Sadik transform and some result in fractional calculus

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Abstract
In this research paper, we present some new properties of Sadik transform which are related to the fractional calculus including Reimann-Liouville fractional operator, then we prove new results of Sadik transform like the infinite series, the convolution theorem and the Mittag-Leffler function. Moreover, it is shown that the Sadik transform method is an efficient technique for obtaining an exact analytic solution of some linear fractional differential equations. Some numerical examples to justify our results are illustrated.

Keywords
Integral transforms, fractional derivative and fractional integral, Sadik transform.

AMS Subject Classification
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1. Introduction

Integral transforms are excessively applied to solve various different type of differential equations. In the literature, there are many integral transforms and all are appropriate to solve various type differential equations. Recently some new integral transforms were introduced, see [1, 6, 10, 11] and applied to solve some ordinary differential equations as well as partial differential equations. Very recently, in the paper series [16–19], S. L. Shaikh introduced a new integral transform so-called Sadik transform and proved the duality theorem, and the convolution theorem of Sadik transform. Moreover, the author have proved that some transforms are particular cases of Sadik transform, exclusively, such as Laplace, Sumudu, Elzaki, Kamal, Tarig and Laplace-Carson transform. Fractional calculus is generalization of calassical differentiation and integration into non-integer order. The fractional derivatives describe the property of memory and heredity of many materials. Fractional differential equations have acquired significance during the past decades due to its applicability in several fields of mathematical applied such as, physics, chemistry biology and engineering and others applications see [2–4, 6, 9, 12–14]. At the outsight, Integral transform method is useful and effective tool for solving fractional differential equations. But it is also true that all types of fractional differential equations are not solvable by integral transform technique see [7, 8, 15] and the references therein.

Motivated by above works, in this paper we introduce new definitions of Sadik's transform of fractional order related to Riemann-Liouville integral and derivative operators with proving of their properties. Further, we give a sufficient condition to guarantee the rationality of solving fractional differential equations by the Sadik transform method.

2. Preliminaries

In this section, we offer some notions, definitions and lem-

mas that used through this paper. let $I = [a,b]$ be a compact interval on $\mathbb{R}$, where $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{C}[a,b]$ be the space of all continuous functions defined on $[a,b]$ with the norm $\|f\| = \max\{|f(t)| : t \in [a,b]\}$ for any $f \in \mathbb{C}[a,b]$. $\mathbb{C}^n[a,b]$ denote the space of all n-times continuously differentiable functions and $L^1[a,b]$ be the labesgue integrable functions with the norm $\|f\|_{L^1} = \int_a^b |f(t)| \, dt < \infty$.

Definition 2.1. *[14] Let $q > 0$ and f be a locally integrable function on $(a, +\infty)$. The left sided Riemann-Liouville integral of order q of the function f is given by*

$$I_{a^+}^q f(t) = \frac{1}{\Gamma(q)} \int_a^t (t - \tau)^{q-1} f(\tau) d\tau$$

here $\Gamma(\cdot)$ denotes the Gamma function of Euler as fallows

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \ z \in \mathbb{C}.$$

We can write the Riemann-Liouville fractional integral by the convolution theorem as follows:

$$I_{a^+}^q f(t) = \phi_q(t) * f(t) = \int_a^t \phi(t - \tau)^{q-1} f(\tau) d\tau$$

where $\phi_q(t) = \frac{t^{q-1}}{\Gamma(q)}$.

Definition 2.2. *[14] Let $n - 1 < q < n$, $n \in \mathbb{N}$, and $f \in \mathbb{C}[a,b]$. Then the left sided Riemann-Liouville fractional derivative of order q of a function f is defined by*

$$\begin{aligned} D_{a^+}^q f(t) &= \left(\frac{d}{dt}\right)^n I_{a^+}^{n-q} f(t) \\ &= \left(\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-q)} \int_a^t (t-\tau)^{n-q-1} f(\tau) d\tau, \end{aligned}$$

where $n = [q] + 1$ and $[q]$ denotes the integer part of the real number q.

Definition 2.3. *[16] (Sadik transform) Assume that*

1) f is piecewise continuous on the interval $[0,A]$ for any $A > 0$.
2) $|f(t)| \leq Ke^{at}$ when $t \geq M$, for any real constant a, and some positive constant K and M. Then Sadik transform of $f(t)$ is defined by

$$F(v, \alpha, \beta) = \mathscr{S}[f(t)] = \frac{1}{v^\beta} \int_0^\infty e^{-tv^\alpha} f(t) dt,$$

where v is complex variable, α is any non zero real number, and β is any real number.

Definition 2.4. *[14](Mittag-Leffler function) Let $\mu, \nu \subset \mathbb{C}$, $Re(\mu) > 0$, $Re(\nu) > 0$, then*

$$E_{\mu,\nu}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\mu k + \nu)}.$$

Lemma 2.5. *[9] Let $q > 0$, $t > a$ and $m > -1$. Then the Riemann-Liouville fractional integral and derivative of power function are given by*

(i) $I_{a^+}^q (t-a)^m = \frac{\Gamma(m+1)}{\Gamma(m+q+1)} (t-a)^{m+q}$,

(ii) $D_{a^+}^q (t-a)^m = \frac{\Gamma(m+1)}{\Gamma(m-q+1)} (t-a)^{m-q}$.

