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Transit index of various graph classes

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Abstract

Transit of a vertex v is a graph invariant which was defined as the sum of the length of all shortest paths with v as an internal vertex. In this paper, transit index for various classes of graph like complete graphs, cycles, wheel graph, friendship graph, crown graph, total graph of a path, comet are computed.

Keywords

Transit of a vertex, Transit Index.

AMS Subject Classification

05C10, 05C12.

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1. Introduction

Graph topological indices are widely studied. They find application in many field of science. Chemical graph Theory and Networking are a few to name. In[8], transit index of a graph was introduced and its correlation with one of the physical property -MON of octane isomers was established. In this paper we compute the transit index for various graph classes and for certain graphs developed from complete graphs.

Throughout G denotes a simple, connected, undirected graph with vertex set V and edge set E. For undefined terms we refer [1].

Preliminaries

Definition 1.1. *[8] Let $v \in V$. Then the transit of v denoted by $T(v)$ is "the sum of the lengths of all shortest path with v as an internal vertex" and the transit index of G denoted by $TI(G)$ is*

$$TI(G) = \sum_{v \in V} T(v)$$

Lemma 1.2. *[8] $T(v) = 0$ iff $\langle N[v] \rangle$ is a clique.*

Theorem 1.3. *[8] For a path P_n, Transit index is*

$$TI(P_n) = \frac{n(n+1)(n^2-3n+2)}{12}$$

Definition 1.4. *Two vertices v_1 and v_2 of a graph are called **transit identical** if the shortest paths passing through them are same in number and length.*

2. Transit index for various graph classes

2.1 Star

Theorem 2.1. *For a star graph S_n, $TI(S_n) = (n-1)(n-2)$*

Proof. In a star graph on n vertices, $n-1$ vertices are pendant vertices. Hence for them $T(v) = 0$. There are $C(n-1,2)$ shortest path of length 2 passing through the center vertex. Hence $TI(S_n) = 2.C(n-1,2) = (n-1)(n-2)$

□

2.2 Complete Graphs

Theorem 2.2. *For the complete graph K_n, transit index is zero.*

Proof. For every vertex v in a complete graph K_n, $\langle N[v]\rangle = K_n$, a clique. Hence by lemma[1.2], $TI(K_n) = 0$ □

Theorem 2.3. *For $n \geq 3$, deleting an edge from K_n, increases the transit index by $2(n-2)$.*

Proof. The deletion of the edge $e = uv$, makes u and v non-adjacent. Hence every other vertex will be an internal vertex of the shortest path between u and v of length 2. Hence $TI(K_n - e) = 2(n-2)$ □

Theorem 2.4. *Let $G = K_{p,q}$ where $V = V_1 \cup V_2$ is the bi-partition with $|V_1| = p, |V_2| = q$. Then $TI(G) = pq[p+q-2]$*

Proof. When $p = 1$ or $q = 1$, the result is obvious.
Let $p, q \geq 2$.
Let $v \in V_1$. Then, $T(v) = 2C(q,2)$.
If $v \in V_2$, then $T(v) = 2C(p,2)$.
Hence

$$\begin{aligned} TI(G) &= \sum_{v\in V} T(v) \\ &= \sum_{v\in V_1} T(v) + \sum_{v\in V_2} T(v) \\ &= 2[\frac{pq(q-1)}{2}] + 2[\frac{pq(p-1)}{2}] \\ &= pq[p+q-2] \end{aligned}$$

□

Theorem[2.4] can be generalised to s-partite graphs as follows.

Theorem 2.5. *Let G be the complete s-partite graph [6]. Then $TI(G) = \sum_{i=1}^{s} 2n_i \left[\sum_{j\neq i} C(n_j, 2)\right]$*

