



Semigroups Of bicomplex linear operators

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Abstract

In this paper, we study bicomplex matrix-valued semigroups and also investigate uniformly continuous semigroups of linear operators with bicomplex scalars.

Keywords

Matrix-valued semigroups, bicomplex modules, hyperbolic modules, bicomplex hermitian and skew-hermitian matrix, Functional Equation, one parameter semigroup, uniformly continuous operator semigroups.

AMS Subject Classification

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1. Introduction and Preliminaries

In 1892, [28] Segre introduced the term bicomplex numbers, which is a generalization of complex numbers. The set of bicomplex numbers form a ring under the usual addition and multiplication of bicomplex numbers. Moreover, \mathcal{BC} is a module over itself.

The set of bicomplex numbers do not form a field, because every non-zero bicomplex number does not have its multiplicative inverse and so the term zero-divisors is introduced in bicomplex numbers, which play a significant role in the idempotent representation of bicomplex numbers. For basis of bicomplex numbers and their properties one can refer to [1], [14] and [22].

Bicomplex numbers can serve as scalars both in the theory of functions and in functional analysis, atleast a reasonable counterpart for the quaternions. From the last two decades, the theory of bicomplex numbers and the theory of bicomplex holomorphic functions have been developed along many interesting directions.

In [7] and [8], R. Gervais Lavoie, L. Marchildon and D. Rochan started the study of bicomplex functional analysis

and introduced the finite and infinite dimensional bicomplex Hilbert spaces. In [10], the concept of topological bicomplex modules is introduced. The theory of functional analysis with bicomplex scalars is a new subject of study. A lot of work is being done in this direction. A recent survey of [1] contains comprehensive and systematic study of bicomplex functional analysis. As it is well known, that the spectrum of bounded linear operators on a Banach space with complex scalars is bounded. But in an interesting paper of F. Colombo, I. Sabadini and D. C Struppa [2], it is proved that the spectrum of bounded \mathcal{BC} -linear operators becomes unbounded due to the presence of zero-divisors in the ring of bicomplex numbers.

Recently, published book [15], gives a complete introduction to the algebra, geometry and analysis of bicomplex numbers. Like bicomplex numbers, the hyperbolic number system has been studied for various reasons, one among which is its commutative property.

The idea to develop hyperbolic numbers is an affordable replacement for the real number system. Recently, in [6], Gargoubi and Kossentini proved that the set of hyperbolic numbers is the natural generalization of real numbers into Archimedean f-algebra of dimension two. However, it was seen that the hyperbolic numbers are subset of bicomplex numbers. But the role of hyperbolic numbers are same for bicomplex numbers as the real numbers for complex numbers. In the last several years, the theory of bicomplex numbers and hyperbolic numbers has found many applications in different areas of mathematics and theoretical physics, (cf. e.g., [2], [13], [18]), and references therein.

We collect some known basic properties of bicomplex numbers. For the detail of the following discussion one can

refer [1], [2], [7], [8] and references therein. The set of bicomplex numbers is denoted by \mathcal{BC} and is defined as the commutative ring whose elements are of form $W = z_1 + jz_2$ where, $z_1 = x_1 + iy_1 \in \mathbb{C}(i)$ and $z_2 = x_2 + iy_2 \in \mathbb{C}(i)$ are complex numbers with imaginary units i and j respectively. Also require $i^2 = j^2 = -1$.

The set \mathcal{D} of hyperbolic numbers is defined as

$$\mathcal{D} = \{ \alpha = \alpha_1 + k\alpha_2 : \alpha_1, \alpha_2 \in \mathbb{R} \},$$

where k is hyperbolic unit such that $k^2 = 1$.

