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Fixed points of almost generalized weakly contractive maps with rational expressions in S-metric spaces

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Abstract
In this paper, we prove the existence and uniqueness of fixed points of (φ,ψ)-almost generalized weakly contractive maps with rational expressions in S-metric spaces. Also, we prove the existence and uniqueness of fixed points of α-admissible almost weak ψ-contraction maps with rational expressions in S-metric spaces. Our results extend the results of Jaggi [16] , Dass and Gupta [10] to S-metric spaces. Also our results extend and generalize the results of Sedghi, Shobe and Aliouche [21]. Supporting examples are provided to our results.

Keywords
S-metric space, fixed point, almost generalized weakly contractive maps, α-admissible maps.

AMS Subject Classification
47H10, 54H25.

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1. Introduction and Preliminaries

The study of fixed point theory in metric spaces is very interesting area in Analysis. Several generalizations of metric spaces have been obtained by many authors who established the existence of fixed points and common fixed points of various contractive and contraction maps. For more works on this literature, we refer [2], [7], [8], [11], [14].

In 1975, Dass and Gupta [10] extended the Banach contraction principle using rational expressions as follows.

Theorem 1.1. [10] *Let (X,d) be a complete metric space and $T: X \to X$ be a self map. If there exists $\alpha,\beta \geq 0$ with $\alpha+\beta<1$ satisfying*
$d(Tx,Ty) \leq \alpha \frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)} + \beta d(x,y)$
for all $x,y \in X$, then T has a unique fixed point in X.

In 1977, Jaggi [16] introduced rational type contraction mappings and proved the existence of fixed points of such mappings.

Theorem 1.2. [16] *Let T be a continuous self map defined on a complete metric space (X,d). Suppose that T satisfies the following condition: there exist $\alpha,\beta \in [0,1)$ with $\alpha+\beta<1$ such that*

$$d(Tx,Ty) \leq \alpha \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x,y) \tag{1.1}$$

for all $x,y \in X$ with $x \neq y$. Then T has a fixed point in X. The map T which satisfies (1.1) is called as 'Jaggi contraction map' on X.

Later, many authors worked in this direction to establish fixed points and common fixed points of mappings involving rational expressions. These are some of the references in this direction [1], [3], [4], [9], [15].

In 2012 Samet, Vetro and Vetro [20] introduced α-admissible maps on metric spaces as follows.

Definition 1.3. [20] *Let (X,d) be a metric space and $T : X \to X$ be a self map and $\alpha : X\times X \to [0,\infty)$. We say that T is α-admissible if $x,y \in X$, $\alpha(x,y) \geq 1 \implies \alpha(Tx,Ty) \geq 1$.*

Let $\Psi_1 = \{\psi : [0,\infty) \to [0,\infty)/$ (i) ψ is continuous
(ii) ψ is non-decreasing, and
(iii) $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for each $t > 0$ $\}$.

Remark 1.4. [4] *Any function $\psi \in \Psi_1$ satisfies $\lim_{n\to\infty} \psi^n(t) = 0$ and $\psi(t) < t$ for any $t > 0$.*

Theorem 1.5. [20] *Let (X,d) be a complete metric space and $T : X \to X$ be a self map. Suppose that there exists $\psi \in \Psi_1$ such that*

$$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y)) \text{ for all } x,y \in X.$$

Suppose that

(i) T is α-admissible

(ii) there exists $x_0 \in X$ such that $x_0 \geq Tx_0$ and

(iii) T is continuous.

Then T has a fixed point in X.

Definition 1.6. [17] *Let X be a nonempty set and $T : X \to X$ be a self map with nonempty fixed point set $F(T)$. Then T is said to satisfy property(P) if $F(T) = F(T^n)$ for all $n \in \mathbb{N}$.*

In 2012, Sedghi, Shobe and Aliouche [21] introduced the concept of S-metric spaces.

Definition 1.7. [21] *Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions: for each $x,y,z,a \in X$*

(S1) $S(x,y,z) \geq 0$,

(S2) $S(x,y,z) = 0$ *if and only if* $x = y = z$ *and*

(S3) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) is called an S-metric space.

The following are the examples of S-metric space.

Example 1.8. [21] $S(x,y,z) = ||x-z|| + ||y-z||$ for all $x, y, z \in \mathbb{R}^n$ and $||.||$ be a norm on $\mathbb{R}^n$.

Example 1.9. [5] $S(x,y,z) = \max\{|x-z|, |y-z|\}$ for all $x, y, z \in \mathbb{R}$.

Example 1.10. [21] $S(x,y,z) = ||y+z-2x|| + ||y-z||$ for all $x,y,z \in \mathbb{R}^n$ and $||.||$ be a norm on $\mathbb{R}^n$.

Example 1.11. [21] $S(x,y,z) = d(x,z) + d(y,z)$ for all $x,y,z \in X$ where d is a metric on a nonempty set X.

Example 1.12.

$$S(x,y,z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x,y,z\} & \text{otherwise.} \end{cases}$$

for all $x,y,z \in \mathbb{R}^+$, the set of all positive real numbers.

Definition 1.13. [21] *Let (X,S) be an S-metric space. Then we have the following:*

(i) $S(x,x,y) = S(y,y,x)$, for all $x,y \in X$.

(ii) a sequence $\{x_n\}$ in X converges to a point $x \in X$ if and only if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n \geq n_0$, $S(x_n,x_n,x) < \varepsilon$ and we denote it by $\lim_{n\to\infty} x_n = x$.

(iii) a sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in N$ such that $S(x_n,x_n,x_m) < \varepsilon$ for all $n,m \geq n_0$.

(iv) (X,S) is said to be complete if each Cauchy sequence in X is convergent.

(v) if the sequence $\{x_n\}$ in X converges to x, then x is unique.

(vi) if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n,x_n,y_n) = S(x,x,y)$

Lemma 1.14. [6] *Let (X,S) be an S-metric space. If a sequence $\{x_n\}$ in X converges to x and $S(x_n,x_n,y_n) \to 0$ then $y_n \to x$.*

Theorem 1.15. [21] *Let (X,S) be an S-metric space. A map $F : X \to X$ be a contraction. i.e., there exists a constant $0 \leq L < 1$ such that*

$$S(Fx,Fx,Fy) \leq LS(x,x,y) \tag{1.2}$$

for all $x,y \in X$. Then F has a unique fixed point u in X.

For more literature on S-metric spaces, we refer [5], [6], [12], [13], [19], [21], [22].

