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Introduction to *Riesz* $lG-$ *module*

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Abstract
Action of an l-group on a vector lattice (Riesz space) is defined and the abstract structure of the space thus formed is termed as a *Riesz* $lG-$ *module*. Submodules namely, *Riesz* $lG-$ *submodule*, ***convex*** *Riesz* $lG-$ *submodule* are defined and properties are studied. A homomorphism called $RlG-$ *module homomorphism* between two *Riesz* $lG-$ *modules* is defined and properties are studied. An isomorphism namely, $RlG-$ *module isomorphism* is also defined.

Keywords
Riesz $lG-$ *module*; *Riesz* $lG-$ *submodule*; ***convex*** *Riesz* $lG-$ *submodule*; $RlG-$ *module homomorphism*; $RlG-$ *module isomorphism*; Riesz space.

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1. Introduction

Group action, defined as the action of a group on a set has many practical applications in the physical world [4]. The concept of group action is explained in [3, 4, 7]. Lattice ordered algebraic structures such as lattice ordered groups, lattice ordered rings, lattice ordered fields, lattice ordered vector spaces are studied in [1, 5, 6, 8, 10, 11]. Representation theory based on G - modules evoked much interest among researchers for, the group in action is studied by means of linear transformations on a vector space [2, 9]. An l-group action on a lattice ordered vector space (vector lattice) was introduced by Ursala [12] . A modified structure leads to the definition of a *Riesz* $lG-$ *module*.

2. Preliminaries

In this section, some basic definitions and results are reviewed.

Through out this paper, e denotes the identity element in the group G with binary operation $*$ and 0 denotes the identity element in the vector space V over the set of reals $\mathbf{R}$.

Definition 2.1. [3] Let X be a set and G be a group. For $g \in G$ and $x \in X$, an *action* of G on X, denoted as $g.x$ is such that

i) : $e.x = x$ for all $x \in X$

ii) : $(g_1 * g_2).x = g_1.(g_2.x)$ for all $x \in X$ and $g_1, g_2 \in G$.

Definition 2.2. [11] A *partial order* on a non empty set L is a binary relation on L that is reflexive, anti-symmetric, and transitive. A set in which a partial order is defined is termed as a *partially ordered set* or a *poset*.

Definition 2.3. [11] Let L be a poset and $x, y \in L$ are such that $x \leq y$, then an *interval* is defined by $[x,y] = \{z \in L : x \leq z \leq y\}$. A non-empty subset C of a poset L is said to be *convex* if $[a,b] \subseteq C$ for all $a, b \in C$ with $a \leq b$.

Definition 2.4. [11] A *Lattice* L is a poset in which the infimum $x \wedge y$ and supremum $x \vee y$ exist for any pair of elements x and y in L.

Definition 2.5. [11] A non empty subset M of a lattice L is said to be a *sublattice* of L if $x \wedge y$ and $x \vee y \in M$ for all $x, y \in M$.

Definition 2.6. [11] A function f between two lattices L and L' is a *lattice homomorphism* if for all $x, y \in L$, we have $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$.

Definition 2.7. [1] Let G be a group and $\leq$ be a partial order on it. Then G is a *lattice ordered group* or an *l-group* if for all $x, y, g_1, g_2 \in G$.

i) : $(G, \leq)$ is a lattice.

ii) : $g_1 \leq g_2 \implies x * g_1 * y \leq x * g_2 * y$

Definition 2.8. [1] A subgroup of G which is also a sublattice of G is called an *l-subgroup* of G.

Definition 2.9. [1] Let G be an l- group. The *positive cone* of G is the set $G^+ = \{g \in G : g \geq e\}$ whose elements are termed as positive elements of G and the *negative cone* of G is the set $G^- = \{g \in G : g \leq e\}$ which contains all negative elements of G.

Definition 2.10. [1] Let G be an l- group. Then for $g \in G$ the *positive part* of g is $g^+ = g \vee e \in G^+$, and the *negative part* is $g^- = g^{-1} \vee e \in G^+$. The *absolute value* of g is $|g| = g \vee g^{-1} = g^+ * g^-$ and $|g| \in G^+$.

Theorem 2.11. [1] Let G be an l-group and $P = G^+$ be the positive cone. Then for all $g \in G$, $g P g^{-1} = P$.

Definition 2.12. [5] A real vector space V is an *ordered vector space* if it satisfies the following conditions

i) : $x \leq y$, then $x + z \leq y + z$

ii) : $x \leq y$, then $\lambda x \leq \lambda y$ for all $x, y, z \in V$ and $\lambda \geq 0$.

