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# Neighbourhood $V_4-$magic labeling of some middle graphs

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**Abstract**

Let $V_4 = \{0, a, b, c\}$ be the Klein-4-group with identity element $0$.A graph $G(V(G), E(G))$ is said to be neighbourhood $V_4$-magic if there exists a labeling $f : V(G) \to V_4\backslash\{0\}$ such that the induced mapping $N_f^+ : V(G) \to V_4$ defined by $N_f^+(v) = \sum_{u \in N(v)} f(u)$ is a constant map. If this constant is $p(p \neq 0)$,we say that $f$ is a $p$-neighbourhood $V_4$-magic labeling of $G$ and $G$ a $p$-neighbourhood $V_4$-magic graph. If this constant is zero, we say that $f$ is a $0$-neighghbourhood $V_4$-magic labeling of $G$ and $G$ a $0$-neighbourhood $V_4$-magic graph. In this paper we investigate middle graph of some special graphs that are $a$-neighbourhood $V_4$-magic, $0$-neighbourhood $V_4$-magic and both $a$-neighbourhood and $0$-neighbourhood $V_4$-magic.

**Keywords**

Klein-4-group, a-neighbourhood $V_4$-magic graphs, 0-neighbourhood $V_4$-magic graphs.

**AMS Subject Classification**

05C78, 05C25.

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## Contents



## 1. Introduction

Through out this paper we shall consider only connected, finite, simple and undirected graphs. All notations and definitions not given here can be found in [2, 3]. The Klein 4-group, denoted by $V_4$ is the abelian group of order 4. It has elements $V_4 = \{0, a, b, c\}$,with $a + a = b + b = c + c = 0$ and $a + b = c, b + c = a, c + a = b$. Obviously $V_4$ is not cyclic since every element is of order 2 (except possibly the identity). The $V_4$-magic graphs were introduced by S. M. Lee et al. in 2002 [7]. We say that, a graph $G = (V(G), E(G))$, with vertex set $V(G)$ and edge set $E(G)$ is Neighbourhood $V_4$-magic if there exists a labeling $f : V(G) \to V_4\backslash\{0\}$ such that the induced mapping $N_f^+ : V(G) \to V_4$ defined by $N_f^+(v) = \sum_{u \in N(v)} f(u)$ is a constant map. If this constant is $p$,where $p$ is any non zero element in $V_4$,then we say that $f$ is a $p$-neighbourhood $V_4$-magic labeling of $G$ and $G$ is said to be a $p$-neighbourhood $V_4$-magic graph. If this constant is 0,then we say that $f$ is a 0-neighbourhood $V_4$-magic labeling of $G$ and $G$ is said to be a 0-neighbourhood $V_4$-magic graph. In this paper, we study a class of graphs in the following categories:

(i) $\Omega_a :=$ the class of all $a$-neighbourhood $V_4$-magic graphs,

(ii) $\Omega_0 :=$ the class of all 0-neighbourhood $V_4$-magic graphs,

(iii) $\Omega_{a,0} := \Omega_a \cap \Omega_0$.

**Definition 1.1.** [1] The middle graph of a graph $G$, denoted by $M(G)$, is the graph obtained from $G$ by inserting a new vertex into every edge of $G$ and by joining those pairs of these new vertices with edges which lie on adjacent edges of $G$.

**Definition 1.2.** [4] A complete bipartite graph of the form $K_{1,n}$ is called a star. A star $K_{1,n}$ is sometimes called an $n-$star.

**Definition 1.3.** [5] The friendship graph or the Dutch windmill graph, denoted by $F_m$ (or $D_3^{(m)}$) is obtained by taking $m$ copies of $C_3$ with one vertex in common.

**Definition 1.4.** [6] The Bistar $B_{m,n}$ is the graph obtained by joining the central vertex of $K_{1,m}$ and $K_{1,n}$ by an edge.

## 2. Main Results

**Theorem 2.1.** $M(C_n) \in \Omega_a$ *if and only if* $n \equiv 0(\mod 2)$.

*Proof.* Consider $M(C_n)$ with vertex set $V = \{u_i, v_i : 1 \le i \le n\}$ labeled as in figure 1. Suppose that $M(C_n) \in \Omega_a$ with a labeling $f$. Then $N_f^+(u_2) = a$ implies that $f(v_1) + f(v_2) = a$, which implies that either $f(v_1) = b$ or $f(v_1) = c$. Without loss of generality we can assume that $f(v_1) = b$. Then $f(v_2) = c$, $f(v_3) = b$, $f(v_4) = c$, etc. and so on. Now $N_f^+(u_1) = a$ implies that $f(v_1) + f(v_n) = a$, which again implies that $f(v_n) = c$. Hence $n \equiv 0(\bmod 2)$. Conversely, assume that $n \equiv 0(\bmod 2)$. Then define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 0(\bmod 2) \\ c & \text{if} \quad i \equiv 1(\bmod 2) \end{cases}$$

Obviously, $f$ is an $a$-neighbourhood $V_4$-magic labeling of $M(C_n)$. This completes the proof of the theorem. □

