Malaya Journal of Matematik, Vol. 8, No. 2, 551-555, 2020

https://doi.org/10.26637/MJM0802/0039

An evaluation of mixed type polynomial approximation with certain condition on the roots of Hermite polynomial

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Abstract

The purpose of this paper is to find a polynomial $R_n(x)$ of degree$\leq (3n-1)$ satisfying (1,0;0) interpolation under certain condition at given knots, also explicit representation of fundamental polynomials and convergence theorem of $R_n(x)$ has been analyzed.

Keywords

Approximation on real line, Hermite polynomial, Explicit representation, Estimation.

AMS Subject Classification

41A10, 33C45, 33C52.

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Article History: Received **17** January **2020**; Accepted **28** April **2020**

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1. Introduction

In 1975 L.G. P*á*l[1] introduced the following interpolation process, he consider

$$-\infty < x_1 < x_2 < .... < x_k < ...x_n < +\infty \tag{1.1}$$

be the set of knots which generates the polynomial

$$W_n(x) = \prod_{k=1}^{n}(x-x_k) \tag{1.2}$$

and roots $\{y_k\}_{k=1}^{n-1}$ of $W_n{}'(x)$ are interscaled between the roots of $W_n(x)$ such that,

$$-\infty < x_1 < y_1 < x_2 < ..... < x_k < y_k .... < y_{n-1} < x_n < +\infty \tag{1.3}$$

He proved that, there exist a unique polynomial $P_n(x)$ of degree (2n-1) satisfying the following condition

$$\begin{cases} P_n(x_k) = \alpha_k & (k = 1,2....,n), \\ P_n'(y_k) = \beta_k & (k = 1,2.,..,n-1), \end{cases} \tag{1.4}$$

with initial condition $P_n(x_0) = 0$,where x_0 is a given point differ from the nodal point (1.3) and $\{\alpha_k\}_{k=1}^{n}$ and $\{\beta_k\}_{k=1}^{n-1}$ are arbitrary numbers, whose convergence for $P_n(x)$ has been proved by S. A. Eneduanya[2] on the roots of $\pi_n(x)$.

In 1985, L. Szili[3] was firstly apply this interpolation process by taking the mixed zeroes of $H_n(x)$ and its derivative on infinite interval and shows that ,there exist a unique polynomial $Q_n(x)$ of degree $\leq 2n-1$ which satisfying the condition

$$\begin{cases} Q_n^*(x_k) = \alpha_k^* & (k = 1,2....,n), \\ Q_n^{*\prime}(y_k) = \beta_k^* & (k = 1,2.,..,n-1), \end{cases} \tag{1.5}$$

$$Q_n^*(0) = -2\sum_{i=0}^{n}\alpha_k^*[\frac{H_n(o)}{H_n'(x_k)}]^2 \tag{1.6}$$

and cunstructed a polynomial which is given by

$$Q_n^*(x) = \sum_{i=1}^{n}\alpha_k^* A_k(x) + \sum_{i=1}^{n-1}\beta_k^* B_k(x) \tag{1.7}$$

which uniqueness does not hold for taking n odd. He also proved convergence theorem for $Q_n(x)$ and later, I.Joó[4] improved Szili[3] result by modifying estimate of fundamental polynomial.
In 1999 Z.F. Sebestyen[5] improved the result of L. Szili[3] and I.Zoo[4] by replacing condition with an interpolatory condition $Q_n(0) = \alpha_0$ for n even.

In another paper, Srivastava and Mathur[6] studied mixed (0;0,1) interpolation on the zeroes of $H_n(x)$ and its derivative which means to determine a polynomial $R_n^*(x)$ satisfies the following conditions

$$\begin{cases} R_n{}^*(x_k) = \alpha_k{}^{**} & (k = 0,1....n) \\ R_n{}^*(y_k) = \beta_k{}^{**} & (k = 1....n-1) \\ R_n{}^{*\prime}(y_k) = \beta_k{}^{**} & (k = 1....n-1) \end{cases} \tag{1.8}$$

for n even they proved there exist unique polynomial$R_n^*(x)$ of degree$\leq 3n-2$ satisfying the above condition, for n odd uniqueness does not exist.
Yamini Singh and R. Srivastava[7] also solve (0,1;0) interpolation problem with special type of boundary condition on the roots of Ultraspherical polynomial.
The aim of this paper to extend the study of (0;1) interpolation problem of Z.F Sebestyen[5] to the case (0,1;0) interpolation. We have given the following problem.