Lemma 1.16. [5], [12] *Let (X,S) be an S-metric space and $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = 0. \tag{1.3}$$

If $\{x_n\}$ is not a Cauchy sequence, then there exists an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $m_k > n_k > k$ such that

$$S(x_{m_k},x_{m_k},x_{n_k}) \geq \varepsilon, \quad S(x_{m_k-1},x_{m_k-1},x_{n_k}) < \varepsilon \tag{1.4}$$

and

(i) $\lim_{k\to\infty} S(x_{m_k},x_{m_k},x_{n_k}) = \varepsilon$

(ii) $\lim_{k\to\infty} S(x_{m_k-1},x_{m_k-1},x_{n_k}) = \varepsilon$

(iii) $\lim_{k\to\infty} S(x_{m_k},x_{m_k},x_{n_k-1}) = \varepsilon$

(iv) $\lim_{k\to\infty} S(x_{m_k-1},x_{m_k-1},x_{n_k-1}) = \varepsilon$

(v) $\lim_{k\to\infty} S(x_{n_k-1}, x_{n_k-1}, x_{m_k+1}) = \varepsilon$

(vi) $\lim_{k\to\infty} S(x_{n_k}, x_{n_k}, x_{m_k+1}) = \varepsilon$.

In Section 2 of this paper, we prove the existence and uniqueness of fixed points of (φ,ψ)-almost generalized weakly contractive maps with rational expressions in S-metric spaces (Theorem 2.2). This theorem extends the result of Dass and Gupta [10] to S-metric spaces and generalize the result of Sedghi, Shobe and Aliouche [21]. In Section 3, we define α-admissible maps on S-metric spaces and prove the existence and uniqueness of fixed points of α-admissible almost weak ψ-contraction maps with rational expressions in S-metric spaces (Theorem 3.4). This theorem extends Jaggi's theorem [16] to S-metric spaces. Supporting examples are provided for our results.

2. Fixed points of (φ,ψ)-almost generalized weakly contractive maps with rational expressions

We use the following notation introduced by Chandok, Choudhury and Metiya [9].
Let $\Psi = \{\psi : [0,\infty) \to [0,\infty)$ / for any sequence $\{x_n\}$ in $[0,\infty)$ with $x_n \to t$, $t > 0$, $\liminf_{n\to\infty} \psi(x_n) > 0\}$.
Through out this paper, let Φ denote the class of all altering distance functions [18]. i.e.,
$\Phi = \{\varphi : [0,\infty) \to [0,\infty)$ / (i) φ is continuous
(ii) φ is monotone increasing, and
(iii) $\varphi(t) = 0$ if and only if $t = 0\}$.

Definition 2.1. *Let (X,S) be an S-metric space. Let $T : X \to X$ be a self map. Suppose that there exist $L \geq 0$, $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\varphi(S(Tx,Ty,Tz)) \leq \varphi(M(x,y,z)) - \psi(M(x,y,z)) + LN(x,y,z) \tag{2.1}$$

where

$$M(x,y,z) = \max\Big\{S(x,y,z), \frac{S(y,y,Ty)[1+S(x,x,Tx)]}{1+S(x,y,z)}, \frac{S(z,z,Tz)[1+S(x,x,Tx)]}{1+S(x,y,z)}, \frac{S(z,z,Tz)[1+S(y,y,Ty)]}{1+S(x,y,z)}, \frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)}, \frac{1}{3}\frac{[S(z,z,Ty)+S(y,y,Tz)][1+S(z,z,Tx)]}{1+S(x,y,z)}\Big\}$$

and

$$N(x,y,z) = \min\Big\{S(x,x,Tx), S(y,y,Tx), S(z,z,Tx), \frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)}\Big\},$$

for all $x,y,z \in X$. Then we say that T is (φ,ψ)-almost generalized weakly contractive map on X.

Theorem 2.2. *Let (X,S) be a complete S-metric space. Let $T : X \to X$ be (φ,ψ)-almost generalized weakly contractive map. Then T has a unique fixed point 'u' in X. Moreover, T is continuous at 'u'.*

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for $n = 0,1,2,\ldots$. If $x_n = x_{n+1}$ for some $n = 0,1,2,\ldots$, then x_n is a fixed point of T and hence we are through. Now we assume that $x_n \neq x_{n+1}$ for each $n = 0,1,2,\ldots$. We show that $S(x_{n+1},x_{n+1},x_{n+2}) \leq S(x_n,x_n,x_{n+1})$ for all $n = 0,1,2,\ldots$. Suppose if possible, there exists some $n = 0,1,2,\ldots$ such that

$$S(x_n,x_n,x_{n+1}) < S(x_{n+1},x_{n+1},x_{n+2}). \tag{2.2}$$

Now, we consider

$$\begin{aligned}\varphi(S(x_{n+1},x_{n+1},x_{n+2})) &= \varphi(S(Tx_n,Tx_n,Tx_{n+1}))\\ &\leq \varphi(M(x_n,x_n,x_{n+1})) - \psi(M(x_n,x_n,x_{n+1})) + LN(x_n,x_n,x_{n+1}),\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}M(x_n,x_n,x_{n+1}) &= \max\Big\{S(x_n,x_n,x_{n+1}), \frac{S(x_n,x_n,Tx_n)[1+S(x_n,x_n,Tx_n)]}{1+S(x_n,x_n,x_{n+1})}, \frac{S(x_{n+1},x_{n+1},Tx_{n+1})[1+S(x_n,x_n,Tx_n)]}{1+S(x_n,x_n,x_{n+1})},\\ &\qquad \frac{1}{3}\frac{[S(x_{n+1},x_{n+1},Tx_n)+S(x_n,x_n,Tx_{n+1})][1+S(x_{n+1},x_{n+1},Tx_n)]}{1+S(x_n,x_n,x_{n+1})}\Big\}\\ &= \max\Big\{S(x_n,x_n,x_{n+1}), S(x_{n+1},x_{n+1},x_{n+2}), \frac{1}{3}\frac{S(x_n,x_n,x_{n+2})}{[1+S(x_n,x_n,x_{n+1})]}\Big\}\\ &\leq \max\Big\{S(x_{n+1},x_{n+1},x_{n+2}), \tfrac{1}{3}S(x_n,x_n,x_{n+2})\Big\}\\ &\leq \max\Big\{S(x_n,x_n,x_{n+1}), \tfrac{1}{3}[2S(x_{n+2},x_{n+2},x_{n+1}) + S(x_{n+2},x_{n+2},x_{n+1})]\Big\}\\ &= S(x_{n+2},x_{n+2},x_{n+1})\end{aligned}$$

and $N(x_n,x_n,x_{n+1}) = 0$.
Hence from (2.3), we get

$$\begin{aligned}\varphi(S(x_{n+1},x_{n+1},x_{n+2})) &\leq \varphi(S(x_{n+2},x_{n+2},x_{n+1})) - \psi(M(x_n,x_n,x_{n+1}))\\ &< \varphi(S(x_{n+2},x_{n+2},x_{n+1})),\end{aligned}$$

a contradiction. Hence $S(x_{n+1},x_{n+1},x_{n+2}) \leq S(x_n,x_n,x_{n+1})$ for each $n = 0,1,2,\ldots$. Therefore the sequence $\{r_n\}$, $r_n = S(x_n,x_n,x_{n+1})$ for $n = 0,1,2,\ldots$ is a decreasing sequence of non-negative real numbers. Hence there exists $r \geq 0$ such that $\lim_{n\to\infty} r_n = r$. We now show that $r = 0$. Suppose if possible $r > 0$.
We consider

