f(\{a_i, a_{i\oplus 1}, b_i\}) \\ &= \{a_{i\oplus 1}, a_{i\oplus 2}, b_{i\oplus 1}\}. \end{aligned}$$

Now

$$\begin{aligned} V(f\{a_i\}) &= V(\{a_{i\oplus 1}\}) \\ &= \{a_{i\oplus 1}, a_{i\oplus 2}, b_{i\oplus 1}\} \\ &= f(V(\{a_i\}) \end{aligned}$$

for $i = 1, 2, \ldots, n$. Also

$$\begin{aligned} f(V(\{b_i\}) &= f(\{b_i, b_{i\oplus 1}\}) \\ &= \{b_{i\oplus 1}, b_{i\oplus 2}\} \end{aligned}$$

and

$$\begin{aligned} V(f(\{b_i\})) &= V(\{b_{i\oplus 1}\}) \\ &= \{b_{i\oplus 1}, b_{i\oplus 2}\}. \end{aligned}$$

That is $f(V(\{b_i\})) = V(f(\{b_i\}))$ for each $i = 1, 2, \ldots, n$. Thus f is a closure isomorphism on Y.

Now let h be a closure isomorphism of (Y, V). Then $h(V(A)) = V(h(A))$ for every $A \subseteq Y$. If $h(a_i) = b_k$, then we have

$$\begin{aligned} h(V(\{a_i\})) = V(h(\{a_i\})) &\Rightarrow h(\{a_i, a_{i\oplus 1}, b_i\}) = V(b_k) \\ &\Rightarrow \{h(a_i), h(a_{i\oplus 1}), h(b_i)\} = \{b_k, b_{k\oplus 1}\} \end{aligned}$$

This is not possible. Thus $h(a_i) \in X_1$. Now suppose that $h(a_i) = a_k$. This implies that

$$\begin{aligned} V(h(a_i)) &= V(a_k) \\ &= \{a_k, a_{k\oplus 1}, b_k\} \\ &= \{h(a_i), h(a_{i\oplus 1}), h(b_i)\}. \end{aligned}$$

Then $h(b_i) = b_k$. Thus $h(X_2) = X_2$. Now let $h(a_1) = a_k$ and $h(b_1) = b_k$. Then

$$\begin{aligned} V(h(b_1)) &= V(b_k) \\ &= \{b_k, b_{k\oplus 1}\} \end{aligned}$$

and

$$\begin{aligned} V(h(a_1)) &= V(a_k) \\ &= \{a_k, a_{k\oplus 1}, b_k\}. \end{aligned}$$

We have $h(V(\{b_1\})) = \{h(b_1), h(b_2)\}$ and
$h(V(\{a_1\})) = \{h(a_1), h(a_2), h(b_1)\}$. Since h is a closure isomorphism, $h(V(\{a_1\})) = V(h(a_1))$ and
$h(V(\{b_1\})) = V(h(b_1))$. This implies that
$\{a_k, a_{k\oplus 1}, b_k\} = \{h(a_1), h(a_2), h(b_1)\}$ and
$\{b_k, b_{k\oplus 1}\} = \{h(b_1), h(b_2)\}$. Hence $h(a_2) = a_{k\oplus 1}$ and
$h(b_2) = b_{k\oplus 1}$. Now suppose that $h(b_m) = b_j$ and $h(a_m) = a_j$ where $1 < m, j < n$. Then

$$\begin{aligned} V(h(b_m)) &= V(b_j) \\ &= \{b_j, b_{j\oplus 1}\} \end{aligned}$$

and

$$\begin{aligned} V(h(a_m)) &= V(a_j) \\ &= \{a_j, a_{j\oplus 1}, b_j\}. \end{aligned}$$

Also we have
$h(V(\{b_m\})) = h(\{b_m, b_{m\oplus 1}\})$ and
$h(V(\{a_m\})) = h(\{a_m, a_{m\oplus 1}, b_m\})$. Thus $h(b_{m\oplus 1}) = b_{j\oplus 1}$ and
$h(a_{m\oplus 1}) = a_{j\oplus 1}$. Thus $h = f^{j-1}$. Hence $h \in H$.
Thus $CI(Y, V) = H$. □