2. Problem

Let $y_0 = 0$ be a real number differing from the interscaled system of nodal points (1.3) where $\{x_k\}_{k=1}^n$ and $\{y_k\}_{k=1}^{n-1}$ are the zeroes of $H_n(x)$ and $H_n{}'(x)$ respectively. Therefore find a minimal degree polynomial $R_n(x)$ which satisfies the following interpolation condition

$$\begin{cases} R_n(y_k) = g_k & (k = 0,1....n-1), \\ R_n(x_k) = g_k^* & (k = 1,.2,...,n), \\ R_n{}'(x_k) = g_k^{**} & (k = 1,2,....,n), \end{cases} \tag{2.1}$$

3. Preliminaries

In this section, we gave some well known results, which we shall use in to prove theorem 4.1, lemma 5.1, lemma 5.2, lemma 5.3 and theorem 6.1.

The differential equation satisfied by $H_n(x)$ is given by

$$H_n''(x) - 2xH_n' + 2nH_n(x) = 0 \tag{3.1}$$

$$H_n'(x) = 2nH_{n-1}(x) \tag{3.2}$$

Let $l_k(x)$ and $L_k(x)$ denote the fundamental polynomial of lagrange interpolation corresponding to the nodal point x_k and y_k respectively then

$$l_k(x) = \frac{H_n(x)}{H_n'(x_k)(x-x_k)} \quad k = 1,.....,n \tag{3.3}$$

$$L_k(x) = \frac{H_n'(x)}{H_n''(y_k)(x-y_k)} \quad k = 1,....,n-1 \tag{3.4}$$

and they follows the condition given below

$$l_k(x_j) = \begin{cases} 0 & for\ \ j \neq k \\ 1 & for\ \ j = k \end{cases} \quad for\ \ k = 1,....,n \tag{3.5}$$

$$L_k(y_j) = \begin{cases} 0 & for\ \ j \neq k \\ 1 & for\ \ j = k \end{cases} \quad for\ \ k = 1,....,n-1 \tag{3.6}$$

G. Szegö[11] gave following results
If x_k(k=1, 2, ..., n) are the roots of $H_n(x)$, then

$$x_k^2 \sim \frac{k^2}{n} \tag{3.7}$$

$$H_n(x) = O(n^{\frac{1}{4}}\sqrt{2^n n!}(1+\sqrt[3]{|x|})e^{\frac{x_k^2}{2}}) \ \ x \in R \tag{3.8}$$

$$|H_n'(x_k)| = n^{\frac{1}{4}}\sqrt{2^{n+1}n!}e^{\frac{\delta x_k^2}{2}} \quad (i = 1,....,n) \tag{3.9}$$

$$|H_n(y_k)| = n^{\frac{-1}{4}}\sqrt{2^{n+1}n!}e^{\frac{\delta y_k^2}{2}} \quad (i = 1,....,n-1) \tag{3.10}$$

where $0 < \delta < 1$ is an arbitrarily given real number

$$|l_k(x)| = O(1)\frac{2^{n+1}n!\sqrt{n}e^{\frac{\nu(x^2+x_k^2)}{2}}}{H_n{}'^2(x_k)} \quad \nu > 1 \ \ and \ \ k = 1...n \tag{3.11}$$

$$|L_k(x)| = O(\frac{2^n n! e^{\frac{\nu(x^2+y_k^2)}{2}}}{\sqrt{n}H_n^2(y_k)}) \ \ \nu > 1 \ \ and \ \ k = 1...n-1 \tag{3.12}$$

L.Szili[8] gave following results

$$\sum_{i=0}^{n} e^{-\varepsilon x_k^2} = O(\sqrt{n}) \tag{3.13}$$

$$\sum_{i=0}^{n} \frac{e^{\delta x_k^2}}{H_n{}'^2(x_k)} = O(2^{n+1}n!)^{-1} \tag{3.14}$$

Definition 3.1. *$\omega(f,\delta)$ denotes the special form of modulus of continuity introduced by G.Freud[9] given by*

$$\omega(f,\delta) = \sup \|W(x+t)f(x+t) - W(x)\| + \|\tau(\delta x)W(x)f(x)\| \tag{3.15}$$

$for \quad 0 \le t \le \delta$
where

$$\tau(x) = \begin{cases} |x| & for \ \ |x| \le 1 \\ 1 & for \ \ |x| > 1 \end{cases}$$

and $\|.\|$ *denotes the sup-norm in* $C(R)$ *and* $\lim_{|x|\to\infty} W(x)f(x) = 0$ *then* $\lim_{|x|\to 0} \omega(f,g) = 0.$