$$\begin{aligned}\varphi(S(x_{n+1},x_{n+1},x_{n+2})) &\leq \varphi(M(x_n,x_n,x_{n+1}))\\ &\quad -\psi(M(x_n,x_n,x_{n+1}))\\ &\quad +LN(x_n,x_n,x_{n+1})\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}M(x_n,x_n,x_{n+1}) &= \max\Big\{S(x_n,x_n,x_{n+1}), S(x_{n+1},x_{n+1},x_{n+2}), \frac{1}{3}\frac{S(x_n,x_n,x_{n+2})}{[1+S(x_n,x_n,x_{n+1})]}\Big\}\\ &\leq \max\Big\{S(x_n,x_n,x_{n+1}), \frac{1}{3}\frac{[2S(x_n,x_n,x_{n+1})+S(x_{n+1},x_{n+1},x_{n+2})]}{[1+S(x_n,x_n,x_{n+1})]}\Big\}\\ &\leq \max\Big\{S(x_n,x_n,x_{n+1}), \frac{1}{3}\frac{3S(x_n,x_n,x_{n+1})}{[1+S(x_n,x_n,x_{n+1})]}\Big\}\\ &= S(x_n,x_n,x_{n+1})\end{aligned}$$

and hence

$$\lim_{n\to\infty} M(x_n,x_n,x_{n+1}) \leq \lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = r. \tag{2.5}$$

We have
$S(x_n,x_n,x_{n+1}) \leq M(x_n,x_n,x_{n+1})$ for each $n = 0,1,2,\dots$.
On taking limits as $n \to \infty$, we get
$\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) \leq \lim_{n\to\infty} M(x_n,x_n,x_{n+1})$. Thus

$$r \leq \lim_{n\to\infty} M(x_n,x_n,x_{n+1}). \tag{2.6}$$

From (2.5) and (2.6), we get
$\lim_{n\to\infty} M(x_n,x_n,x_{n+1}) = r$ and $N(x_n,x_n,x_{n+1}) = 0$.
On taking limit supremum as $n \to \infty$, in (2.4), we get
$\varphi(r) \leq \varphi(r) - \liminf_{n\to\infty} \psi(M(x_n,x_n,x_{n+1}))$.
By using the property of ψ, we get $\varphi(r) < \varphi(r)$, a contradiction. Hence $r = 0$. i.e., $\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = 0$.
Now we show that $\{x_n\}$ is a Cauchy sequenc in X. Suppose that $\{x_n\}$ is not a Cauchy sequence in X. Then there exists an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k$ such that

$$S(x_{m_k},x_{m_k},x_{n_k}) \geq \varepsilon, \quad S(x_{m_k-1},x_{m_k-1},x_{n_k}) < \varepsilon. \tag{2.7}$$

Now, we consider
$\varphi(S(x_{m_k},x_{m_k},x_{n_k})) = \varphi(S(Tx_{m_k-1},Tx_{m_k-1},Tx_{n_k-1}))$ and using the inequality (2.1) we get

$$\begin{aligned}\varphi(S(x_{m_k},x_{m_k},x_{n_k})) \leq \varphi(S(x_{m_k-1},x_{m_k-1},x_{n_k-1}))\\ -\psi(S(x_{m_k-1},x_{m_k-1},x_{n_k-1}))\\ +LN(x_{m_k-1},x_{m_k-1},x_{n_k-1})\end{aligned} \tag{2.8}$$

where
$$\begin{aligned}M(x_{m_k-1},x_{m_k-1},x_{n_k-1}) = \max\{&S(x_{m_k-1},x_{m_k-1},x_{n_k-1}),\\ &\tfrac{S(x_{m_k-1},x_{m_k-1},x_{m_k})[1+S(x_{m_k-1},x_{m_k-1},x_{m_k})]}{1+S(x_{m_k-1},x_{m_k-1},x_{n_k-1})},\\ &\tfrac{S(x_{n_k-1},x_{n_k-1},x_{n_k})[1+S(x_{m_k-1},x_{m_k-1},x_{m_k})]}{1+S(x_{m_k-1},x_{m_k-1},x_{n_k-1})},\\ &\tfrac{1}{3}\tfrac{[S(x_{n_k-1},x_{n_k-1},x_{m_k})+S(x_{m_k-1},x_{m_k-1},x_{n_k})][1+S(x_{n_k-1},x_{n_k-1},x_{m_k})]}{1+S(x_{m_k-1},x_{m_k-1},x_{n_k-1})}\}.\end{aligned}$$
On letting $k \to \infty$, we get
$\lim_{k\to\infty} M(x_{m_k-1},x_{m_k-1},x_{n_k-1}) = \max\{\varepsilon,0,0,\frac{2\varepsilon(1+\varepsilon)}{3(1+\varepsilon)}\} = \varepsilon$ and
$$\begin{aligned}N(x_{m_k-1},x_{m_k-1},x_{n_k-1}) = \min\{&S(x_{m_k-1},x_{m_k-1},x_{m_k}),\\ &S(x_{n_k-1},x_{n_k-1},x_{m_k}),\\ &\tfrac{S(x_{m_k-1},x_{m_k-1},x_{m_k})[1+S(x_{m_k-1},x_{m_k-1},x_{m_k})]}{1+S(x_{m_k-1},x_{m_k-1},x_{n_k-1})}\}.\end{aligned}$$
On letting $k \to \infty$, we get $\lim_{k\to\infty} N(x_{m_k-1},x_{m_k-1},x_{n_k-1}) = 0$.
On taking limit supremum as $k \to \infty$ in (2.8), we get
$$\begin{aligned}\limsup_{k\to\infty} \varphi(S(x_{m_k},x_{m_k},x_{n_k})) \leq &\limsup_{k\to\infty} \varphi(M(x_{m_k-1},x_{m_k-1},x_{n_k-1}))\\ &-\liminf_{k\to\infty} \psi(M(x_{m_k-1},x_{m_k-1},x_{n_k-1}))\\ &+L\limsup_{k\to\infty} N(x_{m_k-1},x_{m_k-1},x_{n_k-1}).\end{aligned}$$
By the property of ψ, we get $\varphi(\varepsilon) < \varphi(\varepsilon)$ which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. We now show that $Tu = u$. Suppose that $u \neq Tu$.
We consider
$$\begin{aligned}\varphi(S(x_{n+1},x_{n+1},Tu)) &= \varphi(S(Tx_n,Tx_n,Tu))\\ &\leq \varphi(M(x_n,x_n,u)) - \psi(M(x_n,x_n,u))\\ &\quad +LN(x_n,x_n,u)\end{aligned} \tag{2.9}$$
where
$$\begin{aligned}M(x_n,x_n,u) = \max\{&S(x_n,x_n,u), \tfrac{S(x_n,x_n,Tx_n)[1+S(x_n,x_n,Tx_n)]}{1+S(x_n,x_n,u)},\\ &\tfrac{S(u,u,Tu)[1+S(x_n,x_n,Tx_n)]}{1+S(x_n,x_n,u)}, \tfrac{S(u,u,Tu)[1+S(x_n,x_n,Tx_n)]}{1+S(x_n,x_n,u)},\\ &\tfrac{S(x_n,x_n,Tx_n)[1+S(x_n,x_n,Tx_n)]}{1+S(x_n,x_n,u)},\\ &\tfrac{1}{3}\tfrac{[S(u,u,Tx_n)+S(x_n,x_n,Tu)][1+S(u,u,Tx_n)]}{1+S(x_n,x_n,u)}\}.\end{aligned}$$
On taking limits as $n \to \infty$, we get
$\lim_{n\to\infty} M(x_n,x_n,u) = S(u,u,Tu)$ and
$$\begin{aligned}N(x_n,x_n,u) = \min\{&S(x_n,x_n,Tx_n), S(x_n,x_n,Tx_n), S(u,u,Tx_n),\\ &\tfrac{S(x_n,x_n,Tx_n)[1+S(x_n,x_n,Tx_n)]}{1+S(x_n,x_n,u)}\}.\end{aligned}$$
On letting $n \to \infty$, we get $\lim_{n\to\infty} N(x_n,x_n,u) = 0$.
On taking limit supremum as $n \to \infty$ in (2.9), we get
$$\begin{aligned}\limsup_{n\to\infty} \varphi(S(x_{n+1},x_{n+1},Tu)) \leq &\limsup_{n\to\infty} \varphi(M(x_n,x_n,u))\\ &-\liminf_{n\to\infty} \psi(M(x_n,x_n,u))\\ &+L\limsup_{n\to\infty} N(x_n,x_n,u).\end{aligned}$$
This implies $\varphi(S(u,u,Tu)) < \varphi(S(u,u,Tu))$, a contradiction. Hence $Tu = u$. That is u is a fixed point of T.
We now prove that T is continuous at 'u'. We consider the sequence $\{x_n\}$ in X such that $x_n \to u$ as $n \to \infty$. Then

