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MAP/M/1 queue with M-policy of transfer of customers from pool to buffer

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Abstract

In this paper we describe an MAP/M/1 queuing system. The customers arrive according to Markovian Arrival Process and are served in first in first out order with exponential service time distribution. The arrived customers wait in a buffer of finite capacity and postponed customers or postponed work can either join pool of infinite capacity or balk. The customers waiting in the pool are transferred to buffer according to an M- Policy.The probabilistic behaviour of the system in steady state is analyzed with the help of Quasi- Birth-Death process(QBD) and matrix geometric method in continuous time Markov chain. We perform numerical calculations to evaluate some queuing measures and this queuing system can thus be analysed.

Keywords

Matrix Geometric method, Quasi- Birth Death Process, M-policy, pool, buffer, Markovian arrival process.

AMS Subject Classification

60K25, 90B22, 90B25.

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1. Introduction

Queuing systems with limited buffer capacity have great applications in many fields like telecommunications, computer networks, manufacturing systems and in many real life situations. In real life the customers may leave the system when buffer reaches the capacity. This results in the loss of customers. Such customers called postponed customers or postponed work are retained in a pool of infinite capacity. The concept of postponed wok began to appear by Deepak et.al [1]. They discussed the long run behaviour of the postponed work queuing system with Poisson arrival and service having PH - distribution using matrix geometric method. The matrix geometric method was first introduced and studied by Neuts [6][8][2][7][9]. A Krishnamoorthy et.al [5] extended the model by introducing N-policy for pooled customers. They remarked that through this model the pooled customers can reduce the waiting time. A numerical solution for level dependent queue with quasi-birth-death process was developed by Hendrick[3]. Qi Ming He et.al [10] observed that transfer of customers decreases the mean queue length, increases idle probability and increases the efficiency of the queuing system. A Krishnamoorthy et. al [4] introduced queuing -inventory models with different transfer mechanisms for pooled customers. This motivated us to study further about single server queuing system of postponed work .

In this paper, the queuing system in MAP/M/1 model is mathematically formulated and analyzed in section 2. The steady state analysis of the system which includes stability condition is obtained in section 3. Section 4 provides iteration algorithm for computing the rate matrix R. Performnce characteristics are discussed in section 5. Numerical results are provided in section 6 to illustrate the effect of certain parameters on the performance of the queuing system under study. At the end conclusion is given in section 7.

2. Model Description

In this model a single server queuing system with buffer of restricted capacity N (including customer being served) is considered. The customers arrive in accordance aith Markovian Arrival Process (MAP) are served according to exponential distribution. If an arriving customer meets the idle server, then the server starts service. When a customer on arrival finds that the server is not free, the customer will enter the buffer to wait for service. When the buffer reaches its capacity, the newly arriving customers may leave the system. In order to reduce the loss of such customers, the customers are directed to a pool of infinite capacity. They may either enter the pool with known probability q; $0 < q < 1$ or balks with complementary probability.

The customers waiting in the pool are transferred back to the buffer in the following manner; At service completion epoch, if the number of customers in the buffer is reduced to a preassigned level L or less; $1 \leq L < N$, then the customers present in the pool which is a minimum of (M,i) (where M is any number greater than L and $i \geq 1$ is the customers present in the pool) are transferred to the buffer with probability one provided that $L+M \leq N$. The transferred customers occupy the position at the head of the queue in the buffer. If there is $L+1$ or more customers in the buffer at a service completion epoch no such transfer will take place. At service completion epoch, when the server is free and there is not less than one customer in pool, then the customer waiting at the head of the pool gets transferred to buffer with probability one. Here we use the following notations suitable for this model.

At time t, let

- P(t) be the number of customers in the pool.
- B(t) be the number of customers in the buffer which includes customer in service.
- A(t) be the phase of the arrival process.