G.Freud, (Theorem 1[9]and Theorem 4[10]) gave the following results:
Let $f : R \to R$ be continuously diffrentiable.Further, let

$$\begin{cases} \lim_{|x|\to+\infty} x^{2k} f(x)\rho(x) = 0 \quad (k = 0,1,....) \\ and \lim_{|x|\to+\infty} f(x)'\rho(x) = 0 \end{cases} \tag{3.16}$$

then there exist polynomial $Q_n(x)$ of degree $\le$ such that

$$\rho(x)|f(x) - Q_n(x)| = O(\frac{1}{\sqrt{n}})\omega(f;\frac{1}{\sqrt{n}}) \tag{3.17}$$

where ω stands for modulus of continuity defined by (6.1) and $\rho(x)$ the weight function.
Szili[3](Lemma 4, Theorem 4) established the following

$$\rho(x)|Q_n^{(r)}| = O(1) \tag{3.18}$$

4. Explicit Representation of interpolatory polynomial

In this section we have proved explicit representation of fundamental polynomials.

Theorem 4.1. *There exist a polynomial*

$$R_n(x) = \sum_{k=0}^{n-1} g_k A_k(x) + \sum_{k=1}^{n} g_k^* B_k(x) + \sum_{k=1}^{n} g_k^{**} C_k(x) \tag{4.1}$$

of degree $(3n-1)$ *satisfying condition (2.1), where* $A_k(x)$ $(k = 0,1,2....,n-1)$ *and* $B_k(x)$ $(k = 1,2.....,n)$ *are the fundamental polynomial of first kind and* $C_k(x)$ $(k = 1,2.....,n)$ *are fundamental polynomial of second kind of (1,0;0) interpolation. Each such fundamental polynomial of degree at most* $3n-1$ *is given by*

$$A_0(x) = \frac{H_n{}'(x)H_n{}^2(x)}{H_n{}'(0)H_n{}^2(0)} \tag{4.2}$$

$$A_k(x) = \frac{xH_n{}^2(x)L_k(x)}{y_k H_n{}^2(y_k)} \tag{4.3}$$

$$B_k(x) = \frac{xH_n'(x)l_k^2(x)\{1 - \frac{(1+4x_k^2)}{x_k}(x-x_k)\}}{x_k H_n{}'(x_k)} \tag{4.4}$$

$$C_k(x) = \frac{xH_n(x)H_n{}'(x)l_k(x)}{x_k(H_n'(x_k))^2} \tag{4.5}$$

Proof. It is enough to show that the polynomials $A_k(x)$ $(k = 0,1,2....,n-1)$, $B_k(x)$ $(k = 0,1,2....,n)$, and $C_k(x)$ $(k = 0,1,2....,n)$ have the following properties:

$$A_k(y_j) = \begin{cases} 0 & for \ \ j \ne k \\ 1 & for \ \ j = k \end{cases} \quad for \ \ (j,k = 0....n-1) \ A_k(x_j) = 0 \tag{4.6}$$

for $(j = 1...n, k = 0,...n-1)$

$$A_k'(x_j) = 0 \quad for \ \ (j = 1,...n, k = 0,...n-1)$$

$$B_k(x_j) = \begin{cases} 0 & for \ \ j \ne k \\ 1 & for \ \ j = k \end{cases} \quad for \ \ (j,k = 1,....n) \ B_k'(x_j) = 0 \tag{4.7}$$

for $\quad (j,k = 1,...n)$

$$B_k(y_j) = 0 \ \ for \ \ (j = 0,...n-1, k = 1,...n)$$

and

$$C_k'(x_j) = \begin{cases} 0 & for \ \ j \ne k \\ 1 & for \ \ j = k \end{cases}$$

for $\quad (j,k = 1...n)$

$$C_k(x_j) = 0 \ \ (j,k = 1,...n)$$

$$C_k(y_j) = 0, \quad (j = 0,...n-1, k = 1,...n) \tag{4.8}$$

First, we construct the polynomials $C_k(x)$, let k be fixed ($k \in${1,...n}), from (4.8) it follows that

