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Continuous linear transformations on generalized fuzzy normed spaces

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Abstract

In this study the definition of bounded linear transformation and continuous linear transformation in a Generalized Fuzzy normed space is introduced. Also classical principals such as open mapping theorem and closed graph theorem are established in Generalized Fuzzy settings. Finally we introduce contraction of a linear operator on Generalized Fuzzy normed space and Banach fixed point theorem is proved in Generalized Fuzzy Banach space.

Keywords

Fuzzy normed space, Generalized Fuzzy normed space, Generalized Fuzzy Banach space, Generalized Fuzzy bounded linear transformation, Generalized Fuzzy Continuous linear transformation.

AMS Subject Classification

03E72, 46S40, 47S40.

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Contents

1. Introduction

In 1965 L.A. Zadeh introduced the notion of Fuzzy set [5]. A.George and P.Veeramani defined Fuzzy metric space [2] in 1994. As a continuation of this Magie Jose defined Fuzzy normed space and discuss some of its properties in 2000 [6]. Also she established open mapping theorem and closed graph theorem in Fuzzy context. K A Khan [3] introduced the concept of Generalized normed space in 2014 and Sukanya K P and Sr Magie Jose introduced generalized E-fuzzy metric space in 2017 [7]. Using these concept we developed Generalized Fuzzy normed space. In this study, we discuss some properties of bounded linear transformation and continuous linear transformation on Generalized Fuzzy normed space.

2. Preliminaries

Some basic definition and results are mentioned here that are used for further development of this paper.

Definition 2.1. *[5]A fuzzy set A in a set X is a function mapping the elements of X to the unit interval [0,1].*

Definition 2.2. *[1]:An operation * :[0,1]× [0,1]→[0,1] which is binary, is a continuous t-norm if it has the following properties:*

- *commutativity and associativity,*
- *continuity,*
- *for all a ∈ [0,1], $a*1 = a$,*
- *for all a, b,c,d ∈ [0,1] and a ≤ c and b ≤ d, a*b ≤ c*d .*

Definition 2.3. *[6]: $(X,N,*)$ is said to be a Fuzzy normed space if X is an arbitrary set, * is a continuous t-norm and N is a fuzzy set on $X\times(0,\infty)$ with the following properties:*

- $N(x,t)$ *is greater than 0,*
- $N(x,t) = 1$, *if and only if* $x = 0$,
- $N(kx,t) = N(x,t/|k|)$,
- $N(x,t)*N(y,s) \leq N(x+y,t+s)$
- $N(x,.) : (0,\infty) \to [0,1]$ *is continuous, for all* $x,y,z \in X$, $s,t > 0$ *and k any scalar.*

Definition 2.4. *:For a real or complex linear space X, $(X, G_{FN}, *)$ is a Generalized Fuzzy normed space if * is a continuous t-norm and G_{FN} is a function from $X^3 \times (0,\infty) \to [0,1]$ with the following properties:*

- $G_{FN}(x,y,z,t)$ *is greater than 0,*
- $G_{FN}(x,y,z,t) = 1$ *if and only if* $x = y = z = 0$,
- $G_{FN}(x,y,z,t) = G_{FN}(p(x,y,z),t)$, *(symmetry) where p is a permutation function,*
- $G_{FN}(kx,ky,kz,t) = G_{FN}(x,y,z,t/|k|)$,
- $G_{FN}(x,y,z,t) * G_{FN}(x',y',z',s) \leq G_{FN}(x+x',y+y',z+z',t+s)$,
- $G_{FN}(x,y,z,.) : (0,\infty) \to [0,1]$ *is continuous,*
- $G_{FN}(x+y,0,z,t) \geq G_{FN}(x,y,z,t)$ *for all* $s,t > 0$, $x,y,z,x',y',z' \in X$ *and k any scalar.*

Example 2.5. *For a real or complex linear space X,* $(X, ||.,.,.||)$ *is a G-normed space if we define* $||.,.,.|| : X^3 \to R$ *by* $||x,y,z|| = ||x|| + ||y|| + ||z||$. *Then* $(X, G_{FN}, *)$ *is a Generalized Fuzzy normed linear space, if we define* $a * b = min(a,b)$ *and* $G_{FN}(x,y,z,t) = \frac{t}{t+||x,y,z||}$.

