Malaya Journal of Matematik, Vol. 8, No. 2, 576-580, 2020

https://doi.org/10.26637/MJM0802/0044

On a certain subclass of analytic functions defined by a differential operator

Santosh M. Popade[1]* Rajkumar N. Ingle[2] and P. Thirupati Reddy[3]

Abstract

In this paper, we introduce and study a new subclass of analytic functions which are defined by means of a new differential operator. Some results connected to coefficient estimates, growth and distortion theorems, radii of starlikeness, convexity close-to-convexity and integral means inequalities related to the subclass is obtained.

Keywords

Analytic functions, differential operator, coefficient estimates.

AMS Subject Classification

30C45.

[1] *Department of Mathematics, Sant Tukaram College of Arts and Science, Parbhani-431401, India.*
[2] *Department of Mathematics, Bahirji Smarak Mahavidyalaya, Basmathnagari-431512, India.*
[3] *Department of Mathematics, Kakatiya University, Warangali-506009, India.*
*__Corresponding author__: [1] smpopade2007@gmail.com; [2]ingleraju11@gmail.com; [3]reddypt2@gmail.com
Article History: Received **10** January **2020**; Accepted **21** April **2020**

Contents

1. Introduction

Let A denote the class of functions u of the form

$$u(z) = z + \sum_{\eta=2}^{\infty} a_n z^{\eta} \tag{1.1}$$

which are analytic in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$

A function u in the class A is said to be in the class $ST(\alpha)$ of starlike functions of order α in E, if it satisfy the inequality

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \alpha, \qquad (0 \le \alpha < 1), z \in E \tag{1.2}$$

Note that $ST(0) = ST$ is the class of Starlike functions.
Denote by T the subclass of A consisting of functions u of the form

$$u(z) = z - \sum_{\eta=2}^{\infty} a_{\eta} z^{\eta}, \qquad (a_{\eta} \ge 0) \tag{1.3}$$

This subclass was introduced and extensively studied by Silvermann[4].
Let u be a function in the class A. We define the following differential operator introduced by Deniz and Ozkan [1].

$$D_{\lambda}^{0} u(z) = u(z)$$
$$D_{\lambda}^{1} u(z) = D_{\lambda} u(z) = \lambda z^3 (u(z))^m + (2\lambda+1) z^2 (u(z))^{\eta} + zu'(z)$$
$$D_{\lambda}^{2} u(z) = D_{\lambda}(D_{\lambda}^{1} u(z))$$
$$\vdots$$
$$D_{\lambda}^{m} u(z) = D_{\lambda}(D_{\lambda}^{m-1} u(z))$$

where $\lambda \ge 0$ and $m \in N_0 = N \cup \{0\}$. If u is given by (1.1), then from the definition of the operator $D_{\lambda}^{m} u(z)$, it is to see that

$$D_{\lambda}^{m} u(z) = z + \sum_{\eta=2}^{\infty} \phi^{m}(\lambda, \eta) a_{\eta} z^{\eta} \tag{1.4}$$

where

$$\phi^{m}(\lambda, \eta) = \eta^{2m}[\lambda(\eta - 1) + 1]^{m} \tag{1.5}$$

If $u \in T$ is given by (1.2) then we have

$$D_\lambda^m u(z) = z - \sum_{\eta=2}^{\infty} \phi^m(\lambda,\eta) a_\eta z^\eta \tag{1.6}$$

where $\phi^m(\lambda,\eta)$ is given by (1.5)

Now we define the following new subclass motivated by Murugusunderamoorthy and Magesh [3]

Definition 1.1. *The function u(z) of the form (1.1) is in the class $S_\lambda^m(\mu,\gamma)$, if it satisfies the inequality*

$$\Re\left\{\frac{z(D_\lambda^m u(z))'}{(1-\mu)z+\mu D_\lambda^m u(z)} - \alpha\right\} > \left|\frac{z(D_\lambda^m u(z))'}{(1-\mu)z+\mu D_\lambda^m u(z)} - 1\right|$$

for $0 \le \lambda \le 1$, $0 \le \gamma \le 1$

Further we define $TS_\lambda^m(\mu,\gamma) = S_\lambda^m(\mu,\gamma) \cap T$

The aim of present paper is to study the coefficient bounds, radii of close-to-convex and starlikeness convex linear combinations and integral means inequalities of the $TS_\lambda^m(\mu,\gamma)$