Lemma 2.6. *[16] Sadik transform of derivative (n^{th}) for $f(t)$ is*

$$\mathscr{S}[f^{(n)}(t)] = v^{n\alpha} F(v, \alpha, \beta) - \sum_{k=0}^{n-1} v^{k\alpha - \beta} f^{(n-1-k)}(0).$$

Lemma 2.7. *[9] Let $p, q \geq 0$. Then $I_{a^+}^p I_{a^+}^q = I_{a^+}^{p+q}$.*

Lemma 2.8. *[16] If $f(t) = t^n$, then Sadik transform of f is*

$$\mathscr{S}[t^n] = \frac{n!}{v^{n\alpha + (\alpha+\beta)}}.$$

Lemma 2.9. *[16] If $f(t) = e^{at}$, then Sadik transform of f is*

$$\mathscr{S}[e^{at}] = \frac{v^{-\beta}}{v^\alpha - a}.$$

Lemma 2.10. *Let f and g two functions belong to $L^1(R^+)$. Then the usuall convolution product is given by*

$$(f * g)(t) = \int_{-\infty}^\infty f(\tau) g(t - \tau) d\tau, t > 0.$$

3. Main Results

In this section, we prove the Sadik Transform of infinite series, convolution theorem, Mitage-Leffller function, and some properties of fractional calculus.

Lemma 3.1. *Let $q > 0$ and $f(t) = e^{\lambda t}, a > 0$. Then*

$$D_{a^+}^q e^{\lambda t} = t^{-q} E_{1,q-1}(\lambda t).$$

Proof.

$$\begin{aligned} D_{a^+}^q e^{\lambda t} &= \sum_{k=0}^\infty D_{a^+}^q \frac{(\lambda t)^k}{\Gamma(k+1)} = \sum_{k=0}^\infty \frac{\lambda^k}{\Gamma(k+1)} D_{a^+}^q t^k \\ &= \sum_{k=0}^\infty \frac{\lambda^k}{\Gamma(k+1)} \frac{\Gamma(k+1)}{\Gamma(k-q+1)} t^{k-q} \\ &= t^{-q} \sum_{k=0}^\infty \frac{(\lambda t)^k}{\Gamma(k-(q-1))} = t^{-q} E_{1,q-1}(\lambda t). \end{aligned}$$

$\square$

Theorem 3.2. *Let*

$$g(t) = \sum_{n=0}^{\infty} c_n t^n$$

is a converges for $t \geq 0$, with $|c_n| \leq \frac{M\gamma^n}{n!}$ for all n sufficiently large and $\gamma > 0$, $M > 0$. Then

$$\mathscr{S}[g(t)] = \sum_{n=0}^{\infty} c_n \mathscr{S}[t^n] = \sum_{n=0}^{\infty} \frac{c_n n!}{v^{n\alpha+(\alpha+\beta)}}.$$

Proof. Since g is cotinuous on $[0, \infty)$ because it is represented by convergent power series.

Now our aim to show that the expression

$$\left| \mathscr{S}[g(t)] - \sum_{n=0}^{N} c_n \mathscr{S}[t^n] \right| = \left| \mathscr{S}\left[g(t) - \sum_{n=0}^{N} c_n t^n \right] \right|$$

$$\leq \mathscr{S}_x \left| g(t) - \sum_{n=0}^{N} c_n t^n \right|,$$

converges to zero as $N \longrightarrow \infty$, where $\mathscr{S}_x[h(t)] = \frac{1}{v^\beta} \int_0^\infty e^{-tv^\alpha} h(t) dt$ where x is the $Re(v)$.

Now

$$\left| [g(t) - \sum_{n=0}^{N} c_n t^n] \right| = \left| \sum_{n=N+1}^{\infty} c_n t^n \right|$$

$$\leq k \sum_{n=N+1}^{\infty} \frac{(\gamma t)^n}{n!}$$

$$= k \left(e^{\gamma t} - \sum_{n=0}^{N} \frac{(\gamma t)^n}{n!} \right),$$

where $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$.

When the transform exist, we have

$$\mathscr{S}_x[g_1] \leq \mathscr{S}_x[g_2] \text{ if } g_1 \leq g_2.$$

Thus

$$\mathscr{S}_x \left(\left| [g(t) - \sum_{n=0}^{N} c_n t^n] \right| \right) \leq k \mathscr{S}_x \left(e^{\gamma t} - \sum_{n=0}^{N} \frac{(\gamma t)^n}{n!} \right)$$

$$= k \left(\frac{1}{v - \gamma} - \sum_{n=0}^{N} \frac{\gamma^n}{v^{n+1}} \right)$$

$$= k \left(\frac{1}{v - \gamma} - \frac{1}{v} \sum_{n=0}^{N} \left(\frac{\gamma}{v} \right)^n \right).$$