Then $(X(t);t \geq 0)$ where $X(t) = (P(t),B(t),A(t))$ is a continuous time Markov Chain. The state space consists of

$S = \bigcup(0,0,a); 1 \leq a \leq m \bigcup\{(i,j,a); i \geq 0, 1 \leq j \leq N,$
$1 \leq a \leq m\}$

We redefine the state space to make the Markov chain a Quasi- Birth - Death process(QBD).

The state space $S = \bigcup_{k=0}^{\infty}$ l(k) where k represents the level of the system.

$l(0) = \bigcup\{(0,0,a); 1 \leq a \leq m\}$
$\bigcup\{(i,j,a); 0 \leq i \leq M-1, 1 \leq j \leq N, 1 \leq a \leq m\}$

$l(k) = \{(i,j,a); kM \leq i \leq (k+1)M-1, 1 \leq j \leq N,$
$1 \leq a \leq m\}; k \geq 1$

The infinitesimal generator matrix Q of this continuous time Markov chain is a level independent quasi - birth - process (LIQBD) and is of the form

$$Q = \begin{pmatrix} A_{00} & A_{01} & & & \\ A_{10} & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & A_2 & A_1 & A_0 \\ & & & \ddots & \ddots \end{pmatrix} \tag{2.1}$$

$$A_{00} = \begin{pmatrix} B_{00} & B_{01} & & & & \\ B_{10} & B_1 & C_1 & & & \\ B_{20} & & B_1 & C_1 & & \\ & \ddots & \ddots & \ddots & & \\ B_{M-2,0} & & & & B_1 & C_1 \\ B_{M-1,0} & & & & & B_1 \end{pmatrix}$$

$$A_{10} = \begin{pmatrix} B_{M,0} & & & & \\ & B_2 & & & \\ & & B_2 & & \\ & & & \ddots & \\ & & & & B_2 \end{pmatrix}$$

$$B_{00} = \begin{pmatrix} D_0 & D_1 & & & & \\ \mu I_m & D_0-\mu I_m & D_1 & & & \\ & \mu I_m & D_0-\mu I_m & D_1 & & \\ & & \ddots & & \ddots & \\ & & & \mu I_m & D_0-\mu I_m & D_1 \\ & & & & \mu I_m & F \end{pmatrix}$$

$F = D_0 + (1-q)\Sigma_{i=1}^{m} E_i D_1 v_i - \mu I_m$

E_i - square matrix of order m with 1 on position (i,i) and zeros elsewhere.

v_i - square matrix of order m with 1 on i^{th} column and zeros elsewhere.
All entries in the matrix A_{01} of order $(MN+1)m \times MNm$ and A_0 of order MNm are zero except the south - west corner entry C_1. B_{01} of order $(N+1)m \times Nm$ and C_1 of order Nm are zero other than the south - east corner entry qD_1.

$A_2 = I_M \otimes B_2$

All entries in the matrix A_{01} of order$(N+1)m \times Nm$ and C_1 of order Nm are zero other than the south - east corner entry qD_1.

$$A_1 = \begin{pmatrix} B_1 & C_1 & & & & \\ & B_1 & C_1 & & & \\ & & B_1 & C_1 & & \\ & & & & & \\ & & & & B_1 & C_1 \\ & & & & & B_1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} D_0-\mu I_m & D_1 & & & & & \\ & D_0-\mu I_m & D_1 & & & & \\ & & \ddots & \ddots & & & \\ & & D_0-\mu I_m & D_1 & & & \\ & & \mu I_m & D_0-\mu I_m & D_1 & & \\ & & \ddots & \ddots & & & \\ & & & \mu I_m & D_0-\mu I_m & D_1 \\ & & & & \mu I_m & F \end{pmatrix}$$

$$B_{M,0} = \begin{pmatrix} O & \mu I_m \otimes I_{L+1} \\ O & O \end{pmatrix}$$

$$B_2 = \begin{pmatrix} O & \mu I_m \otimes I_{L+1} \\ O & O \end{pmatrix}$$

For $i = 1, 2, ..M-1$,

$$B_{i0} = \begin{pmatrix} O_{(L+1)m\times im} & \mu I_m \otimes I_{L+1} & O \\ O & O & O \end{pmatrix}$$

The dimensions of the matrices A_{00}, A_{10}, A_2, A_1, A_0, B_{00}, B_{01}, B_{i0}; for $i = 1, 2, ...M$, B_1, B_2, C_1 are $(MN+1)m$, $MNm \times (MN+1)m$, MNm, MNm, MNm, $(N+1)m$, $(N+1)m \times Nm, Nm \times (N+1)m$, Nm, Nm, Nm respectively.