2. Coefficient bounds

Theorem 2.1. *A function u(z) of the form (1.1) is in $S_\lambda^m(\mu,\gamma)$, then*

$$\sum_{\eta=2}^{\infty} [2\eta - \mu(\gamma+1)]\phi^m(\lambda,\eta)|a_\eta| \le 1-\gamma \tag{2.1}$$

where $0 \le \mu \le 1$, $0 \le \gamma \le 1$ and $\phi^m(\lambda,\eta)$ is given by (1.5)

Proof. It suffices to show that

$$\left|\frac{z(D_\lambda^m u(z))'}{(1-\mu)z+\mu D_\lambda^m u(z)} - 1\right| - \Re\left\{\frac{z(D_\lambda^m u(z))'}{(1-\mu)z+\mu D_\lambda^m u(z)} - 1\right\} \le 1-\gamma$$

We have

$$\begin{aligned}
&\left|\frac{z(D_\lambda^m u(z))'}{(1-\mu)z+\mu D_\lambda^m u(z)} - 1\right| - \Re\left\{\frac{z(D_\lambda^m u(z))'}{(1-\mu)z+\mu D_\lambda^m u(z)} - 1\right\} \\
&\le 2\left|\frac{z(D_\lambda^m u(z))'}{(1-\mu)z+\mu D_\lambda^m u(z)} - 1\right| \\
&\le \frac{2\sum_{\eta=2}^{\infty}(\eta-\mu)\phi^m(\lambda,\eta)|a_\eta||z|^{\eta-1}}{1-\sum_{\eta=2}^{\infty}\mu\phi^m(\lambda,\eta)|a_\eta||z|^{\eta-1}} \\
&\le \frac{2\sum_{\eta=2}^{\infty}(\eta-\mu)\phi^m(\lambda,\eta)|a_\eta|}{1-\sum_{\eta=2}^{\infty}\mu\phi^m(\lambda,\eta)|a_\eta|}
\end{aligned}$$

The last expression is bounded above by $(1-\gamma)$, if

$$\sum_{\eta=2}^{\infty} [2\eta - \mu(\gamma+1)]\phi^m(\lambda,\eta)|a_\eta| \le 1-\gamma$$

and the proof is complete. □

Theorem 2.2. *Let $0 \le \mu \le 1$, $0 \le \gamma \le 1$, then a function u of the form (1.3) to be in the class $TS_\lambda^m(\mu,\gamma)$ if and only if*

$$\sum_{\eta=2}^{\infty} [2\eta - \mu(\gamma+1)]\phi^m(\lambda,\eta)|a_\eta| \le 1-\gamma \tag{2.2}$$

where $\phi^m(\lambda,\eta)$ is given by (1.5)

Proof. In view of Theorem (2.1) we need only to prove the necssity. If $u \in TS_\lambda^m(\mu,\gamma)$ and z is real, then

$$\Re\left\{\frac{1-\sum_{\eta=2}^{\infty}\eta\phi^m(\lambda,\eta)a_\eta z^{\eta-1}}{1-\sum_{\eta=2}^{\infty}\mu\phi^m(\lambda,\eta)a_\eta z^{\eta-1}} - \gamma\right\} > \left|\frac{\sum_{\eta=2}^{\infty}(\eta-\mu)\phi^m(\lambda,\eta)a_\eta z^{\eta-1}}{1-\sum_{\eta=2}^{\infty}\mu\phi^m(\lambda,\eta)a_\eta z^{\eta-1}}\right|$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{\eta=2}^{\infty} [2\eta - \mu(\gamma+1)]\phi^m(\lambda,\eta)|a_\eta| \le 1-\gamma$$

where $0 \le \mu \le 1$, $0 \le \gamma \le 1$ and $\phi^m(\lambda,\eta)$ is given by (1.5) □

Corollary 2.3. *If $u(z) \in TS_\lambda^m(\mu,\gamma)$, then*

$$|a_\eta| \le \frac{1-\gamma}{[2\eta-\mu(\gamma+1)]\phi^m(\lambda,\eta)} \tag{2.3}$$

where $0 \le \mu \le 1$, $0 \le \gamma \le 1$ and $\phi^m(\lambda,\eta)$ is given by (1.5). Equality holds for the function

$$u(z) = z - \frac{1-\gamma}{[2\eta-\mu(\gamma+1)]\phi^m(\lambda,\eta)} z^\eta \tag{2.4}$$