Take limit to both side, we get

$$\lim_{N \longrightarrow \infty} \mathscr{S}_x \left(\left| [g(t) - \sum_{n=0}^{N} c_n t^n] \right| \right)$$

$$\leq \lim_{N \longrightarrow \infty} k \left[\frac{1}{v - \gamma} - \frac{1}{v} \sum_{n=0}^{N} \left(\frac{\gamma}{v} \right)^n \right]$$

$$= \lim_{N \longrightarrow \infty} k \left[\frac{1}{v} \sum_{n=0}^{\infty} \left(\frac{\gamma}{v} \right)^n - \frac{1}{v} \sum_{n=0}^{N} \left(\frac{\gamma}{v} \right)^n \right], \left| \frac{\gamma}{v} \right| < 1$$

$$\longrightarrow 0 \text{ as } N \longrightarrow \infty.$$

Hence,

$$\mathscr{S}[g(t)] = \lim_{N \longrightarrow \infty} \sum_{n=0}^{N} c_n \mathscr{S}[t^n] = \sum_{n=0}^{\infty} c_n \mathscr{S}[t^n]$$

$$= \sum_{n=0}^{\infty} \frac{c_n n!}{v^{n\alpha+(\alpha+\beta)}}.$$

$\square$

Theorem 3.3. *If $x(t)$, $y(t)$ are infinite series, and $X(v, \alpha, \beta)$, $Y(v, \alpha, \beta)$ are Sadik Transform of $x(t)$, $y(t)$ respectively. Then*

$$\mathscr{S}[x(t) * y(t)] = v^\beta X(v, \alpha, \beta).Y(v, \alpha, \beta),$$

where $*$ denotes convolution.

Proof. we have

$$x(t) = \sum_{n=0}^{\infty} a_n t^n \text{ and } y(t) = \sum_{m=0}^{\infty} b_m t^m, \quad (3.1)$$

which are infinite convergent series for $t \geq 0$, so they are Sadik Transformable. Now by definition of convlution, we have

$$x(t) * y(t) = \int_0^t x(t - \tau) y(\tau) d\tau. \quad (3.2)$$

From Eq.(3.1),

$$x(t) * y(t) = \int_0^t \left(\sum_{n=0}^{\infty} a_n (t - \tau)^n \sum_{m=0}^{\infty} b_m \tau^m \right) d\tau$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m \int_0^t (t - \tau)^n \tau^m d\tau.$$

Expanding $(t - \tau)^n$ by the binomial theorem, we get

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{n} a_n b_m \binom{n}{k} (-1)^k \int_0^t \tau^m \tau^k t^{(n-k)} d\tau$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{n} a_n b_m \binom{n}{k} (-1)^k t^{(n-k)} \int_0^t \tau^{(m+k)} d\tau$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{n} a_n b_m \binom{n}{k} (-1)^k \frac{t^{(m+n+1)}}{(m+k+1)}.$$

The beta function is connected with gamma function if m and n are positive integral by the relation:

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1}{(m+k+1)} = \frac{m! n!}{(m+n+1)!}. \quad (3.3)$$

Therefore, Eq.(3.2) become

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m \frac{m! n!}{(m+n+1)!} t^{(m+n+1)}. \quad (3.4)$$

By apply Sadik Transform for Eq.(3.4), using theorem(3.2) and lemma (2.8), we get

$$
\begin{aligned}
&\mathscr{S}[x(t) * y(t)] \\
&= \mathscr{S}\left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m \frac{m!n!}{(m+n+1)!} t^{(m+n+1)}\right] \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m \frac{m!n!}{(m+n+1)!} \mathscr{S}[t^{(m+n+1)}] \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m \frac{m!n!}{(m+n+1)!} \frac{(m+n+1)!}{v^{(m+n+1)\alpha+(\alpha+\beta)}} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m \frac{m!n!}{v^{(m+n+1)\alpha+(\alpha+\beta)}} \\
&= v^{\beta} X(v,\alpha,\beta).Y(v,\alpha,\beta).
\end{aligned}
$$

$\square$

Theorem 3.4. *Let* $0 < q$ *and* $f \in C[a,b]$. *The Sadik Transform of Riemann-Liouville integral of a function* f *of order* q *is given by*

$$
\mathscr{S}[I_{a^+}^q f(t)] = v^{-\alpha q} F(v,\alpha,\beta), Re(\alpha) > 0, Re(\beta) > 0, v \in \mathbb{C}.
$$

Proof. From definition of Riemann-Liouville fractional integral and by using theorem3.3, we have

$$
\begin{aligned}
\mathscr{S}[I_{a^+}^q f(t)] &= \mathscr{S}\left[\int_{a^+}^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds\right] \\
&= \mathscr{S}\left[\frac{(\tau)^{q-1}}{\Gamma(q)} * f(\tau)\right] \\
&= v^{\beta} \Phi(v,\alpha,\beta).F(v,\alpha,\beta), \quad (3.5)
\end{aligned}
$$

where

$$
\Phi(v,\alpha,\beta) = \mathscr{S}\left[\frac{t^{q-1}}{\Gamma(q)}\right] \text{ and } F(v,\alpha,\beta) = \mathscr{S}[f(t)].
$$

Now we use lemma 2.8, to get

$$
\begin{aligned}
\Phi(v,\alpha,\beta) &= \mathscr{S}\left[\frac{t^{q-1}}{\Gamma(q)}\right] \\
&= \frac{1}{\Gamma(q)} \frac{(q-1)!}{v^{(q-1)\alpha+(\alpha+\beta)}} \\
&= \frac{1}{v^{q\alpha-\alpha+\alpha+\beta}} = v^{-(\alpha q+\beta)}. \quad (3.6)
\end{aligned}
$$