$$C_k(x) = H_n(x)H_n{}'(x)p_k(x) \tag{4.9}$$

where $p_k(x)$ is the polynomial for which

$$p_k(0) = 0 \tag{4.10}$$

by (4.9) we get

$$C_k'(x) = (H_k'(x))^2 p_k(x) + H_n(x)(H_k'(x)p_k(x))' \tag{4.11}$$

according to (4.8), equations

$$C_k'(x_j) = (H_k'(x_j))^2 p_k(x_j) = \begin{cases} 0 & for \ \ j \ne k \\ 1 & for \ \ j = k \end{cases} \quad for \ \ j = (1,...,n) \tag{4.12}$$

must hold for polynomial $p_k(x)$.These equations will be satisfied if

$$p_k(x) = \frac{xl_k(x)}{x_k(H_n'(x_k))^2} \tag{4.13}$$

combining (4.13), (4.9), we obtain (4.5). Obviously, $C_k(x)$ is a polynomial of degree $3n-1$, which satisfy (4.8). Second, we construct $B_k(x)$, k be fixed (k∈{1,...n}). We look for $B_k(x)$ in the following form

$$B_k(x) = \frac{xH_n'(x)l_k^2(x)}{x_kH_n'(x_k)} + H_n'(x)q_k(x) \tag{4.14}$$

where $q_k(x)$ is the suitable polynomial for which

$$q_k(0) = 0 \tag{4.15}$$

According to (4.7) $q_k(x)$ must hold the following conditions: for j≠k

$$q_k(x_j) = 0 \tag{4.16}$$

and for j=k

$$q_k(x) = -\frac{1}{H_n'(x_k)} \tag{4.17}$$

Diffrentiating (4.14),we get
for j≠k

$$2x_jq_(x_j) + q_k'(x_j) = 0 \tag{4.18}$$

for j=k

$$\frac{(1+4x_k^2)}{x_kH_n'(x_k)} + 2x_kq_k(x_k) + q_k'(x_k) = 0 \tag{4.19}$$

From (4.15)-(4.19), we conclude that

$$q_k(x) = \frac{xH_n'(x)l_k^2(x)(1+4x_k^2)(x-x_k)}{x_k^2H_n'(x_k)} \tag{4.20}$$

Combining (4.14) and (4.20), we get (4.4).
Proof of $A_k(x)$ is like proof of $C_k(x)$. □

5. Order of convergence of fundamental polynomials

In this Section we compute order of convergence of fundamental polynomials, which is required to prove Theorem 2

Lemma 5.1. *For $k = 0,1....n-1$ and $x \in (-\infty,+\infty)$*

$$\sum_{i=0}^{n-1} e^{\beta y_k^2}|A_k(x)| = O(1)e^{\nu x^2} \quad where \quad \nu > \frac{3}{2} \tag{5.1}$$

where $A_k(x)$ is given by (4.2).

Proof. from (4.2) we have

$$\sum_{k=0}^{n-1} e^{\beta y_k^2}|A_k(x)| \le \sum_{k=0}^{n-1} e^{\beta y_k^2}\frac{|x|H_n{}^2(x)|L_k(x)|}{|y_k|H_n{}^2(y_k)} \tag{5.2}$$

using (3.7) (3.8), (3.9) and (3.10), (3.12), (3.13) we get the requaired lemma. □

Lemma 5.2. *For $k = 1....n$ and $x \in (-\infty,+\infty)$*

$$\sum_{k=1}^{n} e^{\beta x_k^2}|B_k(x)| = O(\sqrt{n})e^{\nu x^2} \quad where \quad \nu > \frac{3}{2} \tag{5.3}$$

where$B_k(x)$ is given by (3.3)

Proof. from (3.3) we have

$$\sum_{k=1}^{n} e^{\beta x_k^2}|B_k(x)| \le \sum_{k=1}^{n} e^{\beta x_k^2}\frac{|x||H_n'(x)|l_k^2(x)\{1+|\frac{(1+4x_k^2)}{x_k}||(x-x_k)|\}}{|x_k||H_n{}'(x_k)|} \tag{5.4}$$

Using (3.2) and (3.4), we get

$$\sum_{k=1}^{n} e^{\beta x_k^2}|B_k(x)| = \sum_{k=1}^{n} e^{\beta x_k^2}\frac{|x||H_n'(x)|l_k^2(x)}{|x_k||H_n'(x_k)|} \tag{5.5}$$

$$+\sum_{k=1}^{n} e^{\beta y_k^2}\frac{|x||H_n'(x)||H_n(x)||l_k(x)||\frac{(1+4x_k^2)}{x_k}|}{x_kH_n'(x_k)^2}$$