Definition 2.6. *In a Generalized Fuzzy normed space X, for given* $x_0 \in X, t > 0$ *and* $0 < r < 1$, *the open ball* $B(x_0,r,t)$ *is defined as* $B(x_0,r,t) = \{y \in X : G_{FN}(x_0 - y, y - x_0, 0, t) > 1 - r\}$.

Definition 2.7. *A sequence* (x_n) *in* $(X, G_{FN}, *)$ *is a Generalized Fuzzy cauchy sequence if for given r,* $0 < r < 1, t > 0 \ni$ *an integer N such that* $G_{FN}(x_l - x_m, x_m - x_n, x_n - x_l, t) > 1 - r$ *for every* $l,m,n \geq N$.

Remark 2.8. *A sequence* (x_n) *in* $(X, G_{FN}, *)$ *is a Generalized Fuzzy cauchy sequence if given* $0 < r < 1, t > 0 \ni$ *an integer N such that* $G_{FN}(x_n - x_m, x_m - x_n, 0, t) > 1 - r$ *for every* $n,m \geq N$.

Remark 2.9. *A sequence* (x_n) *in* $(X, G_{FN}, *)$ *is a Generalized Fuzzy cauchy sequence if for given r,* $0 < r < 1, t > 0 \ni$ *an integer N such that* $G_{FN}(x_{n+p} - x_n, x_n - x_{n+p}, 0, t) > 1 - r$ *for every* $n \geq N$ *and* $p > 0$.

Definition 2.10. *A Generalized Fuzzy normed space* $(X, G_{FN}, *)$ *is said to be complete if every Generalized Fuzzy cauchy sequence in X converges in X.*

Definition 2.11. *A complete Generalized Fuzzy normed space is called a Generalized Fuzzy Banach space.*

3. Main Results

Generalized Fuzzy bounded linear transformation and Generalized Fuzzy continuous linear transformation are defined. Also we have established boundedness imply continuity for a linear transformation in Generalized Fuzzy normed space.

Definition 3.1. *A linear transformation* $F : (X, G_{FN}, *) \to (Y, G_{FN'}, *)$ *is said to be bounded if there exists* $k > 0$ *such that* $G_{FN'}(F(x), F(y), F(z), t) \geq G_{FN}(x,y,z,t/k)$, $\forall x,y,z \in X$ *and* $t > 0$.

Definition 3.2. *A linear transformation* $F : (X, G_{FN}, *) \to (Y, G_{FN'}, *)$ *is said to be continuous at x if given* $r_1, t_1 > 0$, $0 < r_1 < 1$ *there exists* $r_2, t_2 > 0$, $0 < r_2 < 1$ *such that*
$G_{FN}(x - y, y - x, 0, t_2) > 1 - r_2$
$\implies G_{FN'}(F(x) - F(y), F(y) - F(x), 0, t_1) > 1 - r_1 \ \forall y \in X$

A linear transformation F is continuous on X if it is continuous at every $x \in X$.

Theorem 3.3. *A linear transformation* $F : (X, G_{FN}, *) \to (Y, G_{FN'}, *)$ *is continuous at* $x_0 \in X$ *if and only if the sequence* $(F(x_n))$ *converges to* $F(x_0)$ *in Y for every convergent sequence* (x_n) *converges to* x_0 *in X.*