3. Steady- state Analysis

This section discusses the steady state analysis of this queuing model by first constructing the criterion for stability of the queuing system.

3.1 Criterion for Stability

In this section the stability criterion of this queuing model is obtained using the drift condition developed by Neuts [8]. Let $\pi = (\pi_1, \pi_2, ..., \pi_M)$ be the steady state probability vector of the generator $A = A_0 + A_1 + A_2$

$$A = \begin{pmatrix} B_2^{'} & C_1 & & & & \\ O & B_2^{'} & C_1 & & & \\ O & O & B_2^{'} & C_1 & & \\ & & & & & \\ O & O & O & & B_2^{'} & C_1 \\ C_1 & O & O & O & O & B_2^{'} \end{pmatrix}$$

$B_2^{'} = B_2 + B_1$ is of dimension N.
π can be determined using $\pi A = 0$ and $\pi e = 1$
This queuing system is stable if and only if

$$\pi_{MN}(qD_1)e < \pi(I_M \otimes B_2)e \tag{3.1}$$

3.2 Stationary Probability Vector

Assume that stability criterion holds. Let y be the stationary probability vector of the generator matrix Q given in (2.1)
Then y satisfies the conditions $yQ = 0$, $ye = 1$

y can be partitioned as $y = (y_0, y_1, y_2....)$

$y_0 = (y_{00a}; 1 \leq a \leq m, y_{ija}; 0 \leq i \leq M-1, 1 \leq j \leq N, 1 \leq a \leq m)$

$y_1 = (y_{ija}; M \leq i \leq 2M-1, 1 \leq j \leq N, 1 \leq a \leq m)$

$y_2 = (y_{ija}; 2M \leq i \leq 3M-1, 1 \leq j \leq N)$ and so on.

The subvectors y_i are given by $y_i = y_1 R^{i-1}$; $i \geq 2$ where R is the minimal nonnegative solution to the matrix quadratic equation$R^2 A_2 + RA_1 + A_0 = O$

y_0 and y_1 are obtained by the boundary equations

$y_0 A_{00} + y_1 A_{10} = 0$

$y_0 A_{01} + y_1 (A_1 + RA_2) = 0$

and the normalizing condition $y_0 e + y_1 (I-R)^{-1} e = 1$
To determine the vector y, the rate matrix R can be computed using the following iterative algorithm.

4. Computation of Rate Matrix

4.1 Iterative algorithm

Step 1: $R(0) = 0$

Step 2: $R(n+1) = A_0(-A_1)^{-1} + R^2(n)A_2(-A_1)^{-1}, n = 0, 1, 2...$

Continue **Step 2** until $R(n+1)$ close to $R(n)$. ie., $\|R(n+1) - R(n)\|_\infty < \varepsilon$

5. Performance Characteristics

Some performance measures of the system are listed in this section to bring out the qualitative nature of the system under study.