The hyperbolic numbers e and e^\dagger are defined as

$$e = \frac{1+k}{2} \text{ and } e^\dagger = \frac{1-k}{2}.$$

Here, e and e^\dagger form a pair of idempotents such that their product is zero and sum is equal to 1. Thus these are the zero divisors and we denote the set of zero divisors of \mathcal{BC} by \mathbb{NC} i.e.,

$$\mathbb{NC} = \{ W \mid W \neq 0, z_1^2 + z_2^2 = 0 \}.$$

A bicomplex number $W = z_1 + jz_2$ can be uniquely written as

$$\beta_1 e + \beta_2 e^\dagger,$$

where $\beta_1 = z_1 - iz_2$ and $\beta_2 = z_1 + iz_2 \in \mathbb{C}(i)$.

The set of bicomplex numbers \mathcal{BC} has three conjugations that are given for $W = \beta_1 e + \beta_2 e^\dagger \in \mathcal{BC}$ as follows

- (i) $\overline{W} = \overline{\beta_2} e + \overline{\beta_1} e^\dagger,$
- (ii) $W^\dagger = \beta_2 e + \beta_1 e^\dagger,$
- (iii) $W^* = \overline{\beta_1} e + \overline{\beta_2} e^\dagger,$

The above conjugations commute with each other and the composition of any two of them gives the third one. Each conjugation gives a corresponding moduli but we consider the only $|\cdot|_k$ which is defined as

$$|W|_k = |\beta_1| e + |\beta_2| e^\dagger, \quad |W|_k^2 = WW^*$$

for $W = \beta_1 e + \beta_2 e^\dagger \in \mathcal{BC}$.

The real valued norm on \mathcal{BC} is defined as

$$|w| = \sqrt{|z_1|^2 + |z_2|^2},$$

The norm $|\cdot|$ is such that $|u \cdot v| \leq \sqrt{2} |u| \cdot |v|,$ (cf. [1]).

A hyperbolic number $\alpha = \beta_1 + k\beta_2$ can be written as

$$\alpha = \alpha_1 e + \alpha_2 e^\dagger,$$

where $\alpha_1 = \beta_1 + \beta_2, \alpha_2 = \beta_1 - \beta_2$ are real numbers.

The set of all "positive" hyperbolic numbers are denoted by $\mathcal{D}_+,$ The set

$$\mathcal{D}_+ = \{ x + ky / x^2 - y^2 \succeq 0, x \succeq 0 \},$$

is called the set of positive hyperbolic numbers, (cf. [1], Page 5 and [14], Page 7).

For $P, Q \in \mathcal{D},$ (set of hyperbolic numbers) we define a relation \preceq' on \mathcal{D} by $P \preceq' Q \iff Q - P \in \mathcal{D}_+.$

This relation is reflexive, anti-symmetric as well as transitive and hence defines a partial order on $\mathcal{D},$ (cf. [1]).

A \mathcal{BC} -module (or \mathcal{D} -module) \mathbf{B} can be written as

$$\mathbf{B} = e\mathbf{B}_1 + e^\dagger\mathbf{B}_2, \tag{1.1}$$

where $\mathbf{B}_1 = e\mathbf{B} = e\mathbf{B}_1$ and $\mathbf{B}_2 = e^\dagger\mathbf{B} = e^\dagger\mathbf{B}_2$ are $\mathbb{C}(i)$ -vector (or \mathbb{R} -vector) spaces.

In this paper, we study bicomplex matrix-valued semigroups. We also investigate uniformly and continuous semigroups of linear operators with bicomplex scalars. For above discussion we refer to [1],[14] and [22]. The complex matrix semigroup and the uniformly continuous semigroups of linear operators for complex scalars was proved in [5, Chapter I].

A family of bounded linear operators on a Banach \mathcal{BC} -module \mathbf{B} is called a (one-parameter) semigroup or linear dynamical system on \mathbf{B} if it satisfies the Functional Equation (F.E)

$$\Phi(p+q) = \Phi(p) \cdot \Phi(q) \text{ for all } p, q \succeq '0$$

$$\Phi(0) = I.$$

where, $\Phi(\cdot) : \mathcal{D}_+ \rightarrow L(\mathbf{B}).$

There is a vast literature on the semigroups of linear operators. The theory of semigroups of linear operators is used in the study of linear parabolic, hyperbolic partial differential equations, Ergodic theory, Markov process etc.