Proof. Let $V_1, V_2, \ldots, V_s$ be the partition of the vertex set V. Then no two vertices in V_i are adjacent to each other. But every vertex in V_j, $j \neq i$ is adjacent to all vertices of V_i.The shortest paths passing through v_i are those connecting vertices of the same V_j to itself, of length 2. Hence $T(v_i) = 2\sum_{j\neq i} C(n_j, 2)$

$$\therefore TI(G) = \sum_{v_i\in V_1} T(v_i) + \sum_{v_i\in V_2} T(v_i) + \cdots + \sum_{v_i\in V_s} T(v_i)$$

$$= \sum_{i=1}^{s} 2n_i \left[\sum_{j\neq i} C(n_j, 2)\right]$$ □

Corollary 2.6. *If G is the cocktail party graph [5] , $TI(G) = 4n(n-1)$*

Proof. In the theorem [2.5], take $n_i = 2, \forall i$ and $s = n$ with $|G| = 2n$. □

2.3 Cycle

Theorem 2.7. *Let C_n be a cycle with n even. Then*

i) $TI(C_n) = \frac{n^2(n^2-4)}{24}$

ii) $TI(C_{n+1}) = \frac{n(n^2-4)(n+1)}{24}$

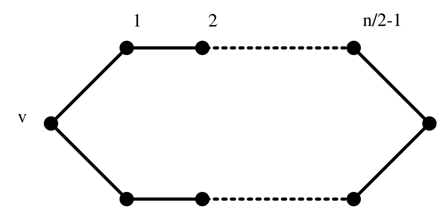


Figure 1. Cycle C_n

Proof.

(i) Consider the vertex v in the figure[1]. The maximum length of the shortest path passing through v is of length $\frac{n}{2}$. The sum length of the length of the shortest paths originating

$$\begin{aligned} &\text{from } 1 \quad is \quad 2+3+\ldots+\frac{n}{2} \\ &\text{from } 2 \quad is \quad 3+4+\ldots+\frac{n}{2} \\ &\quad\vdots \\ &\text{from } \frac{n}{2}-1 \quad is \quad \frac{n}{2} \end{aligned}$$

Hence

$$\begin{aligned} T(v) &= (\frac{n}{2}-1)\frac{n}{2} + (\frac{n}{2}-2)(\frac{n}{2}-1) + \ldots + 2.1 + 1.0 \\ &= \sum_{k=1}^{\frac{n}{2}} (k-1)k \\ &= \frac{(n^2-4)n}{24} \end{aligned}$$

Due to symmetry, every vertex in the cycle are transit identical.

$$\therefore TI(C_n) = \frac{n^2(n^2-4)}{24}, \text{n is even}$$

(ii) Consider C_{n+1}, with n even. The maximum length of the shortest path passing through any vertex v remains to be $\frac{n}{2}$. Hence as in the case of even cycle $T(v) = \frac{(n^2-4)n}{24}$.
$\therefore TI(C_{n+1}) = (n+1)T(v) = (n+1)\frac{(n^2-4)n}{24}$

□

2.4 Wheel Graph

The wheel graph [2], W_{n+1} is the graph obtained from C_n, $n \geq 3$ by adding a new vertex and by making it adjacent to all vertices of C_n.

Theorem 2.8. $TI(W_{n+1}) = n(n-1), n > 3$ *and for* $n = 3, TI(W_{3+1}) = 0$

Proof. Let $n > 3$. In W_{n+1}, the diameter is 2. Hence no shortest path is of length more than 2. The vertices on the outer circle C_n are transit identical. Let v be one such vertex. The only shortest path passing through it is between its adjacent vertices. $\therefore T(v) = 2$, for $v \in C_n$
Consider the center vertex c. To find its transit we consider the contribution of each edge to it. Every edge on C_n contributes 0 to $T(c)$. Consider the edges of the type e, as shown in the figure[2], which are the spokes of the wheel. e will be used only by v to travel to every vertex other than its adjacent ones. Hence the contribution is $(n-3)$. $\therefore T(c) = n(n-3)$
i.e.$TI(W_{n+1}) = 2n + n(n-3) = n(n-1)$
For $n = 3$, we get $W_{3+1} = K_4$. $\therefore$ its transit is zero. □

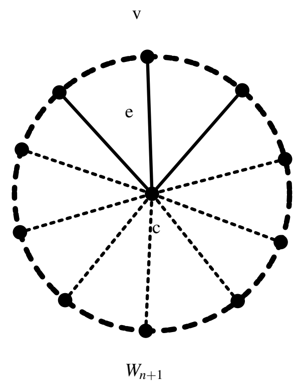


Figure 2. Wheel graph W_{n+1}

2.5 Friendship Graph

The Friendship graph [3], F_n is constructed by coalescence of n copies of cycle C_3 of length 3, with a common vertex.