Proof. Suppose F is continuous at x_0 and (x_n) converges to x_0. Then given $r_1, t_1 > 0, 0 < r_1 < 1 \ni r_2, t_2 > 0, 0 < r_2 < 1$ such that $G_{FN}(y - x_0, x_0 - y, 0, t_1) > 1 - r_1$
$\implies G_{FN'}(F(y) - F(x_0), F(x_0) - F(Y), 0, t_2) > 1 - r_2, y \in Y$.
Since $x_n \to x_0$, for this r_2 and t_2, $\ni$ n such that $G_{FN}(x_n - x_0, x_0 - x_n, 0, t_2) > 1 - r_2 \ \forall n \geq N$. This implies $G_{FN}(F(x_n) - F(x_0), F(x_0) - F(x), 0, t_1) > 1 - r_1$, $\forall n \geq N \implies F(x_n) \to F(x_0)$.
Conversely, let the sequence $(x_n) \to x_0 \implies (F(x_n) \to F(x)$ and F is not continuous at x_0. Then $\ni r_1, t_1 > 0, 0 < r_1 < 1$ such that for any $r_2, t_2 > 0, 0 < r_2 < 1$ there exists $x \in X$ such that $G_{FN}(x_0 - x, x - x_0, 0, t_2) > 1 - r_2$, but $G_{FN'}(F(x_0) - F(x), F(x) - F(x_0), 0, t_1) \leq 1 - r_1$. Take $r_2 = 1/n$ and $t_2 = 1/n$ where $n \in N$. Then for each n there exists an $x_n \in X$ such that $G_{FN}(x_0 - x_n, x_n - x_0, 0, 1/n) > 1 - 1/n$, but $G_{FN'}(F(x_0) - F(x_n), F(x_n) - F(x_0), 0, t_1) \leq 1 - r_1$. Thus $F(x_n)$ does not converges to $F(x_0)$ whereas x_n converges to x_0. This is a contradiction. Hence the result. □

Theorem 3.4. *Let F be a linear transformation from* $(X, G_{FN}, *)$ *to* $(Y, G_{FN'}, *)$. *Then F is continuous on X if and only if F is continuous at a point* x_0 *in X.*

Proof. Let F is continuous on X, then it is continuous at every point in X. Conversely, let F is continuous at a point $x_0 \in X$. Let $y \in X$ is arbitrary. Then from our assumption, given $r_1, t_1 > 0, 0 < r_1 < 1$ there exists $r_2, t_2 > 0, 0 < r_2 < 1$ such that $G_{FN}(x - x_0, x_0 - x, 0, t_1) > r_1 \implies G_{FN'}(F(x) - F(x_0), F(x_0) - F(x), 0, t_2) > r_2 \ \forall x \in X$. Replacing y by $x + x_0 - y$, we get $G_{FN}(x - y, y - x, 0, t_1) > r_1 \implies G_{FN'}(F(x) - F(y), F(y) - F(x), 0, t_2) > r_2 \ \forall x \in X$. Implies F is continuous at $y \in X$. As this $y \in X$ is arbitrary, we have the result. □

Theorem 3.5. *Every bounded linear transformation from* $(X, G_{FN}, *)$ *to* $(Y, G_{FN'}, *)$ *is continuous.*

Proof. Let $F : X \to Y$ be a bounded linear transformation. Then there exists $k > 0$ such that $G_{FN'}(F(x), F(y)F(z), t \geq$

$G_{FN}(x,y,z,t/k)\ \forall x,y,z \in X$, $0<t<1$.
Choose $r_1 < r$ and $t_1 = t/k$. Then
$G_{FN'}(F(x)-F(y),F(y)-F(x),0,t)$
$= G_{FN'}(F(x-y),F(y-x),0,t)$
$\geq GFN(x-y,y-x,0,t/k)$
$\geq G_{FN}(x-y,y-x,0,t_1)$
$> 1-r_1$,
whenever $G_{FN}(x-y,y-x,0,t_1) > 1-r_1, \forall y \in X$.
That is F is continuous at x. Therefore F is continuous on X. □

Theorem 3.6. *Let M be a closed subspace of $(X,G_{FN},*)$ and F be a natuaral mapping of X onto the quotient Space X/M defined by $F(x)=x+M$. Then F is a bounded linear transformation.*

Proof. Since M is closed subspace of a Generalized Fuzzy normed space $(X,G_{FN},*)$, X/M is a Generalized Fuzzy normed space with Generalized Fuzzy norm
$G_{FN'}(x+M,y+M,z+M,t) = Sup\{G_{FN}(x+M,y+M,z+M,t) : m\in M\}$. Then clearly F is linear.
$G_{FN'}(F(x),F(y),F(z),t) = G_{FN'}(x+M,y+M,z+M,t)$
$= sup\{G_{FN}(x+m,y+m,z+m,t) : m \in M\}$
$\geq G_{FN}(x+M,y+M,z+M,t)\ \forall m \in M$.
Since M is a subspace, take m=0.
We get $G_{FN'}(F(x),F(y),F(z),t) \geq G_{FN}(x,y,z,t)\ \forall x,y,z \in X$.
Therefore F is a bounded linear transformation. □

Definition 3.7. *A continuous linear transformation $F: (X,G_{FN},*) \to (Y,G_{FN'},*)$ is said to be open if for every open set A in X the set F(A) is open in Y.*

Proof of open mapping theorem for generalized case will follow readily from the Baire's Theorem for Generalized case and from the following two lemmas.

