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Translations in semirings and semiring of translations

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Abstract

In this paper we describe the inner left [right] translations and bitranslations on a semiring S and it is shown that these translations in semirings provides representations of the semiring. In particular, here it is shown that the translations on a Γ-semiring is again a Γ-semiring.

Keywords

Translations, translational hull, Γ-semiring, ideal.

AMS Subject Classification

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1. Introduction

The translations in semigroups are extensively studied by many authors including A.H.Clifford, M. Petrich (see [1], [2]). The study of translations in semigroups are found significant in the study of extensions of semigroups as is seen in the case of translational hull appearing while one is concerned with the ideal extensions of semigroups. It is observed that in a natural way translational hull of a semigroup is a semigroup, that of a ring is a ring and an algebra is an algebra. This hierarchy has been first noted by B.E.Johnson in 1964. In the survey article [3], Mario Petrich widely describes translational hulls of semigroups and rings. In this paper we describe translations on semirings and it is shown that set of left[right] translations, bitranslations are again semirings.

2. Preliminaries

In the following we briefly recall all basic definitions and results needed in the sequel. A nonempty set S together with an associative binary operation is called a semigroup and is called weakly reductive if for all $x \in S$, $\quad ax = bx \quad and \quad xa = xb$ implies $a = b$.

Definition 2.1. *Let S be a semigroup. A map λ on S is called a left translation of S if $\lambda(xy) = (\lambda x)y$, $x, y \in S$. Similarly, a function ρ is called right translation if $(xy)\rho = x(y\rho)$.*

A left translation λ and a right translation ρ are called linked if $x(\lambda y) = (x\rho)y$ for every $x, y \in S$. Let $\Lambda(S)$ denote the set of all left translations and $P(S)$ denote the set of all right translations. Then the subset $\Omega(S) = \Lambda(S) \times P(S)$ consisting of all linked pair of translations on S is the translational hull of S.

Definition 2.2. *A semiring is a nonempty set R on which operations of addition and multiplication have been defined such that the following conditions are satisfied:*

1. *$(R, +)$ is a commutative semigroup*
2. *$(R, \cdot)$ is a semigroup*
3. *Multiplication distributes over addition from either side;*

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

$$(x + y) \cdot z = x \cdot z + y \cdot z$$

where $x, y, z \in R$.

The set $\mathbb{N}$ of natural numbers with the operations of addition and multiplication is a commutative semiring. Also $\mathbb{Q}^+$ of all nonnegative rationals, $\mathbb{R}^+$ of all nonnegative real numbers are semirings. If $(S,+)$ is a commutative monoid with identity element 0 then the set $End(S)$ of all endomorphisms of S is a semiring under the operations of pointwise addition and composition of functions. For more examples the reader is suggested to refer Jonathan S.Golan [4].

Next we consider a special type of semirings called Γ-semirings, which are introduced by M. Murali Krishna Rao [5] as a generalisation of Γ-rings as well as semirings in 1995. J.Luh studied about Γ-semirings and its sub-Γ-semirings with a left(right) unity. S. Kyuno and M.K. Rao extensively studied about the ideals, prime idels, semiprime ideals, k-ideals of a Γ-semiring, regular Γ-semiring respectively. H.Hedayati and K.P.Shum together deeply studied congruences and ideals.

Definition 2.3. *[6] Let $(M,+)$ and $(\Gamma,+)$ be commutative semigroups. Then $M=(M,\Gamma)$ is called a Γ-semiring if there exists a map $M\times\Gamma\times M\to M$, written (x,γ,y) by $x\gamma y$, such that it satisfies the following axioms for all $x,y,z\in M$ and $\gamma,\beta\in\Gamma$:*

1. $x\gamma(y+z)=x\gamma y+x\gamma z$ *and* $(x+y)\gamma z=x\gamma z+y\gamma z$
2. $x(\gamma+\beta)y=x\gamma y+x\beta y$
3. $(x\gamma y)\beta z=x\gamma(y\beta z)$.