By invoking Eq.(3.6) in Eq.(3.5), we conclude that

$$
\mathscr{S}[I_{a^+}^q f(t)] = v^{-\alpha q} F(v,\alpha,\beta) .
$$

$\square$

Theorem 3.5. *Let* $n-1 < q < n$ *and* $f \in C[a,b]$. *Then the Sadik Transform of left sided Riemann-Liouville derivative of a function* f *of order* q *is given by*

$$
\mathscr{S}[D_{a^+}^q f(t)] = v^{\alpha q} F(v,\alpha,\beta) - \sum_{k=0}^{n-1} v^{k\alpha-\beta} D_{a^+}^{q-k-1} f(0).
$$

Proof. By definition of Riemann-Liouville fractional derivative, we have

$$
\begin{aligned}
\mathscr{S}[D_{a^+}^q f(t)] &= \mathscr{S}\left[\frac{d^n}{dt^n} I_{a^+}^{n-q} f(t)\right] \\
&= \mathscr{S}\left[\frac{d^n}{dt^n} g(t)\right], \quad (3.7)
\end{aligned}
$$

where

$$
g(t) = I_{a^+}^{n-q} f(t), \quad (3.8)
$$

which implies that

$$
\mathscr{S}[D_{a^+}^q f(t)] = \mathscr{S}\left[\frac{d^n}{dt^n} g(t)\right] = \mathscr{S}[g^n(t)].
$$

By lemma 2.6, we obtain

$$
\mathscr{S}[g^{(n)}(t)] = v^{n\alpha} G(v,\alpha,\beta) - \sum_{k=0}^{n-1} v^{k\alpha-\beta} g^{(n-1-k)}(0).
$$

From Theorem 3.4, with the relation $G(v,\alpha,\beta) = \mathscr{S}[g(t)] = \mathscr{S}[I_{a^+}^{n-q} f(t)]$, we get

$$
\mathscr{S}[g^{(n)}(t)] = v^{n\alpha} v^{-\alpha(n-q)} F(v,\alpha,\beta) - \sum_{k=0}^{n-1} v^{k\alpha-\beta} g^{(n-1-k)}(0).
$$
$$(3.9)$$

The hypothesis $Eq.(3.8)$, lead us to

$$
\begin{aligned}
g^{(n-1-k)}(t) &= \frac{d^{n-1-k}}{dt^{n-1-k}} I_{a^+}^{n-q} f(t) \\
&= I_{a^+}^{k-q+1} f(t) \\
&= D_{a^+}^{q-k-1} f(t). \quad (3.10)
\end{aligned}
$$

Substituation Eq.(3.10) into Eq.(3.9), and using Eq.(3.7), we conclude that

$$
\mathscr{S}[D_{a^+}^q f(t)] = v^{\alpha q} F(v,\alpha,\beta) - \sum_{k=0}^{n-1} v^{k\alpha-\beta} D_{a^+}^{q-k-1} f(0).
$$

$\square$

Remark 3.6. *In particular, if* $0 < q < 1$, *then*

$$
\mathscr{S}[D_{a^+}^q f(t)] = v^{\alpha q} F(v,\alpha,\beta) - v^{-\beta} D_{a^+}^{q-1} f(0).
$$

Lemma 3.7. *Assume that linear fractional differential equation*

$$
D_{0^+}^q u(t) = f(t), \quad (3.11)
$$

$$
D_{0^+}^{q-1} u(t)\Big|_{t=0} = u_0, \quad (3.12)
$$

has aunique continuous solution

$$
u(t) = \frac{u_0}{\Gamma(q)} t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau)d\tau, \quad (3.13)
$$

if $f(t)$ *is continuous on* $[0,\infty)$ *and exponentially bounded, then* $u(t)$ *and* $D_{0^+}^q u(t)$ *are both exponentially bounded, thus their sadik Transforms exist.*

Proof. Since $f(t)$ is exponentially bounded, there exist two positive constants M, σ and enough large T such that $\|f(t)\| \leq Me^{\sigma t}$ for all $t \geq T$. It is easy to see that Eq.(3.11) is equivalent to the Volterra integral equation

$$u(t) = \frac{u_0}{\Gamma(q)}t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}f(\tau)d\tau, \ 0 \leq t < \infty.$$

(3.14)

For $t \geq T$, Eq.(3.14) can be rewritten as

$$\begin{aligned} u(t) &= \frac{u_0}{\Gamma(q)}t^{q-1} + \frac{1}{\Gamma(q)}\int_0^T (t-\tau)^{q-1}f(\tau)d\tau \\ &\quad + \frac{1}{\Gamma(q)}\int_T^t (t-\tau)^{q-1}f(\tau)d\tau. \end{aligned}$$

In view of assumptions, $u(t)$ is unique continuous solution on $[0,\infty)$, with $D_{0^+}^{q-1}u(t)\big|_{t=0} = u_0$, then $f(t)$ is bounded on $[0,T]$, i.e. there exists a constant $k > 0$ such that $\|f(t)\| \leq k$. Now, we have

$$\begin{aligned} \|u(t)\| &\leq \frac{\|u_0\|}{\Gamma(q)}T^{q-1} + \frac{k}{\Gamma(q)}\int_0^T (t-\tau)^{q-1}d\tau \\ &\quad + \frac{1}{\Gamma(q)}\int_T^t (t-\tau)^{q-1}\|f(\tau)\|d\tau. \end{aligned}$$