$$\begin{aligned}\varphi(S(Tu,Tu,Tx_n)) \leq \varphi(M(u,u,x_n)) - \psi(M(u,u,x_n))\\ +LN(u,u,x_n)\end{aligned} \tag{2.10}$$

where
$$\begin{aligned}M(u,u,x_n) = \max\{&S(u,u,x_n), \tfrac{S(u,u,Tu)[1+S(u,u,Tu)]}{1+S(u,u,x_n)},\\ &\tfrac{S(x_n,x_n,Tx_n)[1+S(u,u,Tu)]}{1+S(u,u,x_n)}, \tfrac{S(x_n,x_n,Tx_n)[1+S(u,u,Tu)]}{1+S(u,u,x_n)},\\ &\tfrac{S(u,u,Tu)[1+S(u,u,Tu)]}{1+S(u,u,x_n)},\\ &\tfrac{1}{3}\tfrac{[S(x_n,x_n,Tu)+S(u,u,Tx_n)][1+S(x_n,x_n,Tu)]}{1+S(u,u,x_n)}\}.\end{aligned}$$
Now taking the limits as $n \to \infty$, we get
$$\begin{aligned}\lim_{n\to\infty} M(u,u,x_n) &= \max\{0,0,\lim_{n\to\infty} S(Tu,Tu,Tx_n),\\ &\qquad \lim_{n\to\infty} S(Tu,Tu,Tx_n),0,\\ &\qquad \tfrac{1}{3}\lim_{n\to\infty} S(Tu,Tu,Tx_n)\}\\ &= \lim_{n\to\infty} S(Tu,Tu,Tx_n)\end{aligned}$$
and
$$\begin{aligned}N(u,u,x_n) = \min\{&S(u,u,Tu), S(u,u,Tu), S(x_n,x_n,Tu),\\ &\tfrac{S(u,u,Tu)[1+S(u,u,Tu)]}{1+S(u,u,x_n)}\}.\end{aligned}$$
On taking limits as $n \to \infty$, we get $\lim_{n\to\infty} N(u,u,x_n) = 0$.
From (2.10), we get
$$\begin{aligned}\limsup_{n\to\infty} \varphi(S(Tu,Tu,Tx_n)) \leq &\limsup_{n\to\infty} S(Tu,Tu,Tx_n)\\ &-\liminf_{n\to\infty} \psi(S(Tu,Tu,Tx_n))\end{aligned}$$
which implies that

$$\liminf_{n\to\infty} \psi(S(Tu,Tu,Tx_n)) \leq 0. \tag{2.11}$$

Suppose if possible $\lim_{n\to\infty} S(Tu,Tu,Tx_n) > 0$. Then by the property of ψ, we have $\liminf_{n\to\infty} \psi(S(Tu,Tu,Tx_n)) > 0$, a contradiction to (2.11). Therefore $\lim_{n\to\infty} S(Tu,Tu,Tx_n) = 0$.

That is $\lim_{n\to\infty} Tx_n = Tu$. Hence T is continuous at u.
We now prove the uniqueness of fixed point.
Suppose if possible v is another fixed point of T such that $u \neq v$. Then $S(u,u,v) > 0$. We consider

$$\varphi(S(Tu,Tu,Tv)) \leq \varphi(M(u,u,v)) - \psi(M(u,u,v)) + LN(u,u,v) \tag{2.12}$$

where
$M(u,u,v) = S(u,u,v)$ and $N(u,u,v) = 0$.
Hence from (2.12), we get
$$\begin{aligned}\varphi(S(u,u,v)) &\leq \varphi(S(u,u,v)) - \psi(S(u,u,v)) \\ &< \varphi(S(u,u,v)),\end{aligned}$$
a contradiction (since if we define a sequence $\{x_n\}$ by $x_n = S(u,u,v)$ for each $n = 0,1,2,...$ then $x_n \to S(u,u,v) > 0$ as $n \to \infty$. Hence $\liminf_{n\to\infty} \psi(x_n) > 0$.
That is $\liminf_{n\to\infty} \psi(S(u,u,v)) > 0$ so that $\psi(S(u,u,v)) > 0$).
Hence $S(u,u,v) = 0$. Implies $u = v$. □