Example 2.4. *Every semiring A is Γ-semiring with $\Gamma=A$ and the ternary operation $x\gamma y$ as the usual semiring multiplication.*

Example 2.5. *Let S be the additive semigroup of all $m\times n$ matrices over the set of non-negative rational numbers and Γ be the additive semigroup of all $n\times m$ matrices over the set of non-negative integers. Then with respect to usual matrix multiplication S is a Γ-semiring.*

Example 2.6. *Let N be a semiring and $M_{p,q}(N)$ denote the additive abelian semigroup of all $p\times q$ matrices with identity element whose entries are from N. $M_{p,q}(N)$ is a Γ-semiring with $\Gamma=M_{p,q}(N)$ where the composition is defined by $x\alpha z=x(\alpha^t)z$, where α^t denote the transpose of the matrix α, for all x,y and $\alpha\in M_{p,q}(N)$.*

3. Translations in Semirings

Let S be a semiring. The mapping $\lambda_a:S\to S$ is defined by

$$\lambda_a(x)=ax$$

is called the inner left translation on S induced by an element a. Then the mapping λ_a satisfies the following:

1. $\lambda_a(xy)=(\lambda_a x)y$
2. $\lambda_a(x+y)=\lambda_a(x)+\lambda_a(y)$
3. $\lambda_a\lambda_b(x)=\lambda_{ab}(x)$

where $a,b,x,y\in S$. We denote the set $\{\lambda_a : a\in S\}$ as $\Lambda(S)$.

Similarly, the mapping $\rho_a : x\mapsto xa$ is called the inner right translation induced by a, the mapping ρ_a satisfies

1. $(xy)\rho_a=x(y\rho_a)$
2. $(x+y)\rho_a=(x)\rho_a+(y)\rho_a$
3. $(x)\rho_a\rho_b=(x)\rho_{ab}$

where $a,b,x,y\in S$ and the set $\{\rho_a : a\in S\}$ is denoted by $P(S)$. The set $\{(\lambda_a,\rho_a) : a\in S\}$, of all bitranslations is denoted by $\Pi(S)$.

Proposition 3.1. *S be a semiring. Then $\Lambda(S)$ of all inner left translations form a semiring under addition and multiplication defined as follows:*

$$\lambda_{r_1}\lambda_{r_2}=\lambda_{r_1r_2}$$

$$\lambda_{r_1}+\lambda_{r_2}=\lambda_{r_1+r_2}$$

Proof. Let $r_1,r_2,r_3\in S$. Since multiplication is associative in S

$$\begin{aligned}(\lambda_{r_1}\lambda_{r_2})\lambda_{r_3}&=(\lambda_{r_1r_2})\lambda_{r_3}\\&=\lambda_{(r_1r_2)r_3}\\&=\lambda_{r_1}(\lambda_{r_2r_3})\\&=\lambda_{r_1}(\lambda_{r_2}\lambda_{r_3})\end{aligned}$$

Similarly since addition is associative in S, we get

$$\lambda_{r_1}+(\lambda_{r_2}+\lambda_{r_3})=(\lambda_{r_1}+\lambda_{r_2})+\lambda_{r_3}.$$

Also, the multiplication is distributive over addition from left in S implies

$$\begin{aligned}\lambda_{r_1}(\lambda_{r_2}+\lambda_{r_3})&=\lambda_{r_1}(\lambda_{r_2+r_3})\\&=\lambda_{r_1(r_2+r_3)}\\&=\lambda_{r_1r_2+r_1r_3}\\&=\lambda_{r_1}\lambda_{r_2}+\lambda_{r_1}\lambda_{r_3}.\end{aligned}$$

Similarly we get that multiplication is distributive over addition from right side in $\Lambda(S)$. If 0 denote the additive identity in S then λ_0 is the additive identity on $\Lambda(S)$ for $r\in S$

$$\lambda_0+\lambda_r=\lambda_{0+r}=\lambda_r$$

and if 1 is the multiplicative identity of S,

$$\lambda_1\lambda_r=\lambda_r=\lambda_r\lambda_1.$$

Hence $\Lambda(S)$ forms a semiring. ☐

In a similar manner we have the following proposition for the inner right translations.