$=I_1 + I_2$
Owing (3.3), (3.7), (3.8),(3.10), (3.11), (3.13), (3.14) we get

$$I_1 == O(\frac{1}{\sqrt{n}})e^{\nu x^2}e^{\nu x^2} \quad and \quad I_2 = O(\sqrt{n})e^{\nu x^2} \tag{5.6}$$

combining both above we get the required lemma. □

Lemma 5.3. *For $k = 1,2,....,n$ and $x \in (-\infty,+\infty)$*

$$\sum_{k=1}^{n} e^{\beta x_k^2}|C_k(x)| = O(1)e^{\nu x^2} \quad where \quad \nu > \frac{3}{2} \tag{5.7}$$

where $C_k(x)$ is given by (3.4).

Proof. From (3.4) we have

$$\sum_{k=1}^{n} e^{\beta x_k^2}|C_k(x)| = \sum_{k=1}^{n} e^{\beta x_k^2}\frac{|x||H_n(x)||H_n{}'(x)||l_k(x)|}{|x_k||H_n'(x_k)^2|} \tag{5.8}$$

Using (3.1), (3.2), (3.7), (3.8), (3.9), (3.13) we get the required lemma. □

6. Convergence theorem of interpolatory polynomial

In this section we have proved convergence theorem for interpolatory polynomial $R_n(x)$.

Theorem 6.1. *Let the interpolated function $f : R \longrightarrow R$ be continuously differentiable such that*

$$\begin{cases} \lim_{|x|\to+\infty} x^{2k}f(x)\rho(x) = 0 & (k = o,1,....) \\ \lim_{|x|\to+\infty} x^{2k}f(x)\rho(x) = 0 & ,where \ \rho(x) = e^{-\beta x^2}, 0 \le \beta < 1 \end{cases}$$

(6.1)

further taking the number δ_k *such that*

$$\delta_k = O(e^{\delta x_k^2}) = \omega(f'; \frac{1}{\sqrt{n}}) \tag{6.2}$$

where ω *is modulus of continuity of* f'*.Then*

$$R_n(f,x) = \sum_{k=0}^{n-1} f(y_k)A_k(x) + \sum_{k=1}^{n} f(y_k)B_k(x) + \sum_{k=1}^{n} \delta_k C_k(x) \tag{6.3}$$

satisfies the relation

$$e^{-\delta x^2}|f(x) - R_n(x)| = O(\sqrt{n})\omega(f; \frac{1}{\sqrt{n}}) \tag{6.4}$$

Proof. Since $R_n(x)$ given by (4.1) is exact for all polynomials $Q_n(x)$ of degree $\leq$ 3n-1, we have

$$Q_n(x) = \sum_{k=0}^{n-1} Q_n(y_k)A_k(x) + \sum_{k=1}^{n} Q_n(x_k)B_k(x) + \sum_{k=1}^{n} Q_n'(x_k)C_k(x) \tag{6.5}$$

from (10.1) and (5.3), we have

$$|R_n(x) - f(x)| \leq |R_n(x) - Q_n(x)| + |Q_n(x) - f(x)| \tag{6.6}$$

Now

$$\begin{aligned} e^{vx^2}|R_n(x) - f(x)| \leq e^{vx^2}|R_n(x) - f(x)| \\ + e^{vx^2}\sum_{k=0}^{n-1}|f(y_k) - Q_n(y_k)||A_k(x)| \\ + e^{vx^2}\sum_{k=1}^{n}|f(x_k) - Q_n(x_k)||B_k(x)| \\ + e^{vx^2}\sum_{k=1}^{n}\delta_k|C_k(x) \\ + e^{vx^2}\sum_{k=1}^{n}|Q_n'(x_k)||C_k(x)| \end{aligned} \tag{6.7}$$

Using (3.15), (3.16), (3.17), (3.18), (7.2), and lemma 5.1-5.3, theorem is proved.

□

Conclusion

Let $\{x_k\}_{k=1}^{n}$ and $\{y_k\}_{k=1}^{n-1}$ be the roots of hermite polynomial $H_n(x)$ and its derivative $H_n{}'(x)$ respectively. If f(x) be continuously differentiable function on $(-\infty, +\infty)$ satisfying (3.16), then there exist a polynomial $R_n(x)$ satisfying (2.1) which uniformly converges to f(x) on $(-\infty, +\infty)$ as $n \to \infty$

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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