Theorem 2.4. *Let $u_1(z) = z$ and*

$$u_\eta(z) = z - \frac{1-\gamma}{[2\eta-\mu(\gamma+1)]\phi^m(\lambda,\eta)} z^\eta, \quad \eta \ge 2 \tag{2.5}$$

Then $u(z) \in TS_\lambda^m(\mu,\gamma)$, if and only if, it can be expressed in the form

$$u(z) = \sum_{\eta=1}^{\infty} w_\eta u_\eta(z), \;\; w_\eta \ge 0, \;\; \sum_{\eta=1}^{\infty} w_\eta = 1 \tag{2.6}$$

Proof. Suppose $u(z)$ can be written as in (2.6), then

$$u(z) = z - \sum_{\eta=2}^{\infty} w_\eta \frac{1-\gamma}{[2\eta-\mu(\gamma+1)]\phi^m(\lambda,\eta)} z^\eta$$

Now,

$$\begin{aligned}
\sum_{\eta=2}^{\infty} w_\eta \frac{(1-\gamma)[2\eta-\mu(\gamma+1)]\phi^m(\lambda,\eta)}{(1-\gamma)[2\eta-\mu(\gamma+1)]\phi^m(\lambda,\eta)} &= \sum_{\eta=2}^{\infty} w_\eta \\
&= 1 - w_1 \le 1
\end{aligned}$$

Thus $u(z) \in TS_{\lambda}^{m}(\mu,\gamma)$.
Conversely, let $u(z) \in TS_{\lambda}^{m}(\mu,\gamma)$, then by using (2.3), we get

$$w_{\eta} = \frac{[2\eta-\mu(\gamma+1)]\phi^{m}(\lambda,\eta)}{(1-\gamma)}a_{\eta}, \quad \eta \geq 2$$

and $w_1 = 1 - \sum_{\eta=2}^{\infty} w_{\eta}$. Then we have $u(z) = \sum_{\eta=1}^{\infty} w_{\eta}u_{\eta}(z)$ and hence this completes the proof of Theorem. □

Theorem 2.5. *The class $TS_{\lambda}^{m}(\mu,\gamma)$ is a convex set.*

Proof. Let the function

$$u_j(z) = z - \sum_{\eta=2}^{\infty} a_{\eta,j}z^{\eta}, \qquad a_{\eta,j} \geq 0, \quad j=1,2 \tag{2.7}$$

be in the class $TS_{\lambda}^{m}(\mu,\gamma)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \xi u_1(z) + (1-\xi)u_2(z), \ \ 0 \leq \xi < 1,$$

in the class $TS_{\lambda}^{m}(\mu,\gamma)$. Since

$$h(z) = z - \sum_{\eta=2}^{\infty}[\xi a_{\eta,1} + (1-\xi)a_{\eta,2}]z^{\eta},$$

An easy computation with the aid of Theorem (2.2) gives

$$\begin{aligned}\sum_{\eta=2}^{\infty}[2\eta-\mu(\gamma+1)]\xi\phi^{m}(\lambda,\eta)a_{\eta,1} + \\ \sum_{\eta=2}^{\infty}[2\eta-\mu(\gamma+1)](1-\xi)\phi^{m}(\lambda,\eta)a_{\eta,2} \\ \leq \xi(1-\gamma)+(1-\xi)(1-\gamma) \\ \leq (1-\gamma)\end{aligned}$$

which implies that $h \in TS_{\lambda}^{m}(\mu,\gamma)$
Hence $TS_{\lambda}^{m}(\mu,\gamma)$ is convex.

□

3. Radii of Close-to-Convexity, Starlikeness and Convexity

In this section, we obtain the radii of close-to-convexity, starlikeness and convexity for the class $TS_{\lambda}^{m}(\mu,\gamma)$.