Proposition 3.2. *S be a semiring. Then $P(S)$ of all inner right translations form a semiring under addition and multiplication defined by*

$$\rho_{r_1}\rho_{r_2} = \rho_{r_1 r_2}$$

$$\rho_{r_1} + \rho_{r_2} = \rho_{r_1+r_2}.$$

is a semiring.

Lemma 3.3. *S be a semiring. Then $\Pi(S) = \{\pi_r = (\lambda_r, \rho_r), r \in S\}$ is a semiring with respect to the addition and multiplication defined by*

$$(\lambda_{r_1}, \rho_{r_1}) \cdot (\lambda_{r_2}, \rho_{r_2}) = (\lambda_{r_1 r_2}, \rho_{r_1 r_2})$$

$$(\lambda_{r_1}, \rho_{r_1}) + (\lambda_{r_2}, \rho_{r_2}) = (\lambda_{r_1+r_2}, \rho_{r_1+r_2}).$$

Proof. Let $r_1, r_2, r_3 \in S$ then $\lambda_{r_1}, \lambda_{r_2}, \lambda_{r_3} \in \Lambda(S)$ and $\rho_{r_1}, \rho_{r_2}, \rho_{r_3} \in P(S)$. Since $\Lambda(S)$ and $P(S)$ are semirings, it is easy to observe that the binary operations defined above are associative.

Further,

$$\begin{aligned}(\lambda_{r_1}, \rho_{r_1}) \cdot [(\lambda_{r_2}, \rho_{r_2}) + (\lambda_{r_3}, \rho_{r_3})] \\ = (\lambda_{r_1}, \rho_{r_1}) \cdot (\lambda_{r_2+r_3}, \rho_{r_2+r_3}) \\ = (\lambda_{r_1(r_2+r_3)}, \rho_{r_1(r_2+r_3)}) \\ = (\lambda_{r_1 r_2}, \rho_{r_1 r_2}) + (\lambda_{r_1 r_3}, \rho_{r_1 r_3})\end{aligned}$$

hence distributivity follows.

Similarly other distributive law also follows. For the additive identity 0 of S, (λ_0, ρ_0) act as additive identity in $\Pi(S)$ and for the multiplicative identity 1, (λ_1, ρ_1) act as the multiplicative identity in $\Pi(S)$. Hence $\Pi(S)$ is a semiring under componentwise addition and multiplication. □

The mapping $\pi : S \to \Pi(S)$ such that $r \mapsto \pi_r = (\lambda_r, \rho_r)$ is the homomorphism of S into $\Pi(S)$. That is $\Lambda(S), P(S)$ and $\Pi(S)$ are representations of the semiring S.

Proposition 3.4. *S be a semiring. The canonical homomorphism $\pi : S \to \Pi(S)$ is one-one if and only if the multiplicative semigroup $(S, \cdot)$ is weakly reductive.*

Proof. Consider $r_1, r_2 \in S$ and π be one-one, then

$$\begin{aligned}\pi_{r_1} = \pi_{r_2} &\Leftrightarrow (\lambda_{r_1}, \rho_{r_1}) = (\lambda_{r_2}, \rho_{r_2}) \\ &\Leftrightarrow r_1 x = r_2 x \quad \text{and} \quad x r_1 = x r_2\end{aligned}$$

Hence π is one-one if and only if the multiplicative reduct of S is weakly reductive. □