Definition 2.13. [5] A *vector lattice* or a *Riesz space* is an ordered vector space which is also a lattice.

Example 2.14. [5] The Euclidean space $\mathbf{R}^{\mathbf{n}}$ is an example for a vector lattice (Riesz space) under the *product order* given by, $x \leq y$ if $x_i \leq y_i$ for all $i = 1, 2, \ldots, n$. The supremum and infimum of two elements x and y is defined as $x \vee y = x_i \vee y_i$ and $x \wedge y = x_i \wedge y_i$ for all $i = 1, 2, \ldots, n$.

Definition 2.15. [5] A *vector sublattice (Riesz subspace)* of a vector lattice (Riesz space) is a vector subspace which is also a sublattice.

3. Main Results

Definition 3.1. Let G be an l-group. A vector lattice (Riesz space) V is called a *Riesz $lG-$ module* if the group action G on V denoted by $g \circ x \in V$ for all $g \in G$ and $x \in V$ and has the following properties. For all $g, g_1, g_2 \in G$, $x, y \in V$, $r, s \in \mathbf{R}$,

1. $e \circ x = x$
2. $(g_1 * g_2) \circ x = g_1 \circ (g_2 \circ x)$
3. $g \circ (rx + sy) = r(g \circ x) + s(g \circ y)$
4. $|g| \circ (x \wedge y) = (|g| \circ x) \wedge (|g| \circ y)$
   $|g| \circ (x \vee y) = (|g| \circ x) \vee (|g| \circ y)$
   $(g_1 \wedge g_2) \circ |x| = (g_1 \circ |x|) \wedge (g_2 \circ |x|)$
   $(g_1 \vee g_2) \circ |x| = (g_1 \circ |x|) \vee (g_2 \circ |x|)$

Remark 3.2. *Each $g \in G$ give rise to an endomorphism ρ_g on V where $\rho_g(x) = g \circ x$ and the map $\rho : G \to End_{\mathbf{R}}(V)$ is a group homomorphism. Note that ρ_g is a lattice homomorphism when $g \in G^+$.*

Remark 3.3. $g \circ 0 = 0$.

Example 3.4. Consider the action of $\mathbf{R}^+$, the set of positive real numbers on the Euclidean plane $\mathbf{R}^2$, defined by $r \circ (x, y) = (rx, ry)$ for $r \in \mathbf{R}$ and $(x, y) \in \mathbf{R}^2$. Then $\mathbf{R}^2$ is a *Riesz $lG-$ module*. Here, the group action is the scalar multiplication in $\mathbf{R}^2$.

Definition 3.5. Let V be a *Riesz $lG-$ module*. A vector sublattice (Riesz subspace) W of V is a *Riesz $lG-$ submodule* or *$RlG-$submodule* of V if W itself is a *Riesz $lG-$ module* under the action of G same as that on V.

Theorem 3.6. *A vector sublattice W of a Riesz $lG-$ module V is a Riesz $lG-$ submodule of V if and only if W is closed under the group action defined in V. That is, $g \circ x \in W$ for every $g \in G$ and $x \in W$.*

Proof. The proof is straight forward. □

Example 3.7. The line $y = x$ is a vector sublattice of $\mathbf{R}^2$. It is a *Riesz $lG-$ submodule* of $\mathbf{R}^2$ under the group action defined in Example 3.4 .

Example 3.8. Every vector sublattice is not a *Riesz $lG-$ submodule*. If the group action is defined as $r \circ (x, y) = (rx, y)$, then $\mathbf{R}^2$ is a *Riesz $lG-$ module*. But the line $y = x$ is not a *Riesz $lG-$ submodule* of $\mathbf{R}^2$. Note that under this group action the X-axis, that is, the line $y = 0$ is a *Riesz $lG-$ submodule* of $\mathbf{R}^2$.

