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Contribution to pseudo spherical kinematics of pseudo spherical evolutes

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Abstract

In this work, we give some relations about involutes and evolutes of a timelike curve in Lorentz $3-$space. Also, we derive a characterization of pseudo spherical evolutes corresponding to the trajectory of a point in pseudo spherical kinematics. Then, we obtain a transformation matrix from the natural trihedron of a space curve to the geodesic trihedron of its spherical evolutes in Lorentz $3-$space. Finally, we give an example to illustrate our results.

Keywords

Lorentz $3-$space, pseudo spherical evolutes, pseudo spherical kinematics.

AMS Subject Classification

53A04, 53A35, 53A40, 53B30.

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Contents

1. Introduction

The curvature theory of involutes and evolutes curves has been one of the important subject because of having many application area in kinematic and differential geometry. So, there are many important consequences and properties in the curvature theory of the such curves in differential geometry [1, 2, 4, 8, 12]. Also, many paper can be found in the literature for curves and their characterization in Lorentz 3−space [9, 13, 15].

Important contributions to the kinematic geometry of spherical curves have been made by Veldkamp and McCarthy [5, 14]. Veldkamp studied the similarity of spherical kinematics to plane kinematics, [14]. Then, McCarthy and Roth gave some results for the differential kinematics of spherical motion using kinematic mappings, [6]. Also, McCarthy and Ravani presented differential kinematics of spherical and spatial motions using a mapping of spatial kinematics and derived relationships for the intrinsic properties of the image curves corresponding to a mapping of spherical and spatial kinematics, [7]. Schaaf and Yang defined spherical evolutes corresponding to the trajectory of a point in spherical kinematics and derived general expressions for the curvature properties of the n-th spherical evolute with respect to geodesic curvature and the its derivative, [11].

In the current study, we would like to contribute to the study of kinematic geometry of pseudo spherical evolutes in Lorentz 3−space. Firstly, we remind some notations about curves in Lorentz 3−space. After that, we get some relations about involutes and evolutes of the timelike curve. Then, we give a transformation matrix from the natural trihedron of a Lorentz curve to the geodesic trihedron of its pseudo spherical evolutes in Lorentz 3−space.

2. Preliminaries

Let $\mathbb{E}^3_1$ be Lorentz 3−space with the inner product

$$< u,v >= -u_1 v_1 + u_2 v_2 + u_3 v_3$$

and the vector product

$$u \times v = \left(-\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right),$$

where $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in \mathbb{E}^3_1$.

The vector $v \in \mathbb{E}_1^3$ is said to be spacelike, lightlike or timelike whenever $\langle v, v\rangle > 0$ or $v = 0$, $\langle v, v\rangle = 0$ and $v \neq 0$, and $\langle v, v\rangle < 0$, respectively. Similarly, an arbitrary curve $\alpha = \alpha(s)$ can be spacelike, lightlike or timelike, if all of its velocity vectors $\alpha'(s)$ are, respectively, spacelike, lightlike or timelike. The signature of a vector v is defined as

$$\varepsilon = \begin{cases} 1, & v \text{ is a spacelike vector} \\ 0, & v \text{ is a lightlike vector} \\ -1, & v \text{ is a timelike vector,} \end{cases}$$

whereas the norm of the vector $v \in \mathbb{E}_1^3$ is given by $\|v\| = \sqrt{|\langle v, v\rangle|}$, [10].

Theorem 2.1. *Let $\alpha(s)$ be a timelike curve and $\{\vec{T}, \vec{N}, \vec{B}\}$ be the moving Frenet frame along the curve $\alpha(s)$ in $\mathbb{E}_1^3$. The Frenet derivative equations are given by*

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix},$$

where $\left\langle \vec{T}, \vec{T} \right\rangle = -1$, $\left\langle \vec{N}, \vec{N} \right\rangle = \left\langle \vec{B}, \vec{B} \right\rangle = 1$ and $\left\langle \vec{T}, \vec{N} \right\rangle = \left\langle \vec{T}, \vec{B} \right\rangle = \left\langle \vec{N}, \vec{B} \right\rangle = 0$. κ and τ are curvature and torsion of the timelike curve $\alpha(s)$, respectively, [10].