Theorem 2.3. *Under the hypothesis of Theorem 2.2, T has property(P).*

Proof. From Theorem 2.2, T has a fixed point. Therefore $F(T^n) \neq \phi$. Now let $n > 1$ and $u \in F(T^n)$. So $T^n u = u$. We now show that $u \in F(T)$.
We consider
$$\begin{aligned}\varphi(S(u,u,Tu)) &= \varphi(S(T^n u, T^n u, T^{n+1}u)) \\ &= \varphi(S(TT^{n-1}u, TT^{n-1}u, TT^n u)) \\ &\leq \varphi(M(T^{n-1}u, T^{n-1}u, T^n u)) \\ &\quad -\psi(M(T^{n-1}u, T^{n-1}u, T^n u)) \\ &\quad +LN(T^{n-1}u, T^{n-1}u, T^n u).\end{aligned} \tag{2.13}$$

where
$M(T^{n-1}u, T^{n-1}u, T^n u) = \max\{S(T^{n-1}u, T^{n-1}u, u), S(u,u,Tu)\}$ and
$N(T^{n-1}u, T^{n-1}u, u) = 0$.
If maximum is $S(T^{n-1}u, T^{n-1}u, u)$ then we get
$$\begin{aligned}\varphi(S(u,u,Tu)) &\leq \varphi(S(T^{n-1}u, T^{n-1}u, u)) \\ &\quad - \psi(S(T^{n-1}u, T^{n-1}u, u))\end{aligned}$$
which implies that
$$\begin{aligned}\varphi(S(T^n u, T^n u, T^{n+1}u)) &\leq \varphi(S(T^{n-1}u, T^{n-1}u, T^n u)) \\ &\quad - \psi(S(T^{n-1}u, T^{n-1}u, T^n u)) \\ &\leq \varphi(S(T^{n-2}u, T^{n-2}u, T^{n-1}u)) \\ &\quad - \psi(S(T^{n-2}u, T^{n-2}u, T^{n-1}u)) \\ &\quad - \psi(S(T^{n-1}u, T^{n-1}u, T^n u)) \\ &\quad . \\ &\quad . \\ &\quad . \\ &\leq \varphi(S(u,u,Tu)) \\ &\quad - \sum_{k=0}^{n-1} \psi(S(T^{n-k-1}u, T^{n-k-1}u, T^{n-k}u)).\end{aligned}$$
That is $\varphi(S(u,u,Tu)) \leq \varphi(S(u,u,Tu)) - \sum_{k=0}^{n-1} \psi(S(T^{n-k-1}u, T^{n-k-1}u, T^{n-k}u))$.

This implies that $\sum_{k=0}^{n-1} \psi(S(T^{n-k-1}u, T^{n-k-1}u, T^{n-k}u)) = 0$.
Hence $\psi(S(T^{n-k-1}u, T^{n-k-1}u, T^{n-k}u)) = 0$ for all $0 \leq k \leq n-1$. Therefore $\psi(S(u,u,Tu)) = 0$.
If maximum is $S(u,u,Tu)$ then from (2.13), we get $\psi(S(u,u,Tu)) = 0$.
Now suppose $S(u,u,Tu) > 0$ and if $\{x_n\} = \{S(u,u,Tu)\}$ then $x_n \to S(u,u,Tu)$ as $n \to \infty$. By the property of ψ, we get $\liminf_{n\to\infty} \psi(S(u,u,Tu)) > 0$.
That is $\psi(S(u,u,Tu)) > 0$, a contradiction.
Hence $u = Tu$ so that $u \in F(T)$ and T has property(P). □

By choosing $\varphi(t) = t$ for all $t \in [0,\infty)$, in Theorem 2.2, we obtain the following corollary.

Corollary 2.4. *Let (X,S) be a complete S-metric space and $T : X \to X$ be a self map. Suppose that there exist $L \geq 0$ and $\psi \in \Psi$ such that*

$$S(Tx,Ty,Tz) \leq M(x,y,z) - \psi(M(x,y,z)) + L\,N(x,y,z) \tag{2.14}$$

for all $x,y,z \in X$ where $M(x,y,z)$ and $N(x,y,z)$ are given as in the inequality (2.1). Then T has a unique fixed point in X.

By choosing $L = 0$ in Corollary 2.4, we obtain the following corollary.

Corollary 2.5. *Let (X,S) be a complete S-metric space and $T : X \to X$ be a self map. Suppose that there exists $\psi \in \Psi$ such that*

$$S(Tx,Ty,Tz) \leq M(x,y,z) - \psi(M(x,y,z)) \tag{2.15}$$

for all $x,y,z \in X$ where $M(x,y,z)$ is given as in the inequality (2.1). Then T has a unique fixed point in X.

Corollary 2.6. *Let (X,S) be a complete S-metric space and $T : X \to X$ be a self map. Suppose that there exists $k \in [0,1)$ such that*

$$S(Tx,Ty,Tz) \leq kM(x,y,z) \tag{2.16}$$

for all $x,y,z \in X$ where $M(x,y,z)$ is given as in the inequality (2.1). Then T has a unique fixed point in X.

Proof. We define $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = (1-k)t$ for all $t \in [0,\infty)$ in the inequality (2.15). Clearly $\psi \in \Psi$. Now the conclusion follows from Corollary 2.5. □

Remark 2.7. *We obtain Theorem 1.15 as a corollary to Corollary 2.6.*

The following corollary is the version of Dass and Gupta's Theorem (Theorem 1.1) in S-metric spaces.

Corollary 2.8. *Let (X,S) be a complete S-metric space and $T:X\to X$ be a self map. Suppose that there exist α,β,γ and $\eta\in[0,1)$ with $\alpha+\beta+\gamma+\eta<1$ such that*

$$S(Tx,Ty,Tz)\le \alpha S(x,y,z)+\beta\frac{S(y,y,Ty)[1+S(x,x,Tx)]}{1+S(x,y,z)}+\gamma\frac{S(z,z,Tz)[1+S(x,x,Tx)]}{1+S(x,y,z)}+\eta\frac{S(z,z,Tz)[1+S(y,y,Ty)]}{1+S(x,y,z)} \tag{2.17}$$

for all $x,y,z\in X$. Then T has a unique fixed point in X.

Proof. We obtain the inequality (2.16) from the inequality (2.17) by choosing $k=\alpha+\beta+\gamma+\eta$. Hence the conclusion follows from Corollary 2.6. □

The following example is in support of Theorem 2.2.