Multiply the last inequality by $e^{-\sigma t}$ then from fact that $e^{-\sigma t} \leq e^{-\sigma T}$, $e^{-\sigma t} \leq e^{-\sigma \tau}$, and $\|f(t)\| \leq Me^{\sigma t}$ $(t \geq T)$, we obtain

$$\begin{aligned} \|u(t)\|e^{-\sigma t} &\leq \frac{\|u_0\|}{\Gamma(q)}T^{q-1}e^{-\sigma t} + \frac{ke^{-\sigma t}}{\Gamma(q)}\int_0^T (t-\tau)^{q-1}d\tau \\ &\quad + \frac{e^{-\sigma t}}{\Gamma(q)}\int_T^t (t-\tau)^{q-1}\|f(\tau)\|d\tau \\ &\leq \frac{\|u_0\|}{\Gamma(q)}T^{q-1}e^{-\sigma T} + \frac{ke^{-\sigma T}}{\Gamma(q+1)}[(t)^q - (t-T)^q] \\ &\quad + \frac{M}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}e^{\sigma(\tau-t)}d\tau \\ &\leq \frac{\|u_0\|}{\Gamma(q)}T^{q-1}e^{-\sigma T} + \frac{ke^{-\sigma T}}{\Gamma(q+1)}T^q \\ &\quad + \frac{M}{\Gamma(q)}\int_0^t s^{q-1}e^{-\sigma s}ds \\ &\leq \frac{\|u_0\|}{\Gamma(q)}T^{q-1}e^{-\sigma T} + \frac{ke^{-\sigma T}}{\Gamma(q+1)}T^q \\ &\quad + \frac{M}{\Gamma(q)}\int_0^\infty s^{q-1}e^{-\sigma s}ds \\ &\leq \frac{\|u_0\|}{\Gamma(q)}T^{q-1}e^{-\sigma T} + \frac{ke^{-\sigma T}}{\Gamma(q+1)}T^q + \frac{M}{\sigma^q}. \end{aligned}$$

Denote

$$A = \frac{\|u_0\|}{\Gamma(q)}T^{q-1}e^{-\sigma T} + \frac{ke^{-\sigma T}}{\Gamma(q+1)}T^q + \frac{M}{\sigma^q},$$

we get

$$\|u(t)\| \leq Ae^{\sigma t}, \ t \geq T.$$

□

From Eq.(3.11) and hypothesis of f, we conclude that

$$\left\|D_{0^+}^q u(t)\right\| = \|f(t)\| \leq Me^{\sigma t} \ t \geq T.$$

Applying Sadik transform on both sides of Eq.(3.11) and using Theorem 3.5, we have

$$v^{\alpha q}U(v,\alpha,\beta) - v^{-\beta}D_{0^+}^{q-1}u(0) = F(v,\alpha,\beta).$$

Since $D_{0^+}^{q-1}u(0) = u_0$, it follows

$$U(v,\alpha,\beta) = u_0\frac{1}{v^{\alpha q+\beta}} + \frac{F(v,\alpha,\beta)}{v^{\alpha q}}.$$

Take the inverse of Sadik transform to both sides of the above equation, and using Lemma 2.8, we get

$$\begin{aligned} u(t) &= u_0\mathscr{S}^{-1}\left[\frac{1}{v^{\alpha q+\beta}}\right] + \mathscr{S}^{-1}\left[\frac{1}{v^{\alpha q}}F(v,\alpha,\beta)\right] \\ &= \frac{t^{q-1}}{\Gamma(q)}u_0 + \mathscr{S}^{-1}\left[v^\beta\frac{1}{v^{\alpha q+\beta}}F(v,\alpha,\beta)\right] \\ &= \frac{t^{q-1}}{\Gamma(q)}u_0 + \mathscr{S}^{-1}\left[v^\beta\frac{1}{v^{(q-1)\alpha+\alpha+\beta}}F(v,\alpha,\beta)\right] \\ &= \frac{t^{q-1}}{\Gamma(q)}u_0 + (\varphi_1 * \varphi)(t). \end{aligned}$$

(3.15)

Put $F_1(v,\alpha,\beta) := \frac{1}{v^{(q-1)\alpha+\alpha+\beta}}$, such that $\mathscr{S}^{-1}[F_1(v,\alpha,\beta)] = \varphi_1(t)$ and $\mathscr{S}^{-1}[F(v,\alpha,\beta)] = \varphi(t)$. Applying the inverse Sadik transform of $F_1(v,\alpha,\beta)$, with using lemma (L3), we find that

$$\begin{aligned} \mathscr{S}^{-1}[F_1(v,\alpha,\beta)] &= \mathscr{S}^{-1}\left[\frac{1}{v^{(q-1)\alpha+\alpha+\beta}}\right] \\ &= \frac{t^{q-1}}{\Gamma(q)} = \varphi_1(t). \end{aligned}$$

Therefore Eq.(3.15) becomes as follows

$$\begin{aligned} u(t) &= \frac{t^{q-1}}{\Gamma(q)}u_0 + (\varphi_1 * \varphi)(t) \\ &= \frac{t^{q-1}}{\Gamma(q)}u_0 + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}f(\tau)d\tau. \end{aligned}$$

Theorem 3.8. *Let $f(t) = t^{pm+q-1}E_{p,q}^{(m)}(\pm at^p)$. The Sadik Transform of f is given by:*

$$\frac{1}{v^\beta}\int_0^\infty e^{-v^\alpha t}t^{pm+q-1}E_{p,q}^{(m)}(\pm at^p)dt = \frac{m!v^{\alpha p-(\alpha q+\beta)}}{(v^{\alpha p}\mp a)^{m+1}},$$

where $\alpha, \beta \in \mathbb{C}, \mathscr{R}(p) > 0, \mathscr{R}(q) > 0, \mathscr{R}(v) > |a|^{\frac{1}{\mathscr{R}(\alpha p)}}$ and $E_{p,q}^{(m)}(z) = \frac{d^m}{dt^m}E_{p,q}^{(m)}(z)$.