Theorem 2.2. *If the timelike curve $\alpha(t)$ has a non-unit speed, then*

$$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3} \text{ and } \tau(t) = \frac{\det(\alpha'(t), \alpha''(t), \alpha'''(t))}{\|\alpha'(t) \times \alpha''(t)\|^2}.$$

If the timelike curve $\alpha(s)$ has a unit speed, then

$$\kappa(s) = \left\|\alpha''(s)\right\| \text{ and } \tau(s) = \left\|B'(s)\right\|,$$

[10].

3. Involutes and Evolutes of a Timelike Curve

In this section, we study characterizations of involutes-evolutes of a timelike curve. The tangents of the timelike curve C in $\mathbb{E}_1^3$ create a surface called as the tangent surface of C. Curves on the tangent surface which are orthogonal to the creating tangents are called as involutes of the timelike curve C. The equation of an involute I of the timelike curve C is written as follows

$$y = y(s) = \vec{x} + \mathbf{a}\vec{t} \tag{1}$$

where $\vec{x}$ is the position vector of point P on C, $\vec{t}$ is the tangent vector and $\mathbf{a}$ is a scalar function of the arc length s. If we differentiate Equation (1), in terms of the arc length parameter s of the timelike curve C, we get

$$y' = \vec{t} + (\mathbf{a}'\vec{t} + \mathbf{a}\vec{t}') = (1 + \mathbf{a}')\vec{t} + \mathbf{a}\kappa\vec{n}. \tag{2}$$

If we use the definition of involutes

$$< (1 + \mathbf{a}')\vec{t} + \mathbf{a}\kappa\vec{n}, \vec{t} >= 0,$$

we have, $\dfrac{d\mathbf{a}}{ds} = -1$ and $\mathbf{a} = c - s$ where c is an arbitrary constant. So, the vector equation of an involute I (see Figure 1) of the timelike curve C is

$$y = y(s) = \vec{x} + (c - s)\vec{t}. \tag{3}$$

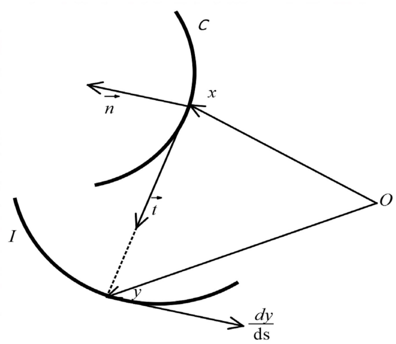

Figure 1. An involute I of the timelike curve C

Every value of c corresponds to one of a single infinity of involutes of the given timelike curve C. So, if C is a plane curve its tangent surface is a plane and from equation (3), we can say that the involutes of a plane curve are also plane curves [11]. Also, the notion of involutes can be used to define the converse problem. For a timelike curve C, determine a curve E which accepts the curve C as an involute. The curve E is called as an evolute of C. The equation of evolute E has the form

$$z = z(s) = \vec{x} + \mathbf{b}\vec{p} \tag{4}$$

where $\mathbf{b}$ is a scalar function of arc length parameter s of the timelike curve C and the unit spacelike vector $\vec{p}$ (see Figure 2) lies in the normal plane of C. $\vec{p}$ is given by

$$\vec{p} = \sin\phi\vec{n} + \cos\phi\vec{b}. \tag{5}$$

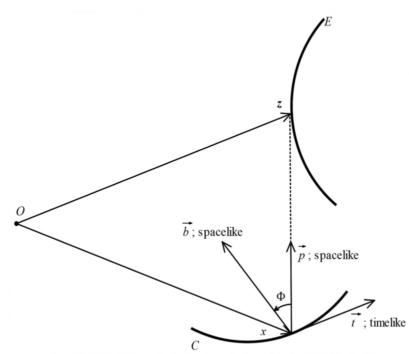

Figure 2. An evolute E of the timelike curve C

The spacelike vector $\vec{p}$ should be tangent to the evolute E. So, we can write

