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Oscillations of delay differential equations with variable coefficients

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Abstract

The purpose of this paper is to obtain a sharp sufficient condition for the oscillation of the delay differential equation.

$$y'(x)+q(x)y(x-\tau)=0, \quad x \geq x_0$$

where $q(x) \in C([x_0,\infty),R^+)$ which improves previously known results.

Keywords

Delay differential equation, Oscillatory property.

AMS Subject Classification

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Contents

1. Introduction

Consider the delay differential equation

$$y'(x)+q(x)y(x-\tau)=0, \quad x \geq x_0 \tag{1.1}$$

where $q(x) \in C([x_0,\infty),R^+)$ and τ is a positive constant. G.Ladas [1], R.G.Koplatadze and T.A.Chanturia [2], have proved that every solution of equation of (1.1) oscillates. If

$$\liminf_{x\to\infty} \int_{x-\tau}^{x} q(t)dt > \frac{1}{e} \tag{1.2}$$

R.G.Koplatadze and T.A.Chanturia [2], proved that differential inequation.

$$y'(x)+q(x)y(x-\tau) \geq 0, \quad x \geq x_0 \tag{1.3}$$

has no eventually positive solution if (1.2) holds.

This observation has been extensively exploited in the study of the oscillatory properties of solutions of various functional differential equations. See for example [3–5].

From [[6], Corollary 3.2.2], inequality (1.3) has no eventually positive solution if and only if Equation (1.1) has no eventually positive solution. By obtaining sharper sufficient conditions for oscillation of (1.1), we expect many of the above mentioned results can be improved.

Li [7] obtained a sharper sufficient condition by improving condition (1.2).

Theorem 1.1. *Let $q(x) \in C([x_0,\infty),R^+)$ and let τ is a positive constant. Suppose that there exists a $\bar{x} > x_0+\tau$ such that*

$$\int_{x-\tau}^{x} q(t)dt > \frac{1}{e} \geq \bar{x} \tag{1.4}$$

and

$$\int_{x+\tau}^{\infty} q(x)\left[exp\left(\int_{x-\tau}^{x} q(t)dt - \frac{1}{e}\right) - 1\right]dx = \infty \tag{1.5}$$

Then every solution of (1.1) oscillates.

Definition 1.2. *A solution of equation (1.1) is said to oscillate if it has arbitrarily large number of zeros.*

In this paper, we obtain new sufficient conditions for oscillation of solution of (1.1) which improve conditions (1.4) and (1.5).

2. Main Results

Let $q(x) \in C([x_0,\infty),R^+)$ and define the following sequences of functions:

$$q_1(x) = \int_{x-\tau}^{x} q(t)dtx \geq x_0+\tau$$
$$q_{k+1}(x) = \int_{x-\tau}^{x} q(t)q_k(t)dtx \geq x_0+(k+1)\tau \qquad (2.1)$$

$$\bar{q}_1(x) = \int_{x}^{x+\tau} q(t)dtx \geq x_0$$
$$\bar{q}_{k+1}(x) = \int_{x}^{x+\tau} q(t)\bar{q}_k(t)dtx \geq x_0, \quad k=1,2,3,\ldots$$

Theorem 2.1. *Let $q(x) \in C([x_0,\infty),R^+)$ and let τ be a positive constant. Suppose that there exist a $x_1 \geq x_0+\tau$ and a positive integer n such that*

$$q_n(x) \geq \frac{1}{e^n}\bar{q}_n(x) \geq \frac{1}{e^n} x \geq x_1 \qquad (2.2)$$

and

$$\int_{x+n\tau}^{\infty} q(x)\Big[exp\Big(e^{n-1}q_n(t)dt-\frac{1}{e}\Big)-1\Big]dx = \infty \qquad (2.3)$$

where $q_n(x)$ and $\bar{q}_n(x)$ are defined by (2.1). Then every solution of (1.1) oscillates.