Theorem 2.9. $TI(F_n) = 4n(n-1), |V| = 2n+1.$

Proof. In F_n, the diameter is 2. For every vertex v other than the coalescence vertex, $\langle N[v] \rangle$ is a clique. Hence $T(v) = 0$, by lemma[1.2]. Hence $TI(F_n) = T(c)$

The edges of the type e', as in the figure[3] does not contribute to $T(c)$. Hence we count the number of times the edges of the type e is used. The edge e will be used by the vertex v to travel to all vertices other than its adjacent ones. Hence contribution of e is $2n+1-3 = 2(n-1)$. There are $2n$ such edges.$\therefore T(c) = 4n(n-1)$.
i.e. $TI(F_n) = 4n(n-1), |V| = 2n+1.$ □

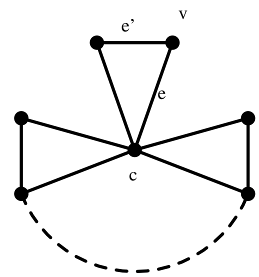


Figure 3. Friendship Graph F_n

2.6 Crown Graph

A crown graph [4] is the unique $n-1$ regular graph with $2n$ vertices, obtained from the complete bipartite graph $K_{n,n}$ by deleting a perfect matching. Or it is the graph with vertices as two sets $\{u_i\}$ and $\{v_i\}$, with an edge from u_i to v_j whenever $i \neq j$

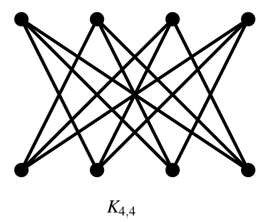


Figure 4. Crown graph

Theorem 2.10. *For the Crown graph G,* $TI(G) = 2n(n^2-1)$.

Proof. Let the bipartition be V, U, with $V = \{v_1, v_2, \ldots, v_n\}$ and $U = \{u_1, u_2, \ldots, u_n\}$. Consider a vertex of V, say v_k. Note that $d(u_i, v_i) = 3$ and $d(u_i, v_j) = 2, i \neq j$. The shortest path through v_k are those connecting v_i to $v_j, i \neq j$ of length 2 and those connecting v_i to u_i of length 3. Hence $T(v_k) = 2C(n-1,2) + 3(n-1) = n^2 - 1$. In this graph every vertex is transit identical. $\therefore TI(G) = 2n(n^2-1)$. □

2.7 Snake Graph

The triangular snake graph can be viewed as the graph formed by replacing every edge of P_n by a triangle, thus adding $n-1$ vertices and $2(n-1)$ edges.

Theorem 2.11. *If G is the triangular snake graph of a path on* $2n-1$ *vertices,* $TI(G) = TI(P_n) + \frac{(n-2)(n-1)n(n+1)}{4}$.

Proof. Let $v_1, v_2, \ldots, v_n$ denote the vertices of the path P_n. The newly added vertices are named as $u_1, u_2, \ldots, u_{n-1}$. For every u_i, $\langle N[u_i] \rangle$ is a clique. Hence $T(u_i) = 0, \forall i$, by lemma[1.2]. Also $\langle N[v_1] \rangle, \langle N[v_n] \rangle$ are cliques. $\therefore T(v_1) = T(v_n) = 0$.

Hence we need to compute only the transit of v_i for $1 < i < n$. The transit of these vertices are due to path connecting v_i among themselves, path connecting u_i among themselves and paths connecting v_i to u_i. i.e. $TI(G) = TI(P_n) + I$,