$$z' = f\vec{p} \tag{6}$$

where f is a scalar function. From the notion of evolutes, the tangents to the evolutes are orthogonal to the given timelike curve C, so we have;

$$<\vec{p},\vec{t}>=0.$$

If we differentiate Equation (4), in terms of the arc length parameter s of the timelike curve C, we get

$$z' = \vec{t} + \mathbf{b}'\vec{p} + \mathbf{b}\vec{p}'. \tag{7}$$

Since $<\vec{p},\vec{p}>=1$ and $<\vec{p},\vec{t}>=0$, we obtain, from Equations (6) and (7) that $\mathbf{b}'=f$ and

$$\vec{t} + \mathbf{b}\vec{p} = 0. \tag{8}$$

If we use the Frenet formula for timelike curve and the derivative of Equation (5) into Equation (8), we get

$$\vec{t} + \mathbf{b}[(\kappa\vec{t} + \tau\vec{b})\sin\phi + \phi'\cos\phi\vec{n} - \cos\phi\,\tau\vec{n} - \phi'\sin\phi\vec{b}] = 0.$$

So, we can write

$$[1+\kappa b\sin\phi]\vec{t} + [(\phi'-\tau)\mathbf{b}\cos\phi]\vec{n} + ((\tau-\phi')\mathbf{b}\sin\phi)\vec{b} = 0. \tag{9}$$

Then, we have the following proposition with the aid of linear independence of $\{\vec{t},\vec{n},\vec{b}\}$:

Proposition 3.1. *Let C be a timelike curve with the curvature κ and the torsion τ in Lorentz $3-$space. Then, the following equations hold*

$$\begin{aligned} 1+\kappa b\sin\phi &= 0, \\ (\phi'-\tau)\cos\phi &= 0, \\ (\tau-\phi')\sin\phi &= 0. \end{aligned}$$

So, we can give relations between the curvature and the torsion of the curve C in the following proposition:

Proposition 3.2. *Let C be a timelike curve with curvature κ and torsion τ in Lorentz $3-$space. The relations between curvature and torsion of the curve C are*

$$b\sin\phi = -\frac{1}{\kappa} = -\rho \tag{10}$$

and

$$\phi' = \tau. \tag{11}$$

If we integrate Equation (11) with respect to s, we get the expression of ϕ

$$\phi = \phi(s) = \int_0^s \tau ds + c_1,\ c_1 = const. \tag{12}$$

Also, with the aid of Equations (5) and (10), the equation of the evolute E can be written as

$$z = z(s) = \vec{x} - \rho(\vec{n} + \lambda\vec{b}), \tag{13}$$

where $\lambda = \lambda(s) = \cot\phi$. From the equation (12), we can say that each value of c_1 corresponds to one of the single infinity of evolutes of the timelike curve C. If $\tau = 0$, the timelike curve C lies in a Lorentz plane. So, we get from Equation (12), $\phi = c_1$. A plane curve has an evolute which lies in the plane, generated by the vectors $\vec{n}$ and $\vec{b}$. This evolute corresponds to $\phi = c_1 = \frac{\pi}{2}$ and is the locus of the centers of curvature for the timelike curve C, [11].

4. Pseudo Spherical Curves

In this section, we give some relations and results for pseudo spherical curves in Lorentz $3-$space. Let $C_s : I \to \mathbb{S}^2_1 \subset \mathbb{E}^3_1$ be a pseudo-spherical curve and the unit vector $\vec{r}$ is the position vector of the point P on C_s (see Figure 3). The timelike tangent vector $\vec{t}$ to C_s at P is given by the derivative of $\vec{r}$ with respect to the arc length s of C_s,