Proof. Assume, for the sake of contradiction, that equation (1.1) has an eventually positive solution $y(x)$. Then there exists a $x_2 \geq x_1$ such that

$$y(x-\tau) \geq y(x) > 0, \quad y'(x) \leq 0, \quad x \geq x_2$$
$$v(x) = \frac{y(x-\tau)}{y(x)}, \quad x \geq x_2 \qquad (2.4)$$

Then

$$v(x) \geq 1, \quad x \geq x_2 \qquad (2.5)$$

Dividing both sides of (1.1) by $y(x)$,for $x \geq x_2$, we obtain

$$\frac{y'(x)}{y(x)} + q(x)v(x) = 0 \quad x \geq x_2 \qquad (2.6)$$

Integrating both sides of (2.6) from $x-\tau$ to x yields

$$\ln y(x) - \ln y(x-\tau) + \int_{x-\tau}^{x} q(t)v(t)dt = 0, \quad x \geq x_2+\tau$$

or

$$v(x) = exp\Big(\int_{x-\tau}^{x} q(t)v(t)dt\Big), \quad x \geq x_2+\tau \qquad (2.7)$$

It is easy to show that $e^c \geq ec$ for all $c \geq 0$, and so

$$v(x) = e\int_{x-\tau}^{x} q(t)v(t)dt, \quad x \geq x_2+\tau \qquad (2.8)$$

Set

$$v_1(x) = \int_{x-\tau}^{x} q(t)v(t)dt, \quad x \geq x_2+\tau$$
$$v_2(x) = \int_{x-\tau}^{x} q(t)v_1(t)dt, \quad x \geq x_2+2\tau$$

$$v_n(x) = \int_{x-\tau}^{x} q(t)v_{n-1}(t)dt, \quad x \geq x_2+n\tau \qquad (2.9)$$

and

$$u(x) = v(x)-1, \quad x \geq x_2$$
$$u_1(x) = \int_{x-\tau}^{x} q(t)u(t)dt, \quad x \geq x_2+\tau$$
$$u_2(x) = \int_{x-\tau}^{x} q(t)u_1(t)dt, \quad x \geq x_2+2\tau$$
$$\vdots$$
$$u_n(x) = \int_{x-\tau}^{x} q(t)u_{n-1}(t)dt, \quad x \geq x_2+n\tau \qquad (2.10)$$

By (2.5),

$$u(x) \geq 0,\ x \geq x_2,\ u_i(x) \geq 0,\ x \geq x_2+i\tau,\ i=1,2,3,\ldots \qquad (2.11)$$

From (2.7) and (2.8), we easily obtain

$$v(x) \geq e^{n-1}v_{n-1}(x), \quad x \geq x_2+(n-1)\tau \qquad (2.12)$$

and

$$v(x) \geq exp\Big(e^{n-1}\int_{x-\tau}^{x} q(t)v_{n-1}(t)dt\Big), \quad x \geq x_2+n\tau \qquad (2.13)$$

In view of (2.1), (2.10) and (2.11), (2.13) can be written as

$$v(x) \geq exp\Big(e^{n-1}\int_{x-\tau}^{x} q(t)u_{n-1}(t)dt + e^{n-1}q_n(x)\Big),$$
$$= exp\Big(e^{n-1}\int_{x-\tau}^{x} q(t)u_{n-1}(t)dt + \frac{1}{e}\Big)$$
$$exp\Big(e^{n-1}q_n(x)-\frac{1}{e}\Big), \quad x \geq x_2+n\tau$$

and so

$$v(x) \geq exp\Big(e^{n}\int_{x-\tau}^{x} q(t)u_{n-1}(t)dt + 1\Big)$$
$$exp\Big(e^{n-1}q_n(x)-\frac{1}{e}\Big), \quad x \geq x_2+n\tau \qquad (2.14)$$