Theorem 3.9. *Let V be a Riesz $lG-$ module. For $x \in V^+$, let $G_x = \{g \in G : g \circ x = x\}$ is an l-subgroup of G.*

Proof. The set G_x is nonempty as $e \in G_x$. Fix $x \in V^+$. Let $g, h \in G_x$.
Now, $(g * h) \circ x = g \circ (h \circ x) = g \circ x = x$. Thus, $g * h \in G_x$.
$g^{-1} \circ x = (g^{-1} \circ (g \circ x)) = (g^{-1} * g) \circ x = e \circ x = x$, shows that $g^{-1} \in G_x$.
Also, $(g \wedge h) \circ x = (g \wedge h) \circ |x| = (g \circ |x|) \wedge (h \circ |x|) = (g \circ x) \wedge (h \circ x) = x \wedge x = x$. Thus $g \wedge h \in G_x$.
Similarly, it can be proved that $g \vee h \in G_x$. Thus, G_x is an l-subgroup of G. □

Theorem 3.10. *Let V be a Riesz $lG-$ module. Then $V_{G^+} = \{ x \in V : \hat{g} \circ x = x$ for all $\hat{g} \in G^+\}$ is a Riesz $lG-$ submodule of V. The set V_{G^+} is called the set of all fixed points of V with respect to G^+.*

Proof. The set V_{G^+} is nonempty, since $0 \in V_{G^+}$.
Let $x,\ y \in V_{G^+}$, $g \in G$ and $\hat{g},\ \hat{h} \in G^+$.
Now, $\hat{g} \circ (x+y) = \hat{g} \circ x + \hat{g} \circ y = x+y$. Therefore $x+y \in V_{G^+}$.
$\hat{g} \circ (rx) = r(\hat{g} \circ x) = rx$ shows that $rx \in V_{G^+}$.
Now, since $\hat{g} \in G^+, |\hat{g}| = \hat{g}$. Condition(4) in the definition of a *Riesz* $lG-$ *module* shows that $\hat{g} \circ (x \wedge y) = (\hat{g} \circ x) \wedge (\hat{g} \circ y) = x \wedge y$. Thus $x \wedge y \in V_{G^+}$. Similarly, $x \vee y \in V_{G^+}$.
Finally, $g \circ x = g \circ (\hat{g} \circ x) = (g * \hat{g}) \circ x = (\hat{h} * g) \circ x = \hat{h} \circ (g \circ x)$, since $g\, G^+ = G^+\, g$ for all $g \in G$. Hence, $g \circ x \in V_{G^+}$. ☐

Theorem 3.11. *Let* V *be a Riesz* $lG-$ *module. Then any lattice* $W = \{\hat{g} \circ x : \hat{g} \in G^+,\ x \in V\}$ *which is closed under vector addition is a Riesz* $lG-$ *submodule of* V.

Proof. Let $g \in G,\ \hat{g},\ \hat{h} \in G^+,\ r \in \mathbf{R},\ x, y \in V$.
We have, $r(\hat{g} \circ x) = \hat{g} \circ (rx) \in W$. Also, $g \circ (\hat{g} \circ x) = (g * \hat{g}) \circ x = (\hat{h} * g) \circ x = \hat{h} \circ (g \circ x) \in W$. ☐

Remark 3.12. *The Riesz* $lG-$ *submodule in the above theorem is called the Riesz* $lG-$ *submodule of* V *generated by* G^+.

Theorem 3.13. *For* $x \in V^+$*, define* $O(x) = \{g \circ x : g \in G\}$. *Then* $O(x)$ *is called the* $lG-$ ***orbit*** *of* x *and is a sublattice of* V.

Proof. Note that $|\,x\,| = x$ for every $x \in V^+$.
By condition (4) in the definition of a *Riesz* $lG-$ *module*, $(g \circ x) \wedge (h \circ x) = (g \wedge h) \circ x \in O(x)$.
Similar reasoning shows that $(g \circ x) \vee (h \circ x) \in O(x)$. This proves the theorem. ☐

Theorem 3.14. *Intersection of any number of Riesz* $lG-$ *submodules is again a Riesz* $lG-$ *submodule.*

Proof. Simple calculations gives the result. ☐

Definition 3.15. A non zero *Riesz* $lG-$ *module* V is called an *irreducible Riesz* $lG-$ *module* or *irreducible* $RlG-$ *module* if its only *Riesz* $lG-$ *submodule*s are 0 and V.

Example 3.16. The real line $\mathbf{R}$ is a *Riesz* $lG-$ *module* under the action of $\mathbf{R}^+$, where the action is the usual multiplication in $\mathbf{R}$. Then $\mathbf{R}$ is an *irreducible Riesz* $lG-$ *module*.

Definition 3.17. Let V and W be two *Riesz* $lG-$ *module*s where the group G in action is the same for both V and W. A linear map $\phi : V \to W$ is a *Riesz* $lG-$ *module homomorphism* or an $RlG-$ *module homomorphism* if it a lattice homomorphism satisfying $\phi(g \circ x) = g \circ \phi(x)$ for all $g \in G,\ x \in V$.