$$\vec{t} = \vec{r'}. \tag{14}$$

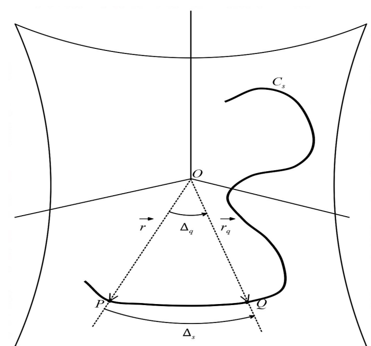

Figure 3. Pseudo spherical curve, C_s

So, $\vec{t}$ is orthogonal to $\vec{r}$ for all points on C_s. If P and Q are two neighboring points on C_s separated by the arc increment Δ_s along the curve and the central angle between the position vectors for P and Q is Δ_q, in the limit when $Q \to P$, we obtain $ds = dq$. Then, the timelike tangent vector can be written as

$$\vec{t} = \frac{d\vec{r}}{dq}. \tag{15}$$

Thus, we can use the central angle q as the parameter of C_s and prime to denote differentiation with respect to q. The spacelike vector $\vec{k} = \vec{t}\times\vec{r}$, $(\varepsilon_r\varepsilon_t\vec{k} = \vec{r}\times\vec{t})$ is called the central normal to C_s at P. The three mutually orthogonal unit vectors $[\vec{r},\vec{t},\vec{k}]$ define the geodesic trihedron of C_s and is denoted $[\mathbf{r}]$. From the Frenet formula of the timelike curve, we can write

$$\vec{t'} = \kappa\vec{n}, \tag{16}$$

where κ is the curvature of C_s at P and $\vec{n}$ is the principal normal of C_s. The three vectors $[\vec{t},\vec{n},\vec{b}]$ define the natural trihedron $[\mathbf{t}]$ of C_s at point P, together with the spacelike binormal $\varepsilon_t\varepsilon_n\vec{b} = \vec{t}\times\vec{n}$. Note that $\vec{r}$ and $\vec{k}$ lie in the normal plane and the plane defined by $\vec{r}$ and $\vec{t}$ is referred to as the central plane of C_s at point P. Since $<\vec{t'},\vec{t}>=0$, we have

$$\vec{t'}=\mu\vec{r}+\sigma\vec{k}. \tag{17}$$

If we differentiate the expression $\varepsilon_r\varepsilon_t\vec{k}=\vec{r}\times\vec{t}$, we obtain

$$\varepsilon_r\varepsilon_t\vec{k'}=\vec{r'}\times\vec{t}+\vec{r}\times\vec{t'}=\vec{r}\times\vec{t'}. \tag{18}$$

Substituting Equation (17) into Equation (18), we obtain

$$\vec{k}'=-\sigma\vec{t}, \tag{19}$$

where σ is the geodesic curvature of the timelike curve C_s at point P. Then, if we take derivative of $\varepsilon_k\varepsilon_r\vec{t}=\vec{k}\times\vec{r}$ with respect to q and using Equations (15), (16), (19), we have

$$\vec{t'}=\sigma\vec{k}+\vec{r}=\kappa\vec{n}. \tag{20}$$

By using (20), it is easy to see that

$$\kappa=\sqrt{1+\sigma^2}. \tag{21}$$

Then, we can give the following proposition with the aid of Equations (15), (17) and (19).

Proposition 4.1. *The geodesic trihedron* $[\mathbf{t}]$ *with respect to* q *is given by*;

$$\begin{bmatrix}\vec{r'}\\ \vec{t'}\\ \vec{k'}\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&\sigma\\0&\sigma&0\end{bmatrix}\begin{bmatrix}\vec{r}\\ \vec{t}\\ \vec{k}\end{bmatrix}$$

or

$$[\mathbf{r}]'=R[\mathbf{r}] \tag{22}$$

where

$$[\mathbf{r}]'=[\vec{r'},\vec{t'},\vec{k'}] \text{ and } R=\begin{bmatrix}0&1&0\\1&0&\sigma\\0&\sigma&0\end{bmatrix}.$$