Theorem 3.1. *Let the function $u(z)$ defined by (1.3) belong to the class $TS_{\lambda}^{m}(\mu,\gamma)$, then $u(z)$ is close-to-convex of order $\delta(0 \leq \delta < 1)$ in the disc $|z| < r_1$, where*

$$r_1 = \inf_{\eta\geq 2}\left[\frac{(1-\delta)\sum_{\eta=2}^{\infty}[2\eta-\mu(\gamma+1)]\phi^{m}(\lambda,\eta)}{\eta(1-\gamma)}\right]^{1/\eta-1}, \eta \geq 2 \tag{3.1}$$

The result is sharp, with the external function $u(z)$ is given by (2.5)

Proof. Given $u \in T$ and u is close-to-convex of order δ, we have

$$|f'(z)-1| < 1-\delta \tag{3.2}$$

For the left hand side of (3.2), we have

$$|u'(z)-1| \leq \sum_{\eta=2}^{\infty}\eta a_{\eta}|z|^{\eta-1}$$

The last expression is less than $1-\delta$

$$\sum_{\eta=2}^{\infty}\frac{\eta}{1-\delta}a_{\eta}|z|^{\eta-1} \leq 1$$

Using the fact, that $u(z) \in TS_{\lambda}^{m}(\mu,\gamma)$ if and only if

$$\sum_{\eta=2}^{\infty}\frac{[2\eta-\mu(\gamma+1)]\phi^{m}(\lambda,\eta)}{1-\gamma}a_{\eta} \leq 1$$

We can see that (3.2) is true, if

$$\frac{\eta}{1-\delta}|z|^{\eta-1} \leq \frac{[2\eta-\mu(\gamma+1)]\phi^{m}(\lambda,\eta)}{1-\gamma}$$

or, equivalently

$$|z| \leq \left\{\frac{(1-\delta)[2\eta-\mu(\gamma+1)]\phi^{m}(\lambda,\eta)}{\eta(1-\gamma)}\right\}^{1/\eta-1}$$

which completes the proof. □

Theorem 3.2. *Let the function $u(z)$ defined by (1.3) belong to the class $TS_{\lambda}^{m}(\mu,\gamma)$. Then $u(z)$ is starlike of order $\delta(0 \leq \delta < 1)$ in the disc $|z| < r_2$, where*

$$r_2 = \inf_{\eta\geq 2}\left[\frac{(1-\delta)\sum_{\eta=2}^{\infty}[2\eta-\mu(\gamma+1)]\phi^{m}(\lambda,\eta)}{(\eta-\delta)(1-\gamma)}\right]^{1/\eta-1} \tag{3.3}$$

The result is sharp, with external function $u(z)$ is given by (2.5)

Proof. Given $u \in T$ and u is starlike of order δ, we have

$$\left|\frac{zu'(z)}{u(z)}-1\right| < 1-\delta \tag{3.4}$$

For the left hand side of (3.4), we have

$$\left|\frac{zu'(z)}{u(z)}-1\right| \leq \sum_{\eta=2}^{\infty}\frac{(\eta-1)a_{\eta}|z|^{\eta-1}}{1-\sum_{\eta=2}^{\infty}a_{\eta}|z|^{\eta-1}}$$

The last expression is less than $1-\delta$ if

$$\sum_{\eta=2}^{\infty}\frac{\eta-\delta}{1-\delta}a_{\eta}|z|^{\eta-1} < 1$$

Using the fact that $u(z) \in TS_{\lambda}^{m}(\mu,\gamma)$ if and only if

$$\sum_{\eta=2}^{\infty} \frac{[2\eta-\mu(\gamma+1)]\phi^{m}(\lambda,\eta)}{1-\gamma} a_{\eta} \leq 1$$

We can say (3.4) is true, if

$$\sum_{\eta=2}^{\infty} \frac{\eta-\delta}{1-\delta}|z|^{\eta-1} \leq \frac{[2\eta-\mu(\gamma+1)]\phi^{m}(\lambda,\eta)}{1-\gamma}$$

or equivalently

$$|z|^{\eta-1} \leq \frac{(1-\delta)[2\eta-\mu(\gamma+1)]\phi^{m}(\lambda,\eta)}{(\eta-\delta)(1-\gamma)}$$

which yields the starlikeness of the family. □

4. Integral Means Inequalities

In [4], Silverman found that the function $u_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality conjunctured [5] and setteled in [6], that

$$\int_0^{2\pi} |u(re^{i\varphi})|^{\tau} d\varphi \leq \int_0^{2\pi} |u_2(re^{i\varphi})^{\eta}|^{\tau} d\varphi$$

for all $u \in T$, $\tau > 0$ and $0 < r < 1$. In [6], he also proved his conjuncture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T.