Theorem 3.18. *The kernel of a* $RlG-$ *module homomorphism is a Riesz* $lG-$ *submodule.*

Proof. Let $\phi : V \to W$ is a $RlG-$ *module homomorphism*. Then *Ker* $\phi = \{x \in V : \phi(x) = 0_W\}$, where 0_W is the zero element in W. Clearly, *Ker* ϕ is a linear subspace of V.
For, $x,\ y \in V$, $\phi(x \wedge y) = \phi(x) \wedge \phi(y) = 0_W$, shows that $x \wedge y \in Ker\phi$. Similar is the case with $x \vee y$.
Also, $\phi(g \circ x) = g \circ \phi(x) = g \circ 0_W = 0_W$. Hence $g \circ x \in$ *Ker* ϕ. ☐

Theorem 3.19. *The Image of a* $RlG-$ *module homomorphism is a Riesz* $lG-$ *submodule.*

Proof. Let $\phi : V \to W$ is a $RlG-$ *module homomorphism*. The image of ϕ, *Im* $\phi = \{\phi(x) \in W : x \in V\}$ is a linear subspace of W. Let $x, y \in V$ and $g \in G$.
Then $\phi(x) \wedge \phi(y) = \phi(x \wedge y) \in Im\,\phi$. Similarly, $\phi(x) \vee \phi(y) \in Im\,\phi$. Finally, $g \circ \phi(x) = \phi(g \circ x) \in Im\,\phi$. ☐

Definition 3.20. An $RlG-$ *module homomorphism* which is both one-to-one and onto is called an $RlG-$ *module isomorphism.*

Theorem 3.21. *Let* V *and* W *be two irreducible Riesz* $lG-$ *modules and let* $\phi : V \to W$ *is a* $RlG-$ *module homomorphism. Then either* $\phi = 0$ *or* ϕ *is a* $RlG-$ *module isomorphism.*

Proof. Theorem 3.18 and Theorem 3.19 together with the fact that V and W are *irreducible Riesz* $lG-$ *module*s proves the theorem. ☐

Definition 3.22. A *Riesz* $lG-$ *submodule* W of a *Riesz* $lG-$ *module* V is a ***convex*** *Riesz* $lG-$ *submodule* or a ***convex*** $RlG-$*submodule* if it is a convex subset (with respect to the order) of V .

Example 3.23. We have seen in example 3.7, that the line $y = x$ is a *Riesz* $lG-$ *submodule* of $\mathbf{R^2}$. But it is not a convex subset of $\mathbf{R^2}$ under the product order in $\mathbf{R^2}$, for $(1,2) \in \mathbf{R^2}$, satisfies $(1,1) \leq (1,2) \leq (2,2)$ but does not lie on the line $y = x$.
But the line $y = 0$ which is a *Riesz* $lG-$ *submodule* of $\mathbf{R^2}$ is a ***convex*** *Riesz* $lG-$ *submodule* with respect to the product order.

Remark 3.24. Every lattice homomorphism is an isotone or order preserving, that is, $x \leq y \implies \phi(x) \leq \phi(y)$ [11]. Hence it is not difficult to prove that the inverse image of a ***convex*** *Riesz* $lG-$ *submodule* under a $RlG-$ *module homomorphism* is a ***convex*** *Riesz* $lG-$ *submodule*.
But the converse is not true, for, $\phi(x) \leq \phi(y)$ need not implies $x \leq y$. An $RlG-$ *module homomorphism* which maps a ***convex*** *Riesz* $lG-$ *submodule* to a ***convex*** *Riesz* $lG-$ *submodule* is termed as a ***convex*** RlG - *module homomorphism*. It can be noted that every $RlG-$*module isomorphism* is a ***convex*** $RlG-$ *module homomorphism*.

4. Conclusion

In this paper, *Riesz* $lG-$ *module*, *Riesz* $lG-$ *submodule*, ***convex*** *Riesz* $lG-$ *submodule*, $RlG-$ *module homomorphism*, $RlG-$ *module isomorphism* etc. are defined and their basic properties are introduced. Further properties of $RlG-$ *module homomorphisms* and $RlG-$ *module isomorphisms* are yet to be explored. The introduction of a *Riesz* $lG-$ *module* is a pathway to further research in this area, especially in representation theory.

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