Since $ds=dq$, the Frenet formula for the natural trihedron $[\mathbf{t}]$ can be given with respect to the central angle parameter q;

$$[\mathbf{t}]'=K[\mathbf{t}] \tag{23}$$

where $[\mathbf{t}]'=[\vec{t'},\vec{n'},\vec{b'}]$ and $K=\begin{bmatrix}0&\kappa&0\\\kappa&0&\tau\\0&-\tau&0\end{bmatrix}$. On the other hand, the Darboux vector associated with the geodesic trihedron is found as follows

$$\mathbf{d}=-\sigma\vec{r}+\vec{k}. \tag{24}$$

Then, from Equations (22) and (23), we can write

$$\begin{aligned}\vec{r'}&=\mathbf{d}\times\vec{r},\\ \vec{t'}&=\mathbf{d}\times\vec{t},\\ \vec{k'}&=\mathbf{d}\times\vec{k}.\end{aligned}$$

The orientation angle between the geodesic trihedron and natural trihedron about the common axis $\vec{t}$ is denoted by q_1. The relation between these trihedrons is given by

$$[\mathbf{r}]=Q[\mathbf{t}], \tag{25}$$

where

$$Q=\begin{bmatrix}0&-\sin q_1&\cos q_1\\1&0&0\\0&\cos q_1&\sin q_1\end{bmatrix}.$$

Since $<\vec{r},\vec{n}>=-\sin q_1$, $<\vec{r},\vec{t}>=0$ and $<\vec{r},\vec{b}>=\cos q_1$, from Equations (23) and (25), we obtain

$$<\vec{r},\vec{t}''>=\kappa\sin q_1 \tag{26}$$

and

$$\kappa=-\frac{1}{\sin q_1}=-\csc q_1. \tag{27}$$

So, we can give the following proposition related with the radius of curvature of C_s:

Proposition 4.2. *The radius of curvature of* C_s *at the point* P *is*

$$\rho=\frac{1}{\kappa}=-\sin q_1.$$

Now, we find an expression for the torsion τ of C_s at the point P. So, if we take first derivative of $<\vec{r},\vec{b}>=\cos q_1$, we get

$$<\vec{r},\vec{b}'>=-q_1'\sin q_1. \tag{28}$$

From the derivative of Equation (20) and Equation (23), the second derivative of the timelike tangent is found as

$$\frac{d^2\vec{t}}{dq^2}=\kappa'\vec{n}+\kappa\tau\vec{b}+\kappa^2\vec{t}. \tag{29}$$

From Equations (21) and (29), we have

$$\sigma'\vec{k}=\kappa'\vec{n}+\kappa\tau\vec{b}.$$

If we use Equation (25) into the above equation, we get

$$\sigma'(\cos q_1\vec{n}+\sin q_1\vec{b})=\kappa'\vec{n}+\kappa\tau\vec{b}. \tag{30}$$

Then, we can give the following proposition related with the torsion of C_s.

Proposition 4.3. *The torsion of C_s at the point P is*

$$\tau = -\sigma'(1+\sigma^2)^{-1}. \quad (31)$$

The equation (24) is normalized by Equation (21) and $\rho = -\sin q_1$. So, the instantaneous axis for the rotation of the geodesic trihedron is express as

$$\tilde{\mathbf{d}} = \rho(-\sigma\vec{r} + \vec{k}). \quad (32)$$

If we use the equality $\rho = -\sin q_1$, the instantaneous axis can be rewritten as

$$\tilde{\mathbf{d}} = \cos q_1 \vec{r} - \sin q_1 \vec{k}. \quad (33)$$

Equations (32) and (33) show that the geodesic curvature σ can be given as

$$\sigma = \sigma(s) = \cot q_1.$$

5. Pseudo Spherical Evolutes

The pseudo spherical evolute of the spherical curve C_s is defined as the locus of points which belong to the set of evolutes of C_s and lies on the pseudo unit sphere. So, the pseudo spherical evolute of C_s denoted by E_s is pseudo spherical curve. $\vec{r}_e(s)$ corresponds to the vector representation of the pseudo spherical evolute. The curvature properties of E_s are important tool to define the higher-order path curvature of E_s at P.