Example 2.9. Let $X=[\frac{1}{4},1]$ and (X,S) be the S-metric space defined in Example 1.12.
Now we define $T:X\to X$ by

$$Tx=\begin{cases}4x^2 & \text{if } x\in[\frac{1}{4},\frac{1}{2}]\\ \frac{1}{2} & \text{if } x\in(\frac{1}{2},1]\end{cases}$$

and $\varphi,\psi:[0,\infty)\to[0,\infty)$ by $\varphi(t)=\frac{t}{2}$ for all $t\ge 0$ and

$$\psi(t)=\begin{cases}0 & \text{if } t=0\\ \frac{t^2}{2} & \text{if } t\in(0,1]\\ \frac{1}{1+2t} & \text{if } t\in(1,\infty)\end{cases}.$$

we show that T satisfies the inequality (2.1).
<u>*Case*</u> (i): Let $x,y,z\in[\frac{1}{4},\frac{1}{2}]$.
We assume without loss of generality, that $x>y>z$.
$S(Tx,Ty,Tz)=4x^2$; $S(x,y,z)=x$; $S(x,x,Tx)=4x^2$;
$S(y,y,Ty)=4y^2$; $S(z,z,Tz)=4z^2$; $S(y,y,Tx)=4x^2$;
$S(x,x,Ty)=\max\{x,4y^2\}$; $S(z,z,Ty)=4y^2$;
$S(y,y,Tz)=\max\{y,4z^2\}$; $S(z,z,Tx)=4x^2$;
Here $M(x,y,z)=\max\{4x^2,\frac{(1+4y^2)4x^2}{1+x}\}$ and
$N(x,y,z)=\min\{4x^2,\frac{4x^2[1+\max\{x,4y^2\}]}{1+x}\}=4x^2$.
In this case, it is easy to verify that the inequality (2.1) holds for any $L\ge 1$.
<u>*Case*</u> (ii): Let $x,y\in[\frac{1}{4},\frac{1}{2}]$ and $z\in(\frac{1}{2},1]$. We assume that $x>y$.
In this case, $S(Tx,Ty,Tz)=\max\{4x^2,\frac{1}{2}\}$; $S(x,y,z)=z$;
$S(x,x,Tx)=4x^2$; $S(y,y,Tx)=4x^2$; $S(z,z,Tx)=\max\{z,4x^2\}$;
$S(x,x,Ty)=\max\{x,4y^2\}$.
Now $N(x,y,z)=\min\{4x^2,z,4x^2[\frac{1+\max\{x,4y^2\}}{1+z}]\}$.
If $x\ge 4y^2$ then $N(x,y,z)=\min\{4x^2,z,\frac{4x^2[1+x]}{1+z}\}=\frac{4x^2[1+x]}{1+z}$ and if $x\le 4y^2$ then $N(x,y,z)=\min\{4x^2,\frac{4x^2(1+4y^2)}{1+z}\}=\frac{4x^2(1+4y^2)}{1+z}$ as $\frac{1+4y^2}{1+z}<1$. In any case we see that the inequality (2.1) holds for any $L\ge\frac{8}{5}$.
<u>*Case*</u> (iii): Let $x\in[\frac{1}{4},\frac{1}{2}]$ and $y,z\in(\frac{1}{2},1]$. We assume that $y\ge z$.
In this case, $S(x,y,z)=y$; $S(x,x,Tx)=4x^2$;
$S(y,y,Tx)=\max\{y,4x^2\}$; $S(x,x,Ty)=\frac{1}{2}$;
$S(z,z,Tx)=\max\{z,4x^2\}$;
Here $N(x,y,z)=\min\{4x^2,\frac{3}{2}[\frac{\max\{y,4x^2\}}{1+y}]\}$.
<u>*Subcase*</u> (i): $y\ge 4x^2$.
If $N(x,y,z)=4x^2$ and $S(Tx,Ty,Tz)=4x^2$ or $\frac{1}{2}$ then the inequality (2.1) holds for any $L\ge 1$.
If $N(x,y,z)=\frac{3}{2}(\frac{y}{1+y})$ and $S(Tx,Ty,Tz)=4x^2$ or $\frac{1}{2}$ then the inequality (2.1) holds for any $L\ge\frac{8}{3}$.
<u>*Subcase*</u> (ii): $y\le 4x^2$.
Here $N(x,y,z)=\min\{4x^2,4x^2.\frac{3}{2}.\frac{1}{1+y}\}=4x^2$. In this case the inequality (2.1) holds for any $L\ge 1$.
<u>*Case*</u> (iv): Let $y,z\in[\frac{1}{4},\frac{1}{2}]$ and $x\in(\frac{1}{2},1]$. We assume that $y\ge z$.
In this case $S(x,y,z)=x$; $S(x,x,Tx)=x$;
$S(y,y,Tx)=\frac{1}{2}$; $S(x,x,Ty)=\max\{x,4y^2\}$; $S(z,z,Tx)=\frac{1}{2}$;
Here $N(x,y,z)=\frac{1}{2}$ and if $S(Tx,Ty,Tz)=4y^2$ or $\frac{1}{2}$ then the inequality (2.1) holds for any $L\ge 1$.
<u>*Case*</u> (v): Let $x,y,z\in(\frac{1}{2},1]$. We assume that $x>y>z$.
$S(Tx,Ty,Tz)=\frac{1}{2}$; $S(x,y,z)=x$; $S(x,x,Tx)=x$;
$S(y,y,Tx)=y$; $S(x,x,Ty)=x$; $S(z,z,Tx)=z$;
Then $N(x,y,z)=z$. Clearly, in this case the inequality (2.1) holds for any $L\ge 1$.
<u>*Case*</u> (vi): Let $z\in[\frac{1}{4},\frac{1}{2}]$ and $x,y\in(\frac{1}{2},1]$. We assume that $x>y$.
In this case $S(x,y,z)=x$; $S(x,x,Tx)=x$; $S(y,y,Tx)=y$;
$S(x,x,Ty)=x$; $S(z,z,Tx)=\frac{1}{2}$;
Here $N(x,y,z)=\frac{1}{2}$ and $S(Tx,Ty,Tz)=\frac{1}{2}$ or $4z^2$. In either of the cases the inequality (2.1) holds for any $L\ge 1$.
<u>*Case*</u> (vii): Let $y\in[\frac{1}{4},\frac{1}{2}]$ and $x,z\in(\frac{1}{2},1]$. We assume that $z>x$.
In this case $S(x,y,z)=z$; $S(x,x,Tx)=x$; $S(y,y,Tx)=\frac{1}{2}$;
$S(x,x,Ty)=\max\{x,4y^2\}$; $S(z,z,Tx)=z$;
Here $N(x,y,z)=\min\{\frac{1}{2},\frac{1}{2}\frac{[1+\max\{x,4y^2\}]}{1+z}\}$.
<u>*Subcase*</u> (i): $x\ge 4y^2$.
In this case $N(x,y,z)=\frac{1}{2}\frac{[1+x]}{1+z}$.
If $S(Tx,Ty,Tz)=4y^2$ or $\frac{1}{2}$ then the inequality (2.1) holds for any $L\ge\frac{4}{3}$.
<u>*Subcase*</u> (ii): $x\le 4y^2$ and $z\ge 4y^2$.

Here $N(x,y,z)=\frac{1}{2}\frac{[1+4y^2]}{1+z}$. If $S(Tx,Ty,Tz)=4y^2$ or $\frac{1}{2}$ then the inequality (2.1) holds for any $L\ge 1$.

By all the above cases, we conclude that the hypothesis of Theorem 2.2 holds for any $L\ge\frac{8}{3}$ and $\frac{1}{4}$ is the unique fixed point of T. Here we note that at $x=\frac{1}{2},y=\frac{1}{4}$, the inequality (1.2) fails to hold.