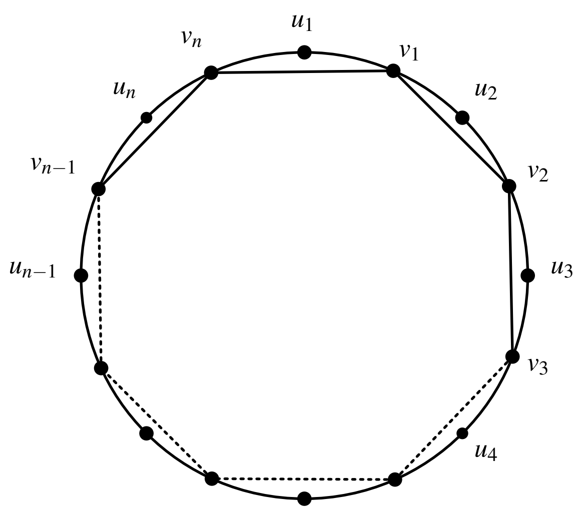


**Figure 1.** The middle graph $M(C_n)$

**Theorem 2.2.** *$M(C_n) \in \Omega_0$ for all $n \ge 3$.*

*Proof.* By labeling all the vertices by $a$, we get $M(C_n) \in \Omega_0$. □

**Corollary 2.3.** *$M(C_n) \in \Omega_{a,0}$ if and only if $n \equiv 0(\bmod 2)$.*

*Proof.* Proof directly follows from Theorem 2.1 and Theorem 2.2. □

**Theorem 2.4.** *$M(P_n) \notin \Omega_a$ for any $n$.*

*Proof.* Consider $M(P_n)$ with vertex set $V = \{u_i, v_j : 1 \le i \le n, 1 \le j \le n-1\}$ and edge set $E = \{u_iv_i : 1 \le i \le n-1\} \cup \{v_iu_{i+1} : 1 \le i \le n-1\} \cup \{v_iv_{i+1} : 1 \le i \le n-2\}$. Assume that $M(P_n) \in \Omega_a$ for some $n$ with a labeling $f$. Then $N_f^+(u_1) = a$ implies that $f(v_1) = a$. Now $N_f^+(u_2) = a$ implies that $f(v_1) + f(v_2) = a$. Hence $f(v_2) = 0$, a contradiction. Hence the theorem is proved. □

**Theorem 2.5.** *$M(P_n) \notin \Omega_0$ for any $n$.*

*Proof.* Proof is obvious, since $M(P_n)$ has pendant vertex in it. □

**Corollary 2.6.** *$M(P_n) \notin \Omega_{a,0}$ for any $n$.*

*Proof.* It directly follows from Theorem 2.4. □

**Theorem 2.7.** *$M(K_{1,n}) \in \Omega_a$ if and only if $n \equiv 1(\bmod 2)$.*

*Proof.* Consider $M(K_{1,n})$ with vertex set $V = \{u, u_i, v_i : 1 \le i \le n\}$ and edge set $E = \{uv_i, u_iv_i : 1 \le i \le n\} \cup \{v_iv_j : 1 \le i, j \le n, i \ne j\}$. Assume that $M(K_{1,n}) \in \Omega_a$ with a labeling $f$. Then, $N_f^+(u_i) = a$ implies that $f(v_i) = a$ for all $1 \le i \le n$. Consequently, $N_f^+(u) = a$ gives $na = a$, which implies that $n \equiv 1(\bmod 2)$. Conversely, assume that $n \equiv 1(\bmod 2)$. Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = b \quad \text{if} \quad i = 1, 2, 3, \ldots, n$$
$$f(v_i) = a \quad \text{if} \quad i = 1, 2, 3, \ldots, n$$
$$f(u) = c$$

Then $f$ is an $a$-neighbourhood $V_4$-magic labeling of $M(K_{1,n})$. □

**Theorem 2.8.** *$M(K_{1,n}) \notin \Omega_0$ for any $n$.*

*Proof.* Proof is obvious due to the presence of pendant vertex in $M(K_{1,n})$. □

**Corollary 2.9.** *$M(K_{1,n}) \notin \Omega_{a,0}$ for any $n$.*

*Proof.* It directly follows from Theorem 2.8. □

**Theorem 2.10.** *$M(F_m) \in \Omega_0$ for all $n$.*

*Proof.* Note that degree of each vertex in $M(F_m)$ is even. By labeling all the vertices by $a$, we get $M(F_m) \in \Omega_0$ for all $m$. □

**Theorem 2.11.** *$M(F_m) \notin \Omega_a$ for any $n$.*

*Proof.* Consider $M(F_m)$ with verex set $V = \{w, u_i, v_i, w_i', u_i', v_i' : 1 \le i \le m\}$ labeled as in figure 3. Suppose that $M(F_m) \in \Omega_a$ with a labeing $f$. Then for each $1 \le i \le m$, $N_f^+(u_i) = a = N_f^+(v_i)$ implies that $f(u_i') = (v_i')$. Hence $N_f^+(w) = \sum f(u_i') + \sum f(v_i') = 0$, which is a contradiction. Hence the theorem is proved. □