By (2.2) and (2.11),

$$q(x)\Big[v(x)-\Big(e^n\int_{x-\tau}^{x} q(t)u_{n-1}(t)dt+1\Big)\Big]$$
$$\geq q(x)\Big[exp\Big(e^{n-1}q_n(x)-\frac{1}{e}\Big)-1\Big], \quad x \geq x_2+n\tau$$

or

$$q(x)\Big[u(x)-e^n u_n(x)\Big]$$
$$\geq q(x)\Big[exp\Big(e^{n-1}q_n(x)-\frac{1}{e}\Big)-1\Big],\quad x\geq x_2+n\tau$$

By integrating both sides from $x_3 \geq x_2+n\tau$ to $X \geq x_3+n\tau$ we obtain

$$\int_{x_3}^{X} q(x)\Big[u(x)-e^n u_n(x)\Big]dx$$
$$\int_{x_3}^{X} q(x)\Big[exp\Big(e^{n-1}q_n(x)-\frac{1}{e}\Big)-1\Big]dx \tag{2.15}$$

From this and (2.3), we have

$$\lim_{x\to\infty}\int_{x_3}^{X} q(x)\Big[u(x)-e^n u_n(x)\Big]dx=\infty \tag{2.16}$$

Since

$$\begin{aligned} e^n\int_{x_3}^{X} q(x)u_n(x)dx &= e^n\int_{x_3}^{X} q(x)dx\int_{x-\tau}^{x} q(t)u_{n-1}(t)dt\\ &\geq e^n\int_{x_3}^{X-\tau} q(t)u_{n-1}(t)dt\int_{x}^{x+\tau} q(x)dx\\ &= e^n\int_{x_3}^{X-\tau} q(x)\bar{q}_1(x)dx\int_{x-\tau}^{x} q(t)u_{n-2}(t)dt\\ &\geq e^n\int_{x_3}^{X-2\tau} q(t)u_{n-2}(t)dt\int_{x}^{x+\tau} q(x)\bar{q}_1(x)dx\\ &= e^n\int_{x_3}^{X-2\tau} q(x)\bar{q}_2(x)u_{n-2}(x)dx\\ &\vdots \end{aligned}$$

we have

$$\begin{aligned} e^n\int_{x_3}^{X} q(x)u_n(x)dx &\geq e^n\int_{x_3}^{X-n\tau} q(x)u(x)\bar{q}_n(x)dx\\ &\geq \int_{x_3}^{X-n\tau} q(x)u(x)dx \end{aligned} \tag{2.17}$$

Thus, we have

$$\begin{aligned} &\int_{x_3}^{X} q(x)\Big[u(x)-e^n u_n(x)\Big]dx\\ &\leq \int_{x_3}^{X} q(x)u(x)dx-\int_{x_3}^{X-n\tau} q(x)u(x)dx\\ &= \int_{X-n\tau}^{X} q(x)u(x)dx \end{aligned}$$

In view of (2.16), we have

$$\lim_{x\to\infty}\int_{X-n\tau}^{X} q(x)u(x)dx=\infty \tag{2.18}$$

This shows that either

$$\lim_{x\to\infty}\int_{X-n\tau}^{X} q(x)dx=\infty \tag{2.19}$$

or

$$\lim_{x\to\infty}\sup u(x)=\infty \tag{2.20}$$

If (2.19) holds, then

$$\lim_{x\to\infty}\sup\int_{X-n\tau}^{X} q(t)dt=\infty$$

By a known result in [8], every solution of (1.1) oscillates.

If (2.20) holds, then

$$\lim_{x\to\infty}\sup v(x)=\infty \tag{2.21}$$

On the other hand, integrating both sides of (1.1) from $x-\tau$ to x we have

$$y(x)-y(x-\tau)+\int_{x-\tau}^{x} q(t)y(t-\tau)dt=0,\quad x\geq x_2$$

and so

$$y(x-\tau)>\int_{x-\tau}^{x} q(t)y(t-\tau)dt=0,\quad x\geq x_2 \tag{2.22}$$

From this, by successively substituting $(n-2)$ times and using the decreasing nature of $y(x)$, it follows that