1. Mean number of customers in the finite buffer

$$E_b = \sum_{i=0}^{\infty} \sum_{j=1}^{N} \sum_{a=1}^{m} j y_{ija}$$

2. Mean number of customers in the pool

$$E_p = \sum_{i=1}^{\infty}\sum_{j=1}^{N}\sum_{a=1}^{m} i y_{ija}$$

3. Mean number of customers in the system

$$= E_b + E_p$$

4. Expected rate of joining pool

$$= q\lambda \sum_{i=0}^{\infty}\sum_{a=1}^{m} y_{iNa}$$

5. Expected rate of transfer from pool to buffer

$$T_t = \mu \sum_{i=1}^{\infty}\sum_{j=1}^{L}\sum_{a=1}^{m} y_{ija}$$

6. Probability that the system is idle

$$= \sum_{a=1}^{m} y_{00a}$$

7. Average departure rate of customers who got service =(rate of departure) Probability of at least one customer in the system

$$\lambda_{out} = \mu \sum_{i=0}^{\infty}\sum_{a=1}^{m} y_{i1a}$$

6. Numerical Results

In order to reveal the qualitative nature of the queuing system under study we provide some examples. For the arrival process we consider the following matrices for D_0 and D_1

- MAP with negative correlation (MNA)

$$D_0 = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -450.5 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} 0 & 0 & 0 \\ .02 & 0 & 1.98 \\ 445.995 & 0 & 4.505 \end{pmatrix}$$

- MAP with positive correlation (MPA)

$$D_0 = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -450.5 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1.98 & 0 & .02 \\ 4.505 & 0 & 445.995 \end{pmatrix}$$

These two arrival processes are normalized so as to have an arrival rate of 1 [11].

Example 6.1. *In this example we analyze the effect of the parameter q on some performance measures with different arrival processes. By fixing the parameters as $N = 12$, $\mu = 4$, $L = 3$ the impact of q for both (MNA) and (MPA) is shown in Table 1. When q increases,*

- The expected number of customers E_b in the buffer decreases for (MNA) and increases for (MPA).
- The expected number of customers E_p in the pool increases for (MNA) and decreases for (MPA).
- The expected rate of customers transferred from pool to buffer T_t increases (MNA) and decreases for (MPA).
- The average rate of customers leaving the system after service completion λ_{out} decreases for(MNA) and increases for(MPA).

	q	E_b	E_p	T_t	λ_{out}
	.10	1.134839	0.000758	0.000137	0.646396
	.20	1.134838	0.000829	0.000142	0.646393
MNA	.30	1.134833	0.000876	0.000144	0.646392
	.40	1.134825	0.000914	0.000146	0.646391
	.50	1.134814	0.000946	0.000147	0.646390
MPA	0.01	0.4298	0.0231	0.00048	0.8466
	.02	0.3699	0.0130	0.00037	0.8511
	.03	0.3495	0.0096	0.00033	0.8527
	.04	0.3392	0.0078	0.00032	0.8534
	.05	0.3330	0.0068	0.00031	0.8539

Table 1. The effect of q on various performance measures

Example 6.2. *In this example we assign following values to the parameters $\mu = 4$, $N = 12$, $q = .5$, $C_s = \$10$, $C_b = \$500$, $C_p = \$10$, $C_t = \$10$. Table 2 shows the optimal values of L and M that minimizes the total expected cost (TEC).*

MNA	L	1	2	3	4	5
	M	2	3	4	5	6
	TEC	573.83	573.81	573.88	573.96	574.04

Table 2. The effect of L and M on Total Expected cost

Example 6.3. *In this example fix the parameters as $\mu = 4$, $N = 12$, $q = .003$, $C_s = \$180$, $C_b = \$10$, $C_p = \$10$, $C_t = \$10$. Table 3 shows the optimal values of L and M that minimizes the total expected cost (TEC).*

MPA	L	1	2	3	4	5
	M	2	3	4	5	6
	TEC	156.510	156.456	156.455	156.502	156.594

Table 3. The effect of L and M on Total Expected cost

7. Conclusion

In this paper, we studied a MAP/M/1 queue with M-Policy of transfer of customers from pool to buffer. Here we analysed the effect of probability of joining the pool on certain performance measures for both MAP with positive correlation and MAP with negative correlation. The optimality of the threshold level is also computed. The M-Policy for transfer of customers from pool to buffer will motivate the customers to join the pool when the buffer reaches its capacity.

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