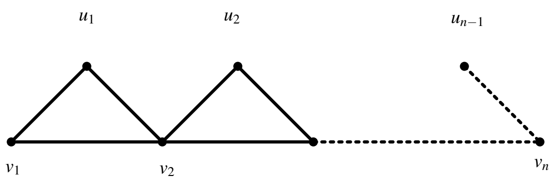


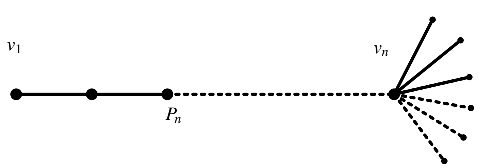
Figure 5. Total graph of a path

where I denote the increase in transit of v_i due to the addition of u_i.
Consider v_k . The increase in its transit is due to
1. Paths connecting u_i to $u_j, i < k, j > k$
2. Paths connecting u_i to $v_j, i < k, j > k$
3. Paths connecting v_i to $u_j, i < k, j > k$
It can be seen that the increase in all the three cases are the same and equal to
$A = (2+3+\ldots+n-k+1)+(3+4+\ldots+n-k+2)+\ldots(k+(k+1)+\ldots+(n-1)$
Hence increase in transit of v_k is $3A$.
If we take $a = 2+3+\ldots+n-k+1$, $A = a+(a+n-k)+(a+2(n-k))+\ldots$
Hence increase in transit of v_k is $= 3[a(k-1)+\frac{(n-k)(k-2)(k-1)}{2}]$
$= \frac{3}{2}(n-k)(k-1)(n+1)$.
$= \frac{3}{2}(n+1)[(n+1)k-k^2-n]$
Hence $I = \sum_{k=1}^{n} \frac{3}{2}(n+1)[(n+1)k-k^2-n]$
$= \frac{(n-2)(n-1)n(n+1)}{4}$.
Hence the proof.

□

2.8 Comet

A comet is formed by appending multiple pendant edges to one end of a path.



Theorem 2.12. *Let G be the graph got by appending m pendant edges to one end of P_n. Then $TI(G) = TI(P_n) + \frac{mn(n^2-1)}{3}$*

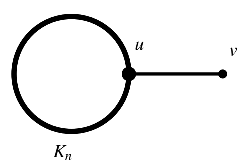
Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n and $u_1, u_2, \ldots, u_m$ the newly appended vertices.
In the graph G, transit is zero for the newly added vertices. Hence $TI(G) = TI(P_n) + I$, where I is the increase in transit of vertices of P_n due to the newly appended edges.
Let v_k be any vertex of P_n. Then the increase in $T(v_k)$ is due to the paths connecting the vertices on the left of it to the newly added vertices. This can be computed as
$= nm+(n-1)m+\ldots+(n-k+2)m$
$= \frac{m}{2}[k(2n+3)-k^2-(2n+2)$, on simplification.
$\therefore I = \sum_{1}^{n} \frac{m}{2}[k(2n+3)-k^2-(2n+2)]$
$= \frac{mn(n^2-1)}{3}$
Hence the theorem. □

Remark 2.13. *Applying the recursive formula for a path, $TI(P_{n+1}) = TI(P_n) + \frac{n(n^2-1)}{3}$, the transit of a comet G of Theorem [2.12] can be expressed as, $TI(G) = mTI(P_{n-1}) - (m-1)TI(P_n)$.*

3. Transit index for some graphs derived from Complete graph

Theorem 3.1. *Let G be the graph obtained by attaching a pendant edge to one of the vertices of a complete graph. i.e. $|V(G)| = |V(K_n)|+1$ and $|E(G)| = |E(K_n)|+1$. Then $TI(G) = 2(n-1)$*

Proof. Let the new vertex be v and the vertex to which it is attached be u. Then for every vertex in G other than u, $\langle N[v_i] \rangle$



is a clique. Hence transit is zero. There are $n-1$ paths of length 2 connecting v to vertices of $K_n - \{u\}$, passing through v.
$\therefore TI(G) = 2(n-1)$ □

Theorem 3.2. *Let G be the graph formed by attaching a pendant edge to every vertex of K_n. Then $TI(G) = 5n(n-1)$*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of K_n and $u_1, u_2, \ldots, u_n$, be the vertices attached to $v_1, v_2, \ldots, v_n$ respectively. Since u_i are pendant vertices $T(u_i) = 0, \forall i$. The shortest path passing through v_i are either $u_i v_j$ paths or $u_i u_j$ paths of length 2 and 3 respectively. Hence $T(v_i) = 2(n-1)+3(n-1)$
$\therefore TI(G) = 5n(n-1)$ □