$$\begin{aligned} y(x-\tau) &> \int_{x-\tau}^{x} q(t)q_{n-2}(t)y(t-\tau)dt\\ &> y(t-\tau)\int_{x-\tau}^{x} q(t)q_{n-2}(t)dt, \end{aligned}$$

and so

$$y(t-\tau)>y(t-\tau)q_{n-2}(x),\quad x\geq x_2+(n-2)\tau \tag{2.23}$$

By (2.2), for any $x\geq x_1+\tau$ there exists a $\xi\in(x-\tau,x)$ such that

$$\int_{\xi}^{x} q(t)q_{n-1}(t)dt\geq\frac{1}{2e^n},\ \int_{x}^{\xi+\tau} q(t)q_{n-1}(t)dt\geq\frac{1}{2e^n} \tag{2.24}$$

By integrating both sides of (1.1) over $[\xi,x]$ and $[x,\xi+\tau]$, we have

$$y(x)-y(\xi)+\int_{\xi}^{x} q(t)y(t-\tau)dt=0,\quad x\geq x_2+(n-1)\tau \tag{2.25}$$

and

$$y(\xi+\tau)-y(x)+\int_{x}^{\xi+\tau} q(t)y(t-\tau)dt=0,\quad x\geq x_2+(n-1)\tau \tag{2.26}$$

Substituting (2.23) into (2.25) and (2.26), omitting the first term in (2.25) and (2.26) and using the decreasing nature of $y(x)$ and (2.24), we see that

$$\begin{aligned} -y(\xi)+\frac{1}{2e^n}y(x-\tau)&<0,\\ -y(x)+\frac{1}{2e^n}y(\xi)&<0 \end{aligned}$$

or

$$y(x) > \frac{1}{2e^n} y(\xi) > \frac{1}{4e^{2n}} y(x-\tau),$$

or

$$v(x) < 4e^{2n}, \quad x \geq x_2 + (n-1)\tau \tag{2.27}$$

This contradicts (2.21) and completes the proof of the theorem. □

Theorem 2.2. *Let $q(x) \in C([x_0,\infty), R^+)$ and let τ is a positive constant. Suppose that there exists a $x > x_0 + \tau$ such that (1.4) and (2.3) hold. Then every solution of (1.1) oscillates.*

Because (1.4) implies (2.2), Theorem 2.1 implies Theorem 2.2.

Remark 2.3. *Theorems 2.1 and 2.2 improve Theorem 1.1.*

Corollary 2.4. *Let $q(x) \in C([x_0,\infty), R^+)$ and let τ is a positive constant. Suppose that, for some positive integer n.*

$$\liminf_{x\to\infty} q_n(x) > \frac{1}{e^n}, \text{ and } \liminf_{x\to\infty} \bar{q}_n(x) > \frac{1}{e^n}, \tag{2.28}$$

where $q_n(x), \bar{q}_n(x)$ are defined by (2.1). Then every solution of (1.1) oscillates.

Remark 2.5. *Condition (2.28) improves (1.2).*

Corollary 2.6. *Let $q(x) \in C([x_0,\infty), R^+)$ and let τ is a positive constant. If (1.4) holds, and for some positive integer n,*

$$\int_{x_0+n\tau}^{\infty} q(x)\left(e^{n-1}q_n(x) - \frac{1}{e}\right)dx = \infty, \tag{2.29}$$

where $q_n(x)$ is defined by (2.1). Then every solution of (1.1) oscillates.