Theorem 3.8. *(Baire's Theorem for Generalized case) If X is a Generalized Fuzzy Banach space, then the intersection of a countable number of dense open subsets of X is dense in X.*

Lemma 3.9. *Let $(X,G_{FN},*)$ be a Generalized Fuzzy normed space. Then*

- $B(x,r,t) = x + B(0,r,t)$
- $B(0,r,nt) = nB(0,r,t)$

Lemma 3.10. *Let F be a continuous linear transformation from $(X,G_{FN1},*)$ onto $(Y,G_{FN2},*)$. Then the image of any open ball centered at x in X will contain an open ball centered at F(x) in Y.*

Proof. For given $r,t>0$, $0<r<1$ let $B(0,r,t)$ be the open ball in X centered at origin and for given $s,k>0$, $0<s<1$, let $B'(0,s,k)$ be the open ball in Y centered at origin. First we prove that $B'(0,s,k) \subset \overline{FB(0,r,t)}$.
Let $x \in X$ be fixed, then there exists some t_0 such tnat $x \in B(0,r,t_0)$. Choose n such that $t_0 < nt$. Then $x \in nB(o,r,t)$. Hence $X = \cup_{n=1}^{\infty} nB(0,r,t)$. Since F is onto and linear $Y = F(X) = \cup_{n=1}^{\infty} F(nB(0,r,t))$. Since Y is complete by Baires Theorem for Generalized case, there exists atleast one n_0 such that $\overline{F(n_0B(0,r,t))}^0 \neq \emptyset$. Let $y \in \overline{F(n_0B(0,r,t))}^0$. That is there exists an open ball containing y contained in $\overline{F(n_0B(0,r,t))}$. Since $\overline{F(n_0B(0,r,t))}$ and $\overline{F(B(0,r,t))}$ are homeomorphic to each other, $\overline{F(B(0,r,t))}$ contains an open ball say $B'(y,s,k)$. Since $y \in B'(y,s,k) \subset \overline{F(B(0,r,t))}$, $y = F(x)$ for some $x \in B(0,r,t)$.
We have $B'(y,s,k) = B'(0,s,k) + y$
$B'(0,s,k) = B'(y,s,k) - y$
$\subset \overline{F(B(0,r,t))} - y$
$\subset \overline{F(B(0,r,t))} - F(x)$
$\subset \overline{F(B(0,r,t)-x)}$.
Let $y_0 \in B(0,r,t)-x \implies y_0 = x_0 - x$ for some $x_0 \in B(0,r,t)$
Now
$G_{FN1}(y_0,-y_0,0,t') =$
$G_{FN}(x_0-x,x-x_0,0,t')$, where $t'=2t$
$\geq G_{FN}(x_0,-x_0,0,t'/2) * G_{FN}(-x,x,0,t'/2)$
$> (1-r)*(1-r)$
$> 1-r'$ for some $r', 0<r'<1$.
$\implies y_0 \in B(0,r',t')$.
Hence $B(0,r,t) - x \subset B(0,r',t')$
$\implies B'(0,s,k) \subset \overline{FB(0,r',t')}$.
That is for given $r,t>0, 0<r<1$ there exists $s,k>0, 0<s<1$ such that $B'(0,s,k) \subset \overline{FB(0,r,t)}$. Now let $x \in X$,
$B'(0,s,k) + F(x) \subset \overline{F(B(0,r,t))} + F(x)$
That is $B'(F(x),s,k) \subset \overline{F(B(x,r,t))}$ Take $B_0 = B(x,r,t)$ and $x_1 = x$. Then there exists $r_1,t_1>0$, $0<r_1<1$ such that $B(x_1,r_1,t_1) \subset B_0$.
Choose $r_1' < r_1$ and $t_1' = min\{t_1,1\}$ such that $B[x_1,r_1',t_1'] \subset B_0$.
Also $B(x_1,r_1',t_1') \subset B[x_1,r_1',t_1']$.
Then there exists $s_1,k_1>0, 0<s_1<1$ such that
$B'(F(x_1),s_1,k_1) \subset \overline{FB(x_1,r_1',t_1')}$.
Let $y \in B'(F(x_1),s_1,k_1) \implies y \in \overline{FB(x_1,r_1',t_1')} \implies$ there exists $x_2 \in B(x_1,r_1',t_1')$ such that $y \to F(x_2)$. That is $F(x_2) \in B'(y,s_2,k_2), 0 < s_2,k_2 < 1/2$ and $B'(F(x_2),s_2,k_2) \subset \overline{FB(x_2,r_2',t_2')}$ where $B[x_2,r_2',t_2'] \subset B(x_1,r_1',t_1,), r_2' < r_2, 0 < r_2 < 1/2$ and $t_2' = min\{t_2,1/2\}$, $t_2>0$ with $B(x_2,r_2,t_2) \subset B(x_1,r_1',t_1')$. Continuing like this , there exists $x_n \in B(x_{n-1},r_{n-1}',t_{n-1}')$ such that $F(x_n) \in B'(y,s_n,k_n), 0 < s_n,k_n < 1/n$ and $B'(F(x_n),s_n,k_n) \subset \overline{FB(x_n,r_n',t_n')}$ where $B[x_n,r_n',t_n'] \subset B(x_{n-1},r_{n-1}',t_{n-1}), r_n' < r_n$, $0<r_n<1/n$ and $t_n' = min\{t_n,1/n\}, t_n>0$ with $B(x_n,r_n,t_n) \subset B(x_{n-1},r_{n-1}',t_{n-1},)$.
Now, for given $r,t>0$,$0<r<1$ choose an integer N such that $1/N < min\{t,r\}$.Then for $n \geq N$ and $l,m \geq n$
$G_{FN1}(x_l-x_m,x_m-x_n,x_n-x_l,t)$
$\geq G_{FN1}(x_l-x_m,x_m-x_n,x_n-x_l,1/N)$
$\geq G_{FN1}(x_l-x_n,0,x_n-x_l,1/N)$
$> (1-1/n)$
$> 1-r$.
Then $\{x_n\}$ is a cauchy sequence in Generalized Fuzzy normed space X. Since X is complete x_n converges to some $x_0 \in X$. Also, since $x_k \in B[x_n,r_n',t_n'] \subset B(x_{n-1},r_{n-1}',t_{n-1})$ for every $k \geq n$ and $B[x_n,r_n',t_n']$ is a closed set, $x_0 \in B[x_n,r_n',t_n'] \subset$