Proposition 3.5. *Let λ and ρ be left and right translations respectively of a semiring S and $a \in S$. Then*

$$\lambda\lambda_a = \lambda_{\lambda a}, \quad \rho_a \rho = \rho_{a\rho}.$$

When λ and ρ are linked, we have

$$\lambda_a \lambda = \lambda_{a\rho}, \quad \rho\rho_a = \rho_{\lambda a}.$$

Proof. Let $a, x \in S$

$$\lambda\lambda_a(x) = \lambda(ax) = (\lambda a)x = \lambda_{\lambda a}(x)$$

$$(x)\rho_a\rho = (xa)\rho = x(a\rho) = (x)\rho_{a\rho}$$

If λ and ρ are linked, then

$$\lambda_a\lambda(x) = \lambda_a(\lambda x) = a(\lambda x) = (a\rho)x = \lambda_{a\rho}(x)$$

$$(x)\rho\rho_a = (x\rho)\rho_a = (x\rho)a = x(\lambda a) = (x)\rho_{\lambda a}$$

□

4. Translations in Γ-Semirings

In the following we proceed to describe translations on a Γ-semiring.

Lemma 4.1. *Let (M, Γ) be a Γ-semiring. Define the map λ_a as follows:*

$$\lambda_a(m) = a + m$$

where $a, m \in M$.Then λ_a is a inner left translation on M.

Proof. Since the operation here is addition we have to show that,

$$\lambda_a(x+y) = \lambda_a(x) + y$$

where $x, y \in M$. Now,

$$\lambda_a(x+y) = a + (x+y)$$

$$= (a+x) + y = \lambda_a(x) + y.$$

Hence λ_a is a left translation on M. □

Let $\Lambda_M = \{\lambda_a, a \in M\}$ denote the set of all inner left translations in M.

Proposition 4.2. *Let (M, Γ) be a Γ-semiring. Then Λ_M is a commutative semigroup under the operation defined as follows:*

$$\lambda_a \oplus \lambda_b = \lambda_{a+b}$$

where $a, b \in M$.

Proof. First we check for the associatiity, since addition is associative in M,

$$(\lambda_a \oplus \lambda_b) \oplus \lambda_c = (\lambda_{a+b}) \oplus \lambda_c$$

$$= \lambda_{(a+b)+c} = \lambda_{a+(b+c)}$$

$$= \lambda_a \oplus (\lambda_{b+c}) = \lambda_a \oplus (\lambda_b \oplus \lambda_c)$$

for $a, b, c \in M$.
Since addition is commutative in M we get,

$$\lambda_a \oplus \lambda_b = \lambda_{a+b} = \lambda_{b+a} = \lambda_b \oplus \lambda_a$$

□

Proposition 4.3. *Let M be a Γ-semiring. Then Λ_M is a Γ-semiring.*

Proof. From Proposition 4.2, Λ_M is an additive commutative semigroup. We define a mapping $\Lambda_M \times \Gamma \times \Lambda_M \to \Lambda_M$ such that $(\lambda_a, \gamma, \lambda_b) \mapsto \lambda_{a\gamma b}$ then

1. $\lambda_a \gamma (\lambda_b \oplus \lambda_c) = \lambda_a \gamma \lambda_{b+c}$
$$= \lambda_{a\gamma(b+c)} = \lambda_{a\gamma b + a\gamma c} = \lambda_{a\gamma b} \oplus \lambda_{a\gamma c}$$

2. $\lambda_a(\gamma + \beta)\lambda_b = \lambda_{a(\gamma+\beta)b} = \lambda_{a\gamma b + a\beta b}$
$$= \lambda_{a\gamma b} \oplus \lambda_{a\beta b}$$

Hence we get Λ_M is a Γ-semiring. □

Remark 4.4. *Since addition is commutative in M the left and right translations are same and hence*

$$\Pi(M) = \{(\lambda_a, \lambda_a), a \in M\}$$

is the set of all bitranslations in M.

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