**Corollary 2.12.** *$M(F_m) \notin \Omega_{a,0}$ for any $n$.*

*Proof.* It directly follows from Theorem 2.11. □

**Theorem 2.13.** *$M(B_{m,n}) \notin \Omega_0$ for all $m$ and $n$.*

*Proof.* Proof is obvious, since $M(B_{m,n})$ has pendant vertex in it. □

**Theorem 2.14.** *$M(B_{m,n}) \in \Omega_a$ if and only if $m$ and $n$ are both even.*

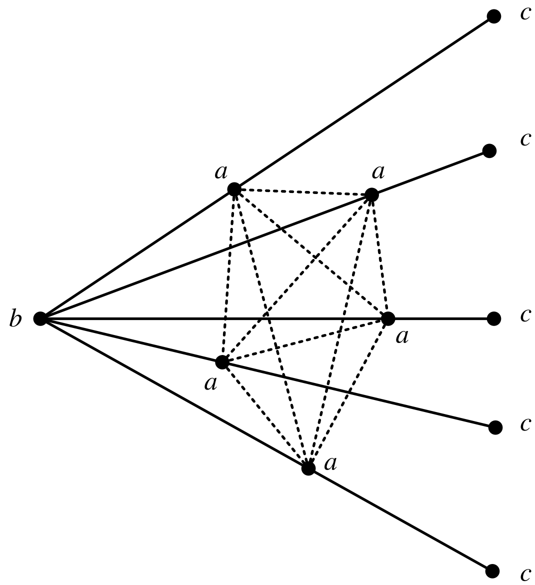


**Figure 2.** An $a$-neighbourhood $V_4$-magic labeling of $M(K_{1,5})$.

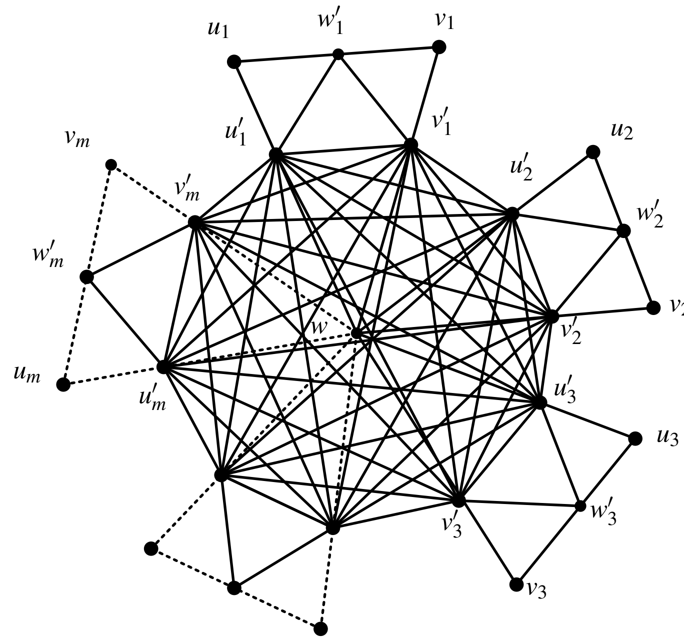


**Figure 3.** The middle graph $M(F_m)$.

*Proof.* Consider $M(B_{m,n})$ with vertex set $V = \{u, v, w, u_i, v_j, u'_i, v'_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set $E = \{u_i u'_i : 1 \leq i \leq m\} \cup \{v_j v'_j : 1 \leq j \leq n\} \cup \{u'_i u'_j : 1 \leq i, j \leq m, i \neq j\} \cup \{v'_i v'_j : 1 \leq i, j \leq n, i \neq j\} \cup \{uu'_i : 1 \leq i \leq m\} \cup \{vv'_j : 1 \leq j \leq n\} \cup \{wu'_i : 1 \leq i \leq m\} \cup \{wv'_j : 1 \leq j \leq n\} \cup \{wu, wv\}$. Assume that $m$ and $n$ are both even. Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = c \quad \text{if} \quad i = 1, 2, 3, \ldots, m$$

$$f(u'_i) = a \quad \text{if} \quad i = 1, 2, 3, \ldots, m$$

$$f(v_j) = b \quad \text{if} \quad j = 1, 2, 3, \ldots, n$$

$$f(v'_j) = a \quad \text{if} \quad j = 1, 2, 3, \ldots, n$$

$$f(u) = b$$

$$f(v) = c$$

$$f(w) = a$$

Then, $f$ is an $a$-neighbourhood $V_4$-magic labeling of $M(B_{m,n})$. Conversely, assume that $m$ and $n$ are not both even. Without loss of generality assume that $m$ is odd. If $M(B_{m,n}) \in \Omega_a$, then we have $f(u'_i) = a$ for $i = 1, 2, 3, \ldots, m$. Now $N_f^+(u) = \sum f(u_i) + f(w) = a$ implies that $na + f(w) = a$, which again implies that $f(w) = 0$, a contradiction. Therefore, $M(B_{m,n}) \notin \Omega_a$. □

**Corollary 2.15.** *$M(B_{m,n}) \notin \Omega_{a,0}$ for any $n$.*

*Proof.* It directly follows from Theorem 2.13. □

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