$B(x_{n-1}, r'_{n-1}, t_{n-1})$ for every n. That is $x_0 \in B_0$. Since F is a continuous linear mapping and $x_n \to x_0 \implies F(x_n) \to F(x_0)$. Now for given $r, t > 0$, $0 < r < 1$. Choose N such that $1/N < min\{t, r\}$. Then for $n \geq N$, $G_{FN2}(F(x_n) - y, y - F(x_n), 0, t) \geq$
$G_{FN2}(F(x_n) - y, y - F(x_n), 0, 1/N)$
$\geq G_{FN2}(F(x_n) - y, y - F(x_n), 0, 1/n)$
$\geq G_{FN2}(F(x_n) - y, y - F(x_n), 0, k_n)$
$> 1 - s_n$
$> 1 - 1/n$
$> 1 - 1/N$
$> 1 - r$,for every $n \geq N$.
That is $F(x_n) \to y$. Therefor $y = F(x_0) \in F(B(x, r, t)) \implies B'(F(x), s_1, k_1) \subset F(B(x, r, t))$. Thus image of an open ball centered at x in X contains an open ball centered at F(x). □

Theorem 3.11. *Open Mapping Theorem*
Let $(X, G_{FN}, *)$ and $(Y, G_{FN}, *)$ be Generalized Fuzzy Banach spaces. Let F be a continuous linear mapping from X to Y. Then F is an open mapping.