If we replace $\vec{x}$ in Equation (13) with $\vec{r}$, we get an evolute of C_s as

$$z = z(s) = \vec{r} - \rho(\vec{n} + \lambda\vec{b}). \quad (34)$$

From the concept of the pseudo spherical evolute, E_s must lie on the unit pseudo sphere. Also, let $\vec{z} = \vec{r}_e$ be the position vector from O to P_e, a point on E_s. The position vector from O to P_e is

$$\vec{r}_e = \vec{r}_e(s) = \vec{r} - \rho(\vec{n} + \lambda\vec{b}). \quad (35)$$

From the transformation given in Equations (25), (35) and $\rho = -\sin q_1$, we can write $\vec{r}_e$ as

$$\begin{aligned} \vec{r}_e &= (\cos q_1 \vec{b} - \sin q_1 \vec{n}) + \sin q_1 \vec{n} + \lambda \sin q_1 \vec{b}, \\ &= (\cos q_1 + \lambda \sin q_1)\vec{b}. \end{aligned}$$

Since $\vec{r}_e$ is a spacelike unit vector it means that

$$\left.\begin{array}{c} \vec{r}_e = \vec{b} \\ \cos q_1 + \lambda \sin q_1 = 1 \end{array}\right\} \quad (36)$$

which represent the position vector of E_s. The parameter λ is a function of the geodesic curvature of the original C_s and defined as follows

$$\lambda = \frac{1}{\sin q_1} - \cot q_1 = -\kappa - \sigma. \quad (37)$$

The arbitrary constant c_1 in Equation (12) used to define λ varies from point to point on the pseudo spherical evolute. But, the pseudo spherical evolute, corresponding to a pseudo spherical curve is similarity with the evolute corresponding to a plane curve. Equation (36) can be considered as a response to the study of Kirson, [3]. Kirson defined a point P_e on the spherical evolute as the intersection of the unit sphere with the binormal vector originating from the center of the sphere O. The spacelike tangent of E_s is

$$\vec{t}_e = \frac{d\vec{r}_e}{ds_e} = \frac{d\vec{b}}{ds_e} \quad (38)$$

where s_e is the arc length of the pseudo evolute E_s. From the Frenet formula for the natural trihedron, we get

$$\vec{t}_e = \frac{d\vec{b}}{dq}\frac{dq}{ds_e} = -\tau\vec{n}\frac{dq}{ds_e}.$$

Then, if we use Equation (31) and $\frac{dq}{ds} > 0$, we have

$$\frac{dq}{ds_e} = \frac{1}{|\tau|} = \frac{dq_1}{dq}.$$

Therefore $ds_e = dq_1$ for $(q_1)' > 0$ and $ds_e = -dq_1$ for $(q_1)' < 0$. The central angle q_1 is parameter for the pseudo evolute E_s. Then, we get

$$\vec{t}_e = -\frac{\tau}{|\tau|}\vec{n} = \pm\vec{n}.$$

The spacelike tangent to E_s is parallel to the spacelike principal normal of C_s and the central normal of E_s is given by

$$\vec{k}_e = \varepsilon_{\vec{r}_e}\varepsilon_{\vec{t}_e}\vec{r}_e \times \vec{t}_e = \vec{b} \times (\pm\vec{n}) = \mp\vec{t}.$$

The set of three unit vectors $[\vec{r}_e, \vec{t}_e, \vec{k}_e]$ is called the geodesic trihedron of the pseudo spherical evolute and shown with $[\mathbf{r}_e]$. So, we can give the following propositions:

Proposition 5.1. *A transformation matrix between the natural trihedron of C_s and the geodesic trihedron of E_s is given by*

$$\begin{bmatrix} \vec{r}_e \\ \vec{t}_e \\ \vec{k}_e \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \mp 1 & 0 \\ \mp 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix}.$$

Proposition 5.2. *The Frenet formula for $[t_e] = [\vec{t}_e, \vec{n}_e, \vec{b}_e]$ is given by*

$$\frac{d}{dq_1}\begin{bmatrix} \vec{t}_e \\ \vec{n}_e \\ \vec{b}_e \end{bmatrix} = \begin{bmatrix} 0 & \pm\kappa_e & 0 \\ \mp\kappa_e & 0 & \tau_e \\ 0 & -\tau_e & 0 \end{bmatrix} \begin{bmatrix} \vec{t}_e \\ \vec{n}_e \\ \vec{b}_e \end{bmatrix}$$

where

$$\vec{n}_e = \pm\frac{\frac{d\vec{t}_e}{dq_1}}{\left|\frac{d\vec{t}_e}{dq_1}\right|} \text{ and } \vec{b}_e = \vec{n}_e \times \vec{t}_e$$

are the timelike normal vector and the spacelike binormal of E_s*, respectively.*