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Proof. In view of Definition of Mittag-leffler function and by using classical calculus, we have

$$\frac{1}{v^\beta}\int_0^\infty e^{-v^\alpha t}t^{pm+q-1}E_{p,q}^{(m)}(\pm at^p)dt$$

$$= \frac{1}{v^\beta}\int_0^\infty e^{-v^\alpha t}t^{pm+q-1}\frac{d^m}{dt^m}\sum_{k=0}^\infty \frac{(\pm at^p)^k}{\Gamma(pk+q)}dt$$

$$= \frac{1}{v^\beta}\int_0^\infty e^{-v^\alpha t}t^{pm+q-1}\sum_{k=0}^\infty \frac{(k+m)!(\pm a)^k t^{pk}}{k!\Gamma(pk+pm+q)}dt$$

$$= \sum_{k=0}^\infty \frac{(k+m)!(\pm a)^k}{k!\Gamma(pk+pm+q)}\frac{1}{v^\beta}\int_0^\infty e^{-v^\alpha t}t^{pm+pk+q-1}dt$$

$$= \sum_{k=0}^\infty \frac{(k+m)!}{k!}\frac{(\pm a)^k}{v^\beta}\frac{(pm+pk+q-1)!}{\Gamma(pk+pm+q)}\frac{1}{v^{\alpha(pm+pk+q)}}$$

$$= \frac{v^{-\alpha(pm+q)}}{v^\beta}\sum_{k=0}^\infty \frac{(k+m)!}{k!}\left(\frac{\pm a}{v^{\alpha p}}\right)^k$$

$$= \frac{v^{-\alpha(pm+q)}}{v^\beta}\sum_{k=0}^\infty (k+m).......(k+1)\left(\frac{\pm a}{v^{\alpha p}}\right)^k,$$

Now let $k = k - m$

$$\frac{1}{v^\beta}\int_0^\infty e^{-v^\alpha t}t^{pm+q-1}E_{p,q}^{(m)}(\pm at^p)dt$$

$$= \frac{v^{-\alpha(pm+q)}}{v^\beta}\sum_{k=m}^\infty (k)(k-1).......(k-m-1)\left(\frac{\pm a}{v^{\alpha p}}\right)^k$$

$$= v^{-\alpha pm-\alpha q-\beta}\frac{d^m}{da^m}\sum_{k=m}^\infty \left(\frac{\pm a}{v^{\alpha p}}\right)^k$$

$$= v^{-\alpha pm-\alpha q-\beta}\frac{d^m}{da^m}\left(\frac{1}{1\mp\frac{a}{v^{\alpha p}}}\right)$$

$$= v^{-\alpha pm-\alpha q-\beta}\frac{m!}{(1\mp\frac{a}{v^{\alpha p}})^{m+1}}$$

$$= \frac{m!v^{\alpha p-(\alpha q+\beta)}}{(v^{\alpha p}\mp a)^{m+1}}. \tag{3.16}$$

Corollary 3.9. *Let* $f(t) = t^{pm+q-1}E_{p,q}^{(m)}(\pm at^p)$. *The Laplace Transform of f is given by:*

$$\int_0^\infty e^{-st}t^{pm+q-1}E_{p,q}^{(m)}(\pm at^p)dt = \frac{m!s^{p-q}}{(s^p\mp a)^{m+1}}, \tag{3.17}$$

where $\mathscr{R}(p) > 0$, $\mathscr{R}(q) > 0, \mathscr{R}(s) > |a|^{\frac{1}{\mathscr{R}(p)}}$ *and* $E_{p,q}^{(m)}(z) = \frac{d^m}{dt^m}E_{p,q}^{(m)}(z)$.

Remark 3.10. *When* $\alpha = 1$ *and* $\beta = 0$, *then Eq.(3.16) implies to Eq.(3.17).*

$\square$

4. Example

In this section, we provide some examples to justify our results.

Example 4.1. *Consider the function* $f(t) = e^{at}$, *then Sadik transform of Riemann-Liouville fractional integral of order q of f is given by*

$$\mathscr{S}[I_{0^+}^q e^{at}] = \frac{v^{-(\alpha q+\beta)}}{v^\alpha - a}.$$

Indeed, according to Theorem 3.4, we have

$$\mathscr{S}[I_{0^+}^q f(t)] = v^{-\alpha q}F(v,\alpha,\beta).$$

and by Lemma 2.9, we see that

$$F(v,\alpha,\beta) = \mathscr{S}[e^{at}] = \frac{v^{-\beta}}{v^\alpha - a},$$

Consequently,

$$\mathscr{S}[I_{0^+}^q e^{at}] = v^{-\alpha q}\frac{v^{-\beta}}{v^\alpha - a} = \frac{v^{-(v\alpha q+\beta)}}{v^\alpha - a}. \tag{4.1}$$