Now, we prove Silverman's conjecture for the class of functions $TS_{\lambda}^{m}(\mu,\gamma)$
We need the concept of subordination between analytic functions and a subordination theorem of Littlewood [2].

Two functions u and v, which are analytic in E, the function u is said to be subordinate to v in E, if there exists a function w analytic in E with $w(0) = 0$, $|w(z) < 1$, $(z \in E)$ such that $u(z) = v(w(z))$, $(z \in E)$. We denote this subordination by $u(z) \prec v(z)$. ($\prec$ denote subordination)

Lemma 4.1. *If the function u and v are analytic in E with $u(z) \prec v(z)$, then for $\tau > 0$ and $z = re^{i\varphi}$, $0 < r < 1$*

$$\int_0^{2\pi} |v(re^{i\varphi})|^{\tau} d\varphi \leq \int_0^{2\pi} |u(re^{i\varphi})|^{\tau} d\varphi$$

Now, we discuss the integral means inequalities for functions u in $TS_{\lambda}^{m}(\mu,\gamma)$

Theorem 4.2. *$u \in TS_{\lambda}^{m}(\mu,\gamma)$, $0 \leq \mu < 1$, $0 \leq \gamma < 1$ and $u_2(z)$ be defined by*

$$u_2(z) = z - \frac{1-\gamma}{\phi_2(\lambda,\gamma)} z^2 \tag{4.1}$$

Proof. For $u(z) = z - \sum_{\eta=2}^{\infty} a_{\eta} z^{\eta}$, (4.1) is equivalent to

$$\int_0^{2\pi} \left|1 - \sum_{\eta=2}^{\infty} a_{\eta} z^{\eta-1}\right|^{\tau} d\varphi \leq \int_0^{2\pi} \left|1 - \frac{1-\gamma}{\varphi_2(\lambda,\gamma)} z\right|^{\tau} d\varphi$$

By Lemma (4.1), it is enough to prove that

$$1 - \sum_{\eta=2}^{\infty} a_{\eta} z^{\eta-1} \prec 1 - \frac{1-\gamma}{\varphi_2(\lambda,\gamma)} z,$$

Assuming

$$1 - \sum_{\eta=2}^{\infty} a_{\eta} z^{\eta-1} \prec 1 - \frac{1-\gamma}{\varphi_2(\lambda,\gamma)} w(z),$$

and using (2.2), we obtain

$$|w(z)| = \left|\sum_{\eta=2}^{\infty} \frac{\varphi_2(\lambda,\gamma)}{1-\gamma} a_{\eta} z^{\eta-1}\right| \leq |z| \sum_{\eta=2}^{\infty} \frac{\varphi_2(\lambda,\gamma)}{1-\gamma} a_{\eta} \leq |z|$$

where

$$\varphi_{\eta}(\lambda,\gamma) = [2\eta - \mu(\gamma+1)\phi^{m}(\lambda,\eta)$$

This completes the proof □

5. Conclusion

This research has introduced a new linear differential operator related to Analytic function and studied some basic properties of geometric function theory . Accordingly, some results related to closure theorems have also been considered, inviting future research for this field of study.

Acknowledgment

The authors would like to acknowledge their sincere gratitude towards the referee's for their studious comments and suggestions for the improvement of our research paper.

References

[1] E.Deniz and Y. Ozkan, Subclasses of analytic functions defined by new differential operator, *Acta.Uni.Apul.* (2014), 85–95.
[2] J.E. Littlewood, On inequalities in the theory of functions,*Proc. London Math. Soc.,* 23(2), (1925), 481-519.
[3] G. Murugusundarmurthy and N. Magesh, Certain subclasses of starlike functions of complex order involving generalized hypergeometric functions , *Int. J. Math. Math. Sci.,*(2010), Art ID 178605, 12pp.
[4] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.,* 51(1975), 109-116.
[5] H. Silverman, A survey with open problems on univalent functions whose coefficient are negative., *Rocky Mountain J. Math.,* 21(3) (1991), 1099-1125.

[6] H. Silvermani, Integral means for univalent functions with negative coefficient, *Houston J. Math.,* 23(1) (1997), 169-174.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