Example 3.12. *As an illustrative example of Theorem 3.11, we consider the following: Let* $X = \{1, 2, 3, 4, 5, 6\}$ *and* $p = (1, 2, 3)(4, 5, 6)$, *which is a product of two cycles of equal length. Then the group generated by p is* $\{(1, 2, 3)(4, 5, 6), (1, 3, 2)(4, 6, 5), I\}$. *Now consider a closure operator* $V : P(X) \to P(X)$ *such that* $V(\{1\}) = \{1, 2, 4\}$, $V(\{2\}) = \{2, 3, 5\}$, $V(\{3\}) = \{3, 1, 6\}$, $V(\{4\}) = \{4, 5\}$, $V(\{5\}) = \{5, 6\}$, $V(\{6\}) = \{6, 4\}$ *and* $V(A) = \cup_{a\in A} V(\{a\})$. *Then the group generated by p is same as the group of closure isomorphisms of* (X, V).

Now we consider the group c-representability of cyclic group generated by an arbitrary product of disjoint cycles having equal length.

Theorem 3.13. *Let X be any set and f be the permutation which is an arbitrary product of disjoint cycles having equal length. Then the cyclic group generated by f is c-representable on X.*

Proof. Proof follows from Theorem 3.2, 3.6, 3.10 and 3.11. □

Corollary 3.14. *Every permutation group of prime order is c-representable.*

Proof. Let X be any set and H be a permutation group on X having order n, where n is a prime number. Then H is a cyclic group generated by a permutation f which is of order n. This implies that f is a product of disjoint cycles having equal length. So by Theorem 3.13, H is c-representable on X. □

We proved that direct sum of c-representable finite permutation groups are c-representable on X[13]. From this result we can deduce that the permutation group generated by two disjoint cycles having lengths m and n where $gcd(m,n) = 1$ is c-representable on X.

Theorem 3.15. *[13] Let $\{(X_i, V_i)\}_{i\in I}$ be an arbitrary family of disjoint closure spaces where each X_i is finite and H_i be c-representable subgroup of $S(X_i)$ for $i \in I$. Then $\underset{i\in I}{\times} H_i$ is c-representable on $X = \bigcup_{i\in I} X_i$.*

Theorem 3.16. *A group generated by a permutation on a finite set X which is a product of two disjoint cycles having lengths n and m respectively where $gcd(n,m) = 1$ is c-representable.*

Proof. Let $X = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m\}$. Let $f = f_1 f_2$ where $f_1 = (a_1, a_2, \ldots, a_n)$ and $f_2 = (b_1, b_2, \ldots, b_m)$. Let H be the group generated by f. Treat X as $X_1 \cup X_2$ where $X_1 = \{a_1, a_2, \ldots, a_n\}$ and $X_2 = \{b_1, b_2, \ldots, b_m\}$. By Theorem 3.2, H_1 is c-representable on X_1 and H_2 is c-representable on X_2. Since m and n are relatively prime, $H = H_1 \times H_2$. Hence H is c-representable on X by Theorem 3.15. □

4. Conclusion

We were in search of c-representable permutation groups. We observed that in order to prove a permutation group H is c-representable, it is enough to prove that H is c-representabile on the set of all points which are moved by the permutations of H. We proved that the dihedral group is c-representable. The c-representability of some cyclic permutation groups are also studied.

Acknowledgment

We express our sincere thanks to Prof. Ramachandran P. T., Department of Mathematics, University of Calicut for his valuable suggestions, help and guidance through out the preparation of this paper.

References

[1] Alas O. T., Wilson R. G., Which Topologies Can Have Immediate Successors in the Lattice of T_1-Topologies?, *Applied General Topology*, Vol. 5, No. 2, (2004), 231–242.