3. Fixed points of α-admissible almost weak ψ-contraction maps with rational expressions

Definition 3.1. *Let (X,S) be an S-metric space. Let $T : X \to X$ and $\alpha : X \times X \times X \to [0,\infty)$. We say that T is α-admissible, if $x,y,z \in X$, $\alpha(x,y,z) \geq 1 \implies \alpha(Tx,Ty,Tz) \geq 1$.*

Example 3.2. Let $X = [0,2]$ and (X,S) be the S-metric space defined in Example 1.12.
Now we define $T : X \to X$ by

$$Tx = \begin{cases} 1 & \text{if } x \in [0,1] \\ 2x-2 & \text{if } x \in (1,2] \end{cases}$$

and let $\alpha : X \times X \times X \to \mathbb{R}$ by

$$\alpha(x,y,z) = \begin{cases} 1 & \text{if } 0 \leq x,y \leq 1, \quad z = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then if $x,y \in [0,1]$ and $z = 1$ then $\alpha(x,y,z) = 1$ which implies that $\alpha(Tx,Ty,Tz) = \alpha(1,1,1) = 1$. Therefore T is α-admissible.

Definition 3.3. *Let (X,S) be an S-metric space. Let $T : X \to X$ be an α-admissible map. If there exist $L \geq 0$ and $\psi \in \Psi_1$ such that*

$$\alpha(x,y,z)S(Tx,Ty,Tz) \leq \psi(M(x,y,z)) + LN(x,y,z) \quad (3.1)$$

where
$$M(x,y,z) = max\{S(x,y,z), \frac{S(x,x,Tx)S(y,y,Ty)}{S(x,y,z)}, \frac{S(x,x,Tx)S(z,z,Tz)}{S(x,y,z)}, \frac{S(y,y,Ty)S(z,z,Tz)}{S(x,y,z)}, \frac{S(x,x,Ty)S(y,y,Tx)}{S(x,y,z)}, \frac{S(y,y,Tz)S(z,z,Ty)}{S(x,y,z)}, \frac{S(z,z,Tx)S(x,x,Tz)}{S(x,y,z)}, \frac{S(y,y,Ty)S(x,x,Ty)}{S(x,y,z)}, \frac{S(x,x,Tx)S(x,x,Ty)}{S(x,y,z)}, \frac{S(z,z,Tz)S(z,z,Tx)}{S(x,y,z)}\}$$
and
$N(x,y,z) = \min\{S(x,x,Tx), S(y,y,Tx), S(z,z,Tx)\}$
for all $x,y,z \in X$ with $x \neq y \neq z$ then we say that T is an α-admissible almost weak ψ-contraction map on X.

Theorem 3.4. *Let (X,S) be a complete S-metric space. Let $T : X \to X$ be a continuous and α-admissible almost weak ψ-contraction map on X. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0,x_0,Tx_0) \geq 1$. Then T has a fixed point in X.*

Proof. We have $x_0 \in X$ such that $\alpha(x_0,x_0,Tx_0) \geq 1$. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for each $n = 0,1,2,\dots$.
If $x_n = x_{n+1}$ for some 'n' then x_n is a fixed point of T.
With out loss of generality suppose that $x_n \neq x_{n+1}$ for each 'n'. We have $\alpha(x_0,x_0,x_1) \geq 1$.
Since T is α-admissible, $\alpha(Tx_0,Tx_0,Tx_1) \geq 1$.
i.e., $\alpha(x_1,x_1,x_2) \geq 1$. Continuing this process, we have $\alpha(x_n,x_n,x_{n+1}) \geq 1$ for each $n \geq 0$.
Now from (3.1), we get

$$\begin{aligned} S(x_{n+1},x_{n+1},x_{n+2}) &= S(Tx_n,Tx_n,Tx_{n+1}) \\ &\leq \alpha(x_n,x_n,x_{n+1})S(Tx_n,Tx_n,Tx_{n+1}) \\ &\leq \psi(M(x_n,x_n,x_{n+1})) + LN(x_n,x_n,x_{n+1}). \end{aligned}$$

That is

$$S(x_{n+1},x_{n+1},x_{n+2}) \leq \psi(M(x_n,x_n,x_{n+1})) + LN(x_n,x_n,x_{n+1}) \quad (3.2)$$

Where
$M(x_n,x_n,x_{n+1}) = \max\{S(x_n,x_n,x_{n+1}), S(x_{n+1},x_{n+1},x_{n+2})\}$ and $N(x_n,x_n,x_{n+1}) = 0$. Hence from 3.2, we have
$S(x_{n+1},x_{n+1},x_{n+2}) \leq \psi(\max\{S(x_n,x_n,x_{n+1}), S(x_{n+1},x_{n+1},x_{n+2})\})$.
If maximum is $S(x_{n+1},x_{n+1},x_{n+2})$ for some 'n' then $S(x_{n+1},x_{n+1},x_{n+2}) \leq \psi(S(x_{n+1},x_{n+1},x_{n+2})) < S(x_{n+1},x_{n+1},x_{n+2})$, a contradiction.
Hence maximum is $S(x_n,x_n,x_{n+1})$.
Therefore for all $n \geq 0$, we have

$$\begin{aligned} S(x_{n+1},x_{n+1},x_{n+2}) &\leq \psi(S(x_n,x_n,x_{n+1})) \\ &\leq \psi^2(S(x_{n-1},x_{n-1},x_n)) \\ &\ \ \vdots \\ &\leq \psi^{n+1}(S(x_0,x_0,x_1)). \end{aligned}$$

Now, we consider

$$\begin{aligned} S(x_n,x_n,x_{n+k}) &\leq S(x_n,x_n,x_{n+1}) + S(x_n,x_n,x_{n+1}) \\ &\quad + S(x_{n+1},x_{n+1},x_{n+k}) \\ &\leq 2S(x_n,x_n,x_{n+1}) + 2S(x_{n+1},x_{n+1},x_{n+2}) \\ &\quad + S(x_{n+2},x_{n+2},x_{n+k}) \\ &\leq 2S(x_n,x_n,x_{n+1}) + 2S(x_{n+1},x_{n+1},x_{n+2}) \\ &\quad + \ . \ . \ . + S(x_{n+k-1},x_{n+k-1},x_{n+k}) \\ &\leq 2\sum_{p=n}^{n+k-1} S(x_p,x_p,x_{p+1}) \\ &< 2\sum_{p=n}^{\infty} S(x_p,x_p,x_{p+1}) \\ &\leq 2\sum_{p=n}^{\infty} \psi(S(x_p,x_p,x_{p+1})) \\ &< 2\sum_{p=0}^{\infty} \psi^p S(x_0,x_0,x_1) \to 0 \text{ as } p \to \infty. \end{aligned}$$

Therefore $S(x_n,x_n,x_{n+k}) \to 0$ as $n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence in X. Since (X,S) is a complete S-metric space there exists $u \in X$ such that $x_n \to u$.
That is $u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tu$.
Therefore $Tu = u$. Hence u is a fixed point of T. □

Condition(U): If $x,y \in F(T)$, the set of all fixed points of T then $\alpha(x,x,y) \geq 1$ or $\alpha(y,y,x) \geq 1$.