For the Frenet formula for $[r_e]$ and transformation matrix between the $[\mathbf{r}]$ and $[\mathbf{r}_e]$, we can give the following proposition:

Proposition 5.3. *The Frenet formula for* $[r_e]$ *is given by*

$$\frac{d}{dq_1}\begin{bmatrix}\vec{r}_e\\ \vec{t}_e\\ \vec{k}_e\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ \mp 1 & 0 & -\sigma_e\\ 0 & \mp\sigma_e & 0\end{bmatrix}\begin{bmatrix}\vec{r}_e\\ \vec{t}_e\\ \vec{k}_e\end{bmatrix}$$

and the transformation matrix between the $[r]$ *and* $[r_e]$ *is*

$$\begin{bmatrix}\vec{r}_e\\ \vec{t}_e\\ \vec{k}_e\end{bmatrix}=\begin{bmatrix}\cos q_1 & 1 & \sin q_1\\ \mp\sin q_1 & 0 & \cos q_1\\ 0 & \mp 1 & 0\end{bmatrix}\begin{bmatrix}\vec{r}\\ \vec{t}\\ \vec{k}\end{bmatrix}.$$

Example 5.4. *Let* $C(s)=(\frac{3}{5}\sinh(\sqrt{5}s),\frac{3}{5}\cosh(\sqrt{5}s),\frac{2}{\sqrt{5}}s)$ *be a unit speed timelike curve such that*

$$\begin{aligned}\vec{t} &= (\frac{3}{\sqrt{5}}\cosh(\sqrt{5}s),\frac{3}{\sqrt{5}}\sinh(\sqrt{5}s),\frac{2}{\sqrt{5}}),\\ \vec{n} &= (\sinh(\sqrt{5}s),\ \cosh(\sqrt{5}s),\ 0),\\ \vec{b} &= (-\frac{2}{\sqrt{5}}\cosh(\sqrt{5}s),-\frac{2}{\sqrt{5}}\sinh(\sqrt{5}s),-\frac{3}{\sqrt{5}})\end{aligned}$$

and $\frac{\kappa}{\tau}=\frac{3}{2}$*. So, from Equation* (3)*, the involutes of the curve* $C(s)$ *can be written as*

$$I(s)=(\frac{3}{5}\sinh(\sqrt{5}s)+(c-s)\frac{3}{\sqrt{5}}\cosh(\sqrt{5}s),\\ \frac{3}{5}\cosh(\sqrt{5}s)+(c-s)\frac{3}{\sqrt{5}}\sinh(\sqrt{5}s),\ \frac{2c}{\sqrt{5}})$$

where c is an arbitrary constant. If $\phi=\frac{\pi}{4}$*, with the aid of Equation* (13)*, the equation of the evolute* $E(s)$ *can be written as follows*

$$E(s)=(\frac{4}{15}\sinh(\sqrt{5}s)+\frac{2}{3\sqrt{5}}\cosh(\sqrt{5}s),\\ \frac{4}{5}\cosh(\sqrt{5}s)+\frac{2}{3\sqrt{5}}\sinh(\sqrt{5}s),\ \frac{2s}{\sqrt{5}}-\frac{1}{\sqrt{5}}).$$