On the other hand, by using the series of exponession function and Lemma 2.5 part (i), we obtain

$$I_{0^+}^q e^{at} = I_{0^+}^q \sum_{k=0}^\infty \frac{(at)^k}{(k)!} = \sum_{k=0}^\infty \frac{(a)^k}{(k)!}I_{0^+}^q t^k$$

$$= \sum_{k=0}^\infty \frac{(a)^k}{(k)!}\frac{\Gamma(k+1)}{\Gamma(k+q+1)}t^{k+q}$$

$$= \sum_{k=0}^\infty \frac{(a)^k}{\Gamma(k+q+1)}t^{k+q} \tag{4.2}$$

Now, we apply Sadik Transform in Eq.(4.2), using Theroem 3.2 and lemma 2.8 we get

$$\mathscr{S}[I_{0^+}^q e^{at}] = \sum_{k=0}^\infty \frac{(a)^k}{\Gamma(k+q+1)}\mathscr{S}[t^{k+q}]$$

$$= \sum_{k=0}^\infty \frac{(a)^k}{\Gamma(k+q+1)}\frac{(k+q)!}{v^{(k+q)\alpha+(\alpha+\beta)}}$$

$$= \sum_{k=0}^\infty \left(\frac{a}{v^\alpha}\right)^k v^{-(\alpha q+\beta)}v^{-\alpha}$$

$$= \left[1+\left(\frac{a}{v^\alpha}\right)+\left(\frac{a}{v^\alpha}\right)^2+...\right][v^{-(\alpha q+\beta)}v^{-\alpha}]$$

$$= \frac{v^{-(\alpha q+\beta)}}{v^\alpha - a}. \tag{4.3}$$

Note that, Eq.(4.1) and Eq.(4.3) are equal. So Theorem 3.4 holds. In particular, if $q = \frac{1}{2}$, *then*

$$\mathscr{S}[I_{0^+}^{\frac{1}{2}} e^{at}] = \frac{v^{-(\frac{\alpha}{2}+\beta)}}{v^\alpha - a}.$$

Example 4.2. *Let* $0 < q < 1$, *and* $f(t) = e^{at}$. *Then Sadik Transform of Riemann-Liouville fractional derivative of order q of f is given by*

$$\mathscr{S}[D_{0^+}^q e^{at}] = \frac{v^{\alpha q-\beta}}{v^\alpha - a}.$$

In fact, by Remark 3.6, we have

$$\mathscr{S}[D_{0^+}^q e^{at}] = v^{\alpha q}\mathscr{S}[e^{at}] - v^{-\beta}D_{0^+}^q e^{at}\Big|_{t=0}.$$

From Lemma 2.9 and Lemma 3.1, we get

$$
\begin{aligned}
\mathscr{S}[D_{0^+}^q e^{at}] &= v^{\alpha q}\frac{v^{-\beta}}{v^\alpha - a} - v^{-\beta}\, t^{-q}E_{1,q-1}(at)\Big|_{t=0} \\
&= \frac{v^{\alpha q - \beta}}{v^\alpha - a}. \qquad (4.4)
\end{aligned}
$$

Then again by serise of function e^{at} and lemma 2.5, we get

$$
\begin{aligned}
D_{0^+}^q e^{at} &= D_{0^+}^q \sum_{k=0}^\infty \frac{(at)^k}{(k)!} = \sum_{k=0}^\infty \frac{(a)^k}{(k)!}D_{0^+}^q t^k \\
&= \sum_{k=0}^\infty \frac{(a)^k}{(k)!}\frac{\Gamma(k+1)}{\Gamma(k-q+1)}t^{k-q} \\
&= \sum_{k=0}^\infty \frac{(a)^k}{\Gamma(k-q+1)}t^{k-q}.
\end{aligned}
$$

An application of Sadik Transform with Theorem 3.2 and lemma 2.8 gives

$$
\begin{aligned}
\mathscr{S}[D_{0^+}^q e^{at}] &= \sum_{k=0}^\infty \frac{(a)^k}{\Gamma(k-q+1)}\mathscr{S}[t^{k-q}] \\
&= \sum_{k=0}^\infty \frac{(a)^k}{\Gamma(k-q+1)}\frac{(k-q)!}{v^{(k-q)\alpha+(\alpha+\beta)}} \\
&= \sum_{k=0}^\infty \left(\frac{a}{v^\alpha}\right)^k v^{(\alpha q-\beta)}v^{-\alpha} \\
&= \left[1+\left(\frac{a}{v^\alpha}\right)+\left(\frac{a}{v^\alpha}\right)^2+...\right]\left[v^{(\alpha q-\beta)}v^{-\alpha}\right] \\
&= \left[\frac{1}{1-\frac{a}{v^\alpha}}\right]\left[v^{(\alpha q-\beta)}v^{-\alpha}\right] \\
&= \frac{v^{\alpha q-\beta}}{v^\alpha - a}. \qquad (4.5)
\end{aligned}
$$

Hence, the Theorem 3.5 is satisfied.

4.1 An application
In this part, we give a model described by a fractional differential equation through Sadik transform.