Theorem 3.5. *Under the hypothesis of Theorem 3.4 and Condition(U), T has a unique fixed point.*

Proof. From Theorem 3.4, T has a fixed point. Therefore $F(T) \neq \emptyset$. Let $x,y \in F(T)$ then by Condition(U), $\alpha(x,x,y) \geq 1$ or $\alpha(y,y,x) \geq 1$. We consider

$$\begin{aligned} S(Tx,Tx,Ty) &\leq \alpha(x,x,y)S(Tx,Tx,Ty) \\ &\leq \psi(M(x,x,y)) + LN(x,x,y) \end{aligned}$$

where
$M(x,x,y) = S(x,x,y)$ and $N(x,x,y) = 0$.

Therefore $S(Tx,Tx,Ty) \leq \psi(S(x,x,y)) + L(0)$.
That is $S(x,x,y) < S(x,x,y)$, a contradiction.
Therefore $x = y$. □

Corollary 3.6. *Let (X,S) be a complete S-metric space. Let $T : X \to X$ be a continuous and α-admissible map on X and there exists $x_0 \in X$ such that $\alpha(x_0,x_0,Tx_0) \geq 1$. Suppose that there exist $k \in [0,1)$ and $L \geq 0$ such that*

$$\alpha(x,y,z)S(Tx,Ty,Tz) \leq kM(x,y,z) + LN(x,y,z) \quad (3.3)$$

for all $x,y,z \in X$ with $x \neq y \neq z$, where $M(x,y,z)$ and $N(x,y,z)$ are given as in the inequality (3.1). Then T has a fixed point in X.

Proof. We define $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = kt$ for all $t \in [0,\infty)$ in the inequality (3.1). Clearly $\psi \in \Psi_1$. Then we have T is an α-admissible almost weak ψ-contraction map. Now by Theorem 3.4 the conclusion follows. □

By choosing $L = 0$ in Corollary 3.6, we obtain the following corollary.

Corollary 3.7. *Let (X,S) be a complete S-metric space. Let $T : X \to X$ be a continuous and α-admissible map on X and there exists $x_0 \in X$ such that $\alpha(x_0,x_0,Tx_0) \geq 1$. Suppose that there exists $k \in [0,1)$ such that*

$$\alpha(x,y,z)S(Tx,Ty,Tz) \leq kM(x,y,z) \quad (3.4)$$

for all $x,y,z \in X$ with $x \neq y \neq z$, where $M(x,y,z)$ is given as in the inequality (3.1). Then T has a fixed point in X.

If $\alpha \equiv 1$ in Corollary 3.7 then we obtain the following corollary.

Corollary 3.8. *Let (X,S) be a complete S-metric space. Let $T : X \to X$ be a continuous mapping. Suppose that there exists $k \in [0,1)$ such that*

$$S(Tx,Ty,Tz) \leq kM(x,y,z) \quad (3.5)$$

for all $x,y,z \in X$ with $x \neq y \neq z$, where $M(x,y,z)$ is given as in the inequality (3.1). Then T has a fixed point in X.

Corollary 3.9. *Let (X,S) be a complete S-metric space. Let $T : X \to X$ be a continuous mapping. Suppose that there exist $\alpha,\beta,\gamma,\eta \in [0,1)$ with $\alpha+\beta+\gamma+\eta < 1$ such that*

$$\begin{aligned} S(Tx,Ty,Tz) \leq \alpha &\frac{S(x,x,Tx)S(y,y,Ty)}{S(x,y,z)} \\ &+\beta\frac{S(y,y,Ty)S(z,z,Tz)}{S(x,y,z)} \\ &+\gamma\frac{S(z,z,Tz)S(x,x,Tx)}{S(x,y,z)} \\ &+\eta S(x,y,z) \end{aligned} \quad (3.6)$$

for all $x,y,z \in X$ with $x \neq y \neq z$. Then T has a fixed point in X. The map T which satisfies the above inequality is called as 'Jaggi type contraction map' on X.

Proof. We obtain the inequality (3.5) from the inequality (3.6) by choosing $k = \alpha+\beta+\gamma+\eta$. Hence the conclusion follows from Corollary 3.8. □

Example 3.10. Let $X = [0,3]$ and (X,S) be an S-metric space defined in Example 1.9.
Now we define $T : X \to X$ by

$$Tx = \begin{cases} 1 & \text{if } x \in [0,1) \\ 2x-1 & \text{if } x \in [1,2] \\ 3 & \text{if } x \in (2,3] \end{cases}$$

and $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \frac{t}{2}$ for all $t \geq 0$ and let $\alpha : X \times X \times X \to \mathbb{R}$ by

$$\alpha(x,y,z) = \begin{cases} 1 & \text{if } \frac{5}{3} \leq x,y \leq 3, \quad z = 3 \\ 0 & \text{otherwise.} \end{cases}$$

<u>*Case*</u> (i): Let $x,y \in [\frac{5}{3},3]$ and $z = 3$.
In this case $\alpha(x,y,z) = 1$ implies $\alpha(Tx,Ty,Tz) = 1$.
Therefore T is α-admissible.
<u>*Subcase*</u> (i): Let $x,y \in [\frac{5}{3},2]$ and $z = 3$.
We assume with out loss of generality that $x < y$.
In this case $S(Tx,Ty,Tz) = S(2x-1,2y-1,3) = 4-2x$ and $S(x,y,z) = \max\{|x-3|,|y-3|\} = 3-x$.
We consider $\alpha(x,y,z)S(Tx,Ty,Tz) = 4-2x$.
In this case the inequality (3.1) holds for any $L \geq 0$.
<u>*Subcase*</u> (ii): Let $x \in [\frac{5}{3},2]$, $y \in (2,3]$ and $z = 3$. We assume that $x < y$.
In this case $S(Tx,Ty,Tz) = S(2x-1,2y-1,3) = 4-2x$ and $S(x,y,z) = 3-x$. Hence the verification is similar as in Subcase(i).
<u>*Subcase*</u> (iii): Let $y \in [\frac{5}{3},2]$, $x \in (2,3]$ and $z = 3$.
Here $S(Tx,Ty,Tz) = S(3,2y-1,3) = 4-2y$ and $S(x,y,z) = 3-y$.
In this case $\alpha(x,y,z)S(Tx,Ty,Tz) = 4-2y$ which shows that the inequality (3.1) holds for any $L \geq 0$.
In all the remaining cases the inequality (3.1) holds trivially.

Hence all the hypotheses of Theorem 3.4 hold and 1,3 are fixed points of T.
Here we observe that the inequality
$S(Tx,Ty,Tz) \leq \psi(M(x,y,z)) + LN(x,y,z)$ fails to hold when $x = 1, y = 1, z = 3$.

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