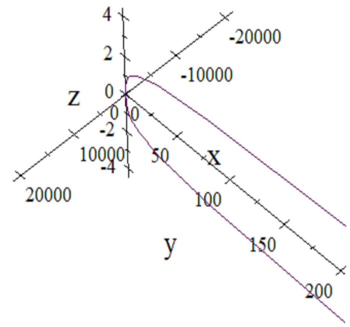

Figure 4. The timelike curve C

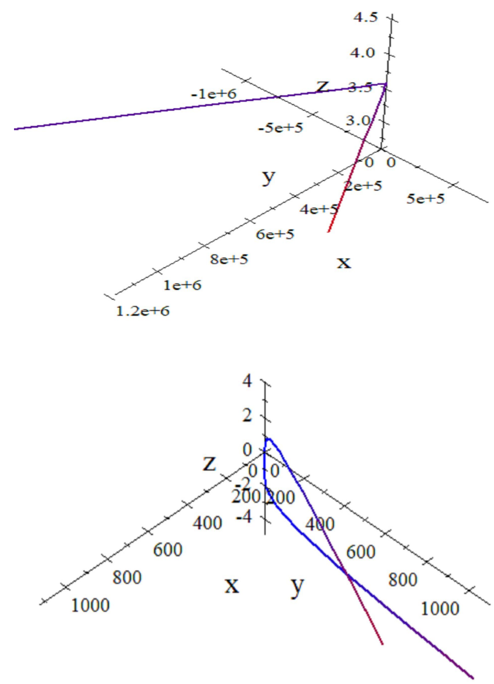

Figure 5. (a) An involute I of the timelike curve C, (b) An evolute E of the timelike curve C

References

[1] Bukcu B, Karacan MK, On the Involute and Evolute Curves of the Timelike Curve in Minkowski 3−space, *Demonstratio Mathematica*, 3(2007), 721–732.

[2] Erisir T, Güngör MA, On the Quaternionic Curves in the Semi-Euclidean Space $\mathbb{E}_2^4$, *Caspian Journal of Mathematical Sciences*, 7(1)(2018), 36–45.

[3] Kirson Y, *Curvature Theory in Space Kinematics*, Doctoral Dissertation, University of California, Berkeley, CA, 1975.

[4] Li Y, Pei D, Evolutes of Dual Spherical Curves for Ruled Surfaces, *Mathematical Methods in the Applied Sciences*, 39(2016), 3005-3015.

[5] McCarthy JM, Planar and Spatial Rigid Motion as Special Cases of Spherical and 3−Spherical Motion, *ASME Journal of Mechanisms, Transmissions, and Automation in Design*, 105(1983), 569-575.

[6] McCarthy JM, Roth B, The Instantaneous Properties of Trajectories Generated by Planar, Spherical, and Spatial Rigid Body Motion, *ASME Journal of Mechanical Design*, 104(1982), 39-51.

[7] McCarthy JM, Ravani B, Differential Kinematics of Spherical and Spatial Motions Using Kinematic Mapping, *ASME Journal of Applied Mechanics*, 53(1986), 15-22.

[8] Nolasco B, Pacheco R, Evolutes of Plane Curves and Null Curves in Minkowski 3−space, *Journal of Geometry*,

108(2017), 195-214.

[9] Ozturk U, Koc, Ozturk, EB, Ilarslan K, On the Involute-Evolute of the Pseudonull Curve in Minkowski 3−space, *Journal of Applied Mathematics*, 2013; doi.org/10.1155/2013/651495.

[10] O'Neill B, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, London, 1983.

[11] Schaaf JA, Yang AT, Kinematic Geometry of Spherical Evolutes, *Journal of Mechanical Design*, 114(1992), 109-116.

[12] Şenyurt S, Bilici M, Çalışkan M, Some Characterizations for the Involute Curves in Dual Space, *International Journal of Mathematical Combinatorics*, 1(2015), 113-125.

[13] Turhan, T, Özdemir V, Ayyıldız N, Some Results on Point-Line Trajectories in Lorentz 3-space, *International Electronic Journal of Geometry*, 9(2)(2016), 44-49.

[14] Veldkamp GR, An Approach to Spherical Kinematics Using Tools Suggested by Plane Kinematics, *Journal of Mechanisms*, 2(1967), 437-450.

[15] Yücesan A, Tükel Özkan G, Turhan T, Curves whose pseudo spherical indicatrices are elastic, *Turkish Journal of Mathematics*, 42(6)(2018), 3123-3132.

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