Corollary 2.7. *Let $q(x) \in C([x_0,\infty), R^+)$ and let τ is a positive constant. If (2.2) and (2.29) hold, then every solution of (1.1) oscillates.*

3. Example

Consider the delay differential equation

$$y'(x) + \frac{1}{2e}(1+cosx)y(x-\pi) = 0, \quad x \geq 0, \tag{3.1}$$

clearly, for, $x \geq \pi$,

$$\begin{aligned} q_1(x) &= \int_{x-\pi}^{x} \frac{1}{2e}(1+cost)dt \\ &= \frac{1}{2e}(\pi + 2sinx) \end{aligned}$$

$$\liminf_{x\to\infty} \int_{x-\pi}^{x} \frac{1}{2e}(1+cost)dt = \frac{1}{2e}(\pi-2) < \frac{1}{e}$$

This shows that (1.2) and (1.4) do not hold. But

$$\begin{aligned} q_2(x) &= \int_{x-\pi}^{x} q(t)q_1(t)dt \\ &= \frac{1}{4e^2}\int_{x-\pi}^{x}(1+cost)(\pi+2sint)dt \\ &= \frac{1}{4e^2}\left(\pi^2 + 2\pi sinx - 4cosx\right) \\ q_3(x) &= \int_{x-\pi}^{x} q(t)q_2(t)dt \\ &= \frac{1}{8e^3}\int_{x-\pi}^{x}\left(1+cost\right)\left(\pi^2+2\pi sint - 4cost\right)dt \\ &= \frac{1}{8e^3}\left(\pi^3 - 2\pi + (2\pi^2-8)sinx - 4\pi cosx\right) \end{aligned}$$

$$\begin{aligned} q_4(x) &= \int_{x-\pi}^{x} q(t)q_3(t)dt \\ &= \frac{1}{16e^4}\int_{x-\pi}^{x}\left(1+cost\right) \\ &\quad \left(\pi^3 - 2\pi + (2\pi^2-8)sint - 4\pi cost\right)dt \\ &= \frac{1}{16e^4}\Big[\pi^4 - 4\pi^2 \\ &\quad - 2(\pi^3-6\pi)sinx - 4(\pi^2-4)cosx\Big] \end{aligned}$$

$$\begin{aligned} \liminf_{x\to\infty} q_4(x) = \frac{1}{16e^4}\Big[\pi^4 - 4\pi^2 \\ - 2\sqrt{(\pi^3-6\pi)^2 + 4(\pi^2-4)^2}\Big] > \frac{22}{16e^4} \end{aligned}$$

and

$$\bar{q}_1(x) = \int_{x}^{x+\pi} \frac{1}{2e}(1+cost)dt = \frac{1}{2e}(\pi - 2sinx)$$

$$\begin{aligned} \bar{q}_2(x) &= \int_{x}^{x+\pi} q(t)\bar{q}_1(t)dt \\ &= \frac{1}{4e^2}\int_{x}^{x+\pi}(1+cost)(\pi-2sint)dt \\ &= \frac{1}{4e^2}\left(\pi^2 - 2\pi sinx - 4cosx\right) \\ \bar{q}_3(x) &= \int_{x}^{x+\pi} q(t)\bar{q}_2(t)dt \\ &= \frac{1}{8e^3}\int_{x}^{x+\pi}\left(1+cost\right)\left(\pi^2 - 2\pi sint - 4cost\right)dt \\ &= \frac{1}{8e^3}\left(\pi^3 - 2\pi - (2\pi^2-8)sinx - 4\pi cosx\right) \end{aligned}$$

$$\bar{q}_4(x) = \int_x^{x+\pi} q(t)\bar{q}_3(t)dt$$
$$= \frac{1}{16e^4}\int_x^{x+\pi}\Big(1+cost\Big)$$
$$\Big(\pi^3-2\pi-(2\pi^2-8)sint-4\pi cost\Big)dt$$
$$= \frac{1}{16e^4}\Big[\pi^4-4\pi^2$$
$$-2(\pi^3-6\pi)sinx-4(\pi^2-4)cosx\Big]$$

$$\lim_{x\to\infty}\inf\bar{q}_4(x) = \frac{1}{16e^4}\Big[\pi^4-4\pi^2$$
$$-2\sqrt{(\pi^3-6\pi)^2+4(\pi^2-4)^2}\Big] > \frac{22}{16e^4}$$

Then, by corollary 2.4, every solution of (3.1) oscillates.

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