Theorem 3.3. *Let G be the graph formed by merging a vertex of K_n and K_m. i.e. $|V(G)| = m+n-1$ and $|E(G)| = |E(K_n)| + |E(K_m)|$. Then $TI(G) = 2(n-1)(m-1)$*

Proof. Let v be coalescence vertex. For every vertex u of G other than v, $T(u) = 0$, as $N[u]$ is a clique. The shortest paths passing through v are those connecting the $n-1$ vertices of K_n with $m-1$ vertices of K_m, each of length 2. Hence $TI(G) = T(v) = 2(n-1)(m-1)$ □

Theorem 3.4. *Let G be the graph formed by merging a vertex of K_n with a vertex of C_m.*
Then $TI(G) = TI(C_m) + \frac{(n-1)(m+4)(m+2)m}{12}$, if m is even and $TI(G) = TI(C_m) + \frac{(n-1)(m-1)(m+1)(m+3)}{12}$, if m is odd.

Proof. Let us denote the coalescence vertex by v
Case 1 [m even]
Clearly, $TI(G) = TI(C_m) + TI(K_n) + I$, where I denote the increment in transit due to merging of graphs. The transit for vertices in K_n remains zero, except for v. The vertex at the distance $\frac{m}{2}$ from v on C_m has no increment. Let v_k denote the kth vertex on P_1, v_1 being v. For $1 < k < \frac{m}{2}$, the increment for v_k is due to the shortest paths from vertices on its right to vertices of K_n including v. This can be computed as

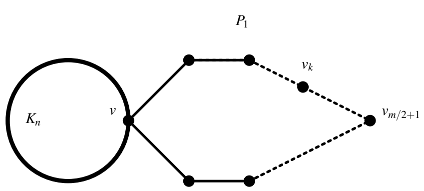


Figure 6. K_n and C_m merged at v, m even.

$= [(k+1) + (k+2) + \ldots + (\frac{m}{2}+1)](n-1)$
$= \left[(\frac{m}{2}+1)(\frac{m}{2}+2) - k - k^2\right] \frac{(n-1)}{2}$
Now due to similar positions, $T(v_k), T(v_{m-k+2})$ are transit identical.
Hence we have $I =$

$$=2\sum_{1}^{\frac{m}{2}} \left[(\frac{m}{2}+1)(\frac{m}{2}+2) - k - k^2\right] \frac{(n-1)}{2}$$

$= \frac{(n-1)(m+4)(m+2)m}{12}$, on simplification.
$\therefore TI(G) = TI(C_m) + \frac{(n-1)(m+4)(m+2)m}{12}$

Case 2[m odd]
Let v_k denote the kth vertex on P_1, $v_1 = v$. For $1 < k < \frac{m-1}{2}$, the increment for v_k is due to the shortest paths from vertices on its right to vertices of K_n including v. This can be computed as

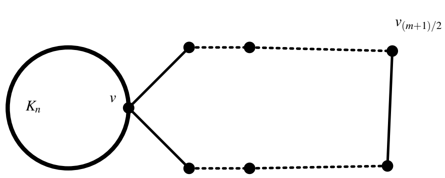


Figure 7. K_n and C_m merged at v, m odd.

$I = (k+1) + (k+2) + \ldots + \frac{m+1}{2}$
$= \frac{(m+1)}{2} \frac{(m+3)}{4} - \frac{k}{2} - \frac{k^2}{2}$
In this case also $T(v_k) = T(v_{m-k+2})$

$$\text{Hence } TI(G) = 2\sum_{1}^{\frac{m-1}{2}} \left[\frac{(m+1)}{2}\frac{(m+3)}{4} - \frac{k}{2} - \frac{k^2}{2}\right]$$

$= \frac{(n-1)(m-1)(m+1)(m+3)}{12}$
$\therefore TI(G) = TI(C_m) + \frac{(n-1)(m-1)(m+1)(m+3)}{12}$. □

4. Conclusion

In this paper, transit index for various graph classes and for graphs obtained from complete graphs are computed. In future, authors are planning to extend the study to sub-division graphs, graph products and various graphs of importance in chemical graph theory and communication networks .

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