[2] Alas O. T., Hernandez S., Sanchis M., Tkachenko M. G., Wilson R. G., Adjacency in Subposets of the Lattice of T_1-Topologies on a Set, *Acta Mathematica Hungarica*, Vol. 112, No. 3, (2006), 199–219.

[3] Boonpok C., Generalized closed sets in Čech Closure Spaces, *Acta Universitatis Apulensis*, Vol.22, No.22,(2010), 133–140.

[4] Boonpok C., Khampakdee J., Between Closed Sets and Generalized Closed Sets in Closure Spaces, *Acta Mathematica Universitatis Ostraviensis*, Vol. 16, No. 3, (2008), 3-14.

[5] Cameron P. J. et al., Product Action, *Discrete Mathematics* 308(2008), 386–394.

[6] Čech E. , Topological Spaces, Rev. Ed. Wiley, New York(1966).

[7] Dixon J. D., Mortimer B. Permutation Groups, Springer-Verlag, New York, Inc. (1996).

[8] Fraleigh J. B., A First Course in Abstract Algebra, Fifth Edition, Addison Wesley Longman Inc., (1994).

[9] Jech T., Set Theory, Academic Press, New York, San Francisco, London, (1978).

[10] Joshi K. D., Introduction to General Topology, New Age International (P) Limited, (2003).

[11] Joseph P. J., Upper Neighbours in the Lattice of Topologies, *Proceedings, International Seminar on Recent Trends in Topology and its Applications*, St. Joseph's College, Irinjalakuda, Kerala, (2009).

[12] Joseph P. J., Neighbours in the Lattice of Topologies, Ph. D. Thesis Submitted to the University of Calicut, (2003).

[13] Kavitha T., Sini P., Ramachandran P. T., Closure Isomorphisms of Čech Closure Spaces, *Journal of Advanced Studies in Topology* 7(2016), 132–136.

[14] Kunheenkutty M., Kavitha T., Ramachandran P. T., Adjacency in the Lattice of Čech Closure Operators, *International Journal of Pure and Applied Mathematics*, Vol. 105, No. 1, (2015), 73–86.

[15] Ramachandran P.T., Some Problems in Set Topology Relating Group of Homeomorphisms and Order, Ph. D. Thesis Submitted to CUSAT, Kerala (1985).

[16] Ramachandran P.T., The Group of Homeomorphisms and the L-fuzzy Homeomrphisms, *Proceedings of the National Conference on Mathematical Modeling*, Kottayam, India (2002), 117–121.

[17] Ramachandran P.T., Complementation in the Lattice of Čech Closure Operators, *Indian Journal of Pure & Applied Mathematics*, Vol. 18, No. 2, (1987), 152–158.

[18] Ramachandran P. T., Groups of Homeomorphisms and Normal Subgroups of the Group of Permutations, *International Journal of Mathematics & Mathematical Sciences*, Vol. 14, No. 3, (1991), 475–480.

[19] Ramachandran P.T., Fixed Points of the Lattice of Čech Closure Operators, *Proceedings, U.G.C. Sponsored National Seminar on Discrete Analysis*, Sri Ramakrishna Mission Vidyalaya College of Arts and Science (Autonomous), Coimbatore 641020, (1999), 29–31.

[20] Sini P., The Group of Homeomorphisms and the Cyclic Group of Permutations, *Palestine Journal of Mathematics* 8(2019),53–60.

[21] Sini P., Studies on the Group of Homeomorphisms and

$L-$fuzzy Homeomorphisms, Ph.D. Thesis Submitted to the University of Calicut (2017).

[22] Sini P., Ramachandran P.T., On the Group of Homeomorphisms, *Bulletin of Kerala Mathematics Association* 9(2012), 55–61.

[23] Sini P., Ramachandran P. T., On t- representability of Cyclic Subgroups of Symmetric Group-1, *International Journal of Pure and Applied Mathematics* 106(2016), 851–857.

[24] Sini P., t-representability of Maximal Subgroups of the Symmetric Group, *Journal of Advanced Studies in Topology*, Vol.9 No.1(2018) ,27–33.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