Example 4.3. *Consider the fractional differential equation*

$$D_{0^+}^q N(t) = \lambda N(t), \qquad (4.6)$$

with the initial condition

$$D_{0^+}^{q-1}N(0^+) = N_0, \qquad (4.7)$$

where $0 < q < 1$ and $N(t)$ is the number of individuals of a population at the time t, λ is the population growth rate, and N_0 denotes the initial population size. Applying the Sadik Transform on both side of Eq.(4.6), using Remark 3.6 and the initial condition Eq.(4.7), we get

$$N(v,\alpha,\beta) = \frac{N_0 v^{-\beta}}{v^{\alpha q} - \lambda}. \qquad (4.8)$$

Applying the inverse of Sadik transform on both side of Eq.(4.8) with Theorem 3.8. The solution of this fractional differential equation, together with the initial condition $D_{0^+}^{q-1}N(0^+) = N_0$ is given by

$$
\begin{aligned}
N(t) &= N_0\mathscr{S}^{-1}\left[\frac{v^{\alpha q-(\beta+\alpha q)}}{v^{\alpha q}-\lambda}\right] \\
&= N_0 t^{\beta+\alpha q-1}E_{\alpha q,\beta+\alpha q}(\lambda t^{\alpha q}).
\end{aligned}
$$

Note that, if $\alpha = 1$ and $\beta = 0$, then Sadik transform reduces to Laplace transform. Hence the solution of (4.6)-(4.6) is given by

$$N(t) = N_0 t^{q-1}E_{q,q}(\lambda t^q). \qquad (4.9)$$

This case was considered in the literature, and it was proved that the fractional differential equation was more efficient in modeling the population growth than the ordinary differential equation.

Example 4.4. *Consider the linear differential equation of fractional order*

$$D_{0^+}^q y(t) = \lambda I_{0^+}^\gamma y(t) + f(t) \qquad (4.10)$$

with the initial condition

$$D_{0^+}^{q-1}y(t)\Big|_{t=0} = K, \qquad (4.11)$$

where $0 < q < 1$, $\gamma > 0$ and $\lambda \in \mathbb{R}$. Applying the Sadik Transform on both side of Eq.(4.10), using Remark 3.6, Theorem 3.4, and the initial condition Eq.(4.11), we get

$$Y(v,\alpha,\beta) = \frac{K v^{\alpha\gamma-\beta}}{(v^{\alpha(q+\gamma)}-\lambda)} + \frac{v^{\alpha\gamma}F(v,\alpha,\beta)}{(v^{\alpha(q+\gamma)}-\lambda)}$$

Applying the inverse Sadik transform, we get

$$
\begin{aligned}
y(t) &= K\mathscr{S}^{-1}\left[\frac{v^{\alpha(q+\gamma)-(\alpha q+\beta)}}{(v^{\alpha(q+\gamma)}-\lambda)}\right] \\
&\quad + \mathscr{S}^{-1}\left[v^\beta\frac{v^{\alpha\gamma}}{(v^{\alpha(q+\gamma)+\beta}-\lambda)}F(v,\alpha,\beta)\right] \\
&= Kt^{\alpha(q+\gamma)+\beta-1}E_{\alpha(q+\gamma),\alpha q+\beta}(\lambda t^{\alpha(q+\gamma)}) \\
&\quad + \mathscr{S}^{-1}\left[v^\beta G(v,\alpha,\beta).F(v,\alpha,\beta)\right],
\end{aligned}
$$

where $G(v,\alpha,\beta) = \left[\frac{v^{\alpha\gamma}}{(v^{\alpha(q+\gamma)+\beta}-\lambda)}\right]$. Note that

$$
\begin{aligned}
g(t) &= \mathscr{S}^{-1}[G(v,\alpha,\beta)] \\
&= \mathscr{S}^{-1}\left[\frac{v^{\alpha\gamma}}{(v^{\alpha(q+\gamma)+\beta}-\lambda)}\right] \\
&= t^{\alpha(q+\gamma)+2\beta-1}E_{\alpha(q+\gamma)+\beta,(\alpha q+\beta)}(\lambda t^{\alpha(q+\gamma)+\beta}).
\end{aligned}
$$

Now, by the convolusion theorem of Sadik transform, we conclude that

$$
\begin{aligned}
y(t) &= Kt^{\alpha(q+\gamma)+\beta-1}E_{\alpha(q+\gamma),\alpha q+\beta}(\lambda t^{\alpha(q+\gamma)}) \\
&\quad + [(g*f)(t)] \\
&= Kt^{\alpha(q+\gamma)+\beta-1}E_{\alpha(q+\gamma),\alpha q+\beta}(\lambda t^{\alpha(q+\gamma)}) \\
&\quad + \int_0^\infty g(t-\tau)f(\tau)d\tau \\
&= Kt^{\alpha(q+\gamma)+\beta-1}E_{\alpha(q+\gamma),\alpha q+\beta}(\lambda t^{\alpha(q+\gamma)}) + \Psi(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi(t) &= \int_0^\infty (t-\tau)^{\alpha(q+\gamma)+2\beta-1} \\
&\quad \times E_{\alpha(q+\gamma)+\beta,(\alpha q+\beta)}(\lambda(t-\tau)^{\alpha(q+\gamma)+\beta})f(\tau)d\tau.
\end{aligned}
$$

5. Conclusion

There are a lot of the integral transforms of exponential type kernels, the Sadik Transform is new and a very powerful among of them. And there are many problems in engineering and applied sciences can be considered by Sadik transform as integral transform to solve it, so we have provided Sadik transform of the Riemann-Liouville fractional calculus, the convolution theorem, and the infinite series. In order to illustrate the efficiency of theoretical results, suitable examples with some applications and models described by a fractional differential equation through Sadik transform.

Acknowledgment

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