Proof. Let A be any open set in X. To show that $F(A)$ is open in Y. Let $F(x) \in F(A)$, where $x \in X$. Since A is open in X, there exists $r, t > 0, 0 < r < 1$ such that $B(x, r, t) \subset A$. Hence by above lemma there exists $s, k > 0, 0 < s < 1$ such that $B'(F(x), s, k) \subset F(B(x, r, t)) \subset F(A)$. Hence $F(A)$ is open. □

Theorem 3.12. *Closed graph theorem for Generalized Fuzzy normed space*
Let $(X, G_{FN1}, *)$ and $(Y, G_{FN2}, *)$ be two Generalized Fuzzy Banach spaces. If F is a closed linear transformation from X into Y, then F is continuous.

Proof. Let X' denote the same space X with another Generalized Fuzzy norm $G_{FN'}$ defined by
$G_{FN'}(x, y, z, t) = G_{FN1}(x, y, z, t) * G_{FN2}(x, y, z, t)$. Then $G_{FN'}$ is a Generalized Fuzzy norm and X' is a Generalized Fuzzy normed space with this norm.
Consider

$$
\begin{aligned}
G_{FN2}(F(x), F(y), F(z), t) &= 1 * G_{FN2}(F(x), F(y), F(z), t) \\
&\geq G_{FN1}(x, y, z, t) * G_{FN2}(F(x), F(y), F(z), t) \\
&\geq G_{FN'}(x, y, z, t).
\end{aligned}
$$

That is $F : X' \to Y$ is bounded and so it is continuous. Now consider $I : X' \to X$ defined by $I(x) = x$. Then I is one one and onto.
Also

$$
\begin{aligned}
G_{FN1}(I(x), I(y), I(z), t) &= 1 * G_{FN1}(x, y, z, t) \\
&\geq G_{FN'}(x, y, z, t) * G_{FN1}(x, y, z, t) \\
&\geq G_{FN'}(x, y, z, t).
\end{aligned}
$$

Therefore I is bounded and so I is continuous. Since I is one one ,onto and continuous, X and X' are homeomorphic. Hence $F : X \to Y$ is continuous. □

Next we define contraction of a mapping on Generalized Fuzzy normed space.

Definition 3.13. *Let $(X, G_{FN}, *)$ be a Generalized Fuzzy normed space. A mapping $F : X \to X$ is called a contraction on X if there exists k, $0 < k < 1$ such that*
$G_{FN}(Fx - Fy, Fy - Fz, Fz - Fx, t) \geq G_{FN}(x - y, y - z, z - x, t/k)$
$\forall t > 0$ and $x, y, z \in X$.

Theorem 3.14. *Banach fixed point theorem*
Let $(X, G_{FN}, *)$ be a Generalized Fuzzy Banach space and $F : X \to X$ be a contraction on X.Then F has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ by $x_n = F^n(x_0)$. For $t > 0$ and $p > 0$
$G_{FN}(x_{n+p} - x_n, x_n - x_{n+p}, 0, t) = G_{FN}(F^{n+p}(x_0) - F^n(x_0),$
$F^n(x_0) - F^{n+p}(x_0), 0, t)$
$\geq G_{FN}(F^{n+p-1}(x_0) - F^{n-1}(x_0), F^{n-1}(x_0) - F^{n+p-1}(x_0), 0, t/k)$
$\vdots$
$\geq G_{FN}(F^p(x_0) - x_0, x_0 - F^p(x_0), 0, t/k^n)$
Sine $0 < k < 1$, $t/k^n \to \infty$ as $n \to \infty$
Therefore $G_{FN}(x_{n+p} - x_n, x_n - x_{n+p}, 0, t) \to 1$ as $n \to \infty$
$\implies \{x_n\}$ is a cauchy sequence in X. Since X is complete, $x_n \to x$ in X.
That is $lim_{n\to\infty} F^n(x_0) = x$
Now $x = lim_{n\to\infty} F^{n+1}(x_0) = lim_{n\to\infty} F(F^n(x_0)) = F(x)$.
Hence x is a fixed point. To show uniqueness, let $y \in X$ such that $y = F(y)$ and $y \neq x$.
As $y \neq x$ $\exists$ $t_1 > 0$ and such that

$$
\begin{aligned}
G_{FN}(x - y, y - z, z - x, 0, t_1) &= G_{FN}(Fx - Fy, Fy - Fx, 0, t_1) \\
&\geq G_{FN}(x - y, y - z, z - x, t_1/k).
\end{aligned}
$$

Which is a contradiction as $0 < k < 1$. Hence $x = y$. □

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