In the first half of the century, the theory of one-parameter semigroups of linear operators on Banach space was initiated, and in 1948 it acquired its core with the Hille-Yosida generation theorem. In 1957, the edition of semigroups and Functional Analysis, by E. Hille and R. S Phillips, the theory of one-parameter semigroups on Banach space attained its first peak. For the detailed study of the theory of semigroups one can refer to [5] and [21].

The exponential function of bicomplex variable is defined as

$$\begin{aligned} e^W &= \lim_{n \rightarrow \infty} \left(1 + \frac{W}{n} \right)^n \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{z_1}{n} \right)^n e + \left(1 + \frac{z_2}{n} \right)^n e^\dagger \\ &= e^{z_1} e + e^{z_2} e^\dagger. \end{aligned}$$

(cf. [16, Theorem 10]).

2. Bicomplex Matrix Semigroups

In this section, we study the bicomplex matrix semigroups.

We start with the problem of finding all the maps satisfying the Functional Equation (F.E)

$$\Phi(p+q) = \Phi(p)\Phi(q), \text{ for all } p, q \succeq '0,$$



$$\Phi(0) = I.$$

where $\Phi(\cdot) : \mathcal{D}_+ \rightarrow M_n(\mathcal{BC})$ is a map.

Firstly we study the matrix-valued linear dynamical system on finite dimensional bicomplex module space. The work of this section is essentially based on [5, Section 2, Page 6].

We consider the space $\mathbf{B} = \mathcal{BC}^n$ and $\mathbf{L}(\mathbf{B}) = M_n(\mathcal{BC})$ acts as \mathcal{BC} module space of \mathcal{BC} -linear operators on \mathbf{B} . We have matrix-valued linear dynamical system on \mathbf{B} , defined as $\Phi(\cdot) : \mathcal{D}_+ \rightarrow M_n(\mathcal{BC})$ satisfying the F.E

$$\Phi(p+q) = \Phi(p)\Phi(q), \text{ for all } p, q \succeq' 0, \Phi(0) = I.$$

Time evolution of a state $x_o \in \mathbf{B}$ is a map $\mathcal{O}_{x_o} : \mathcal{D}_+ \rightarrow \mathbf{B}$ defined by $\mathcal{O}_{x_o}(p) = \Phi(p)x_o$. We call $\{\Phi(p)x_o : p \succeq' 0\}$ orbit of x_o under $\Phi(\cdot)$.

The main problem is to characterize all the maps $\Phi(\cdot) : \mathcal{D}_+ \rightarrow M_n(\mathcal{BC})$ satisfying F. E .

We consider a solution posed by the matrix valued exponential function

$$e^{pA} = \sum_{m=0}^{\infty} \frac{p^m A^m}{m!}, \tag{2.1}$$

where $A \in M_n(\mathcal{BC})$.

For example, if $A = [a_{ij}]_{n \times n} = [z_{ij}^1 + jz_{ij}^2]_{n \times n} \in M_n(\mathcal{BC})$, then its bicomplex Hermitian adjoint is defined as $A^* = [a_{ij}^*]_{n \times n}$, where $z_{ij}^1, z_{ij}^2 \in \mathbb{C}(i)$ and $*$ denotes the i and j conjugate of the bicomplex numbers a_{ij} . We can also write $A = A_1 + jA_2$, where $A_1, A_2 \in M_n(\mathbb{C}(i))$.

Now we discuss some properties of matrices with bicomplex entries.

Proposition 2.1. *The bicomplex matrix A is skew-Hermitian if and only if the complex matrices A_1 is skew-Hermitian and A_2 is Hermitian.*

Proof. We have, $A = A_1 + jA_2, A_1, A_2 \in M_n(\mathbb{C}(i))$.

First suppose that A_1 and A_2 are skew-hermitian and hermitian matrices respectively. We have to show that A is skew-hermitian.

Since $A_1^* = -A_1, A_2^* = A_2$,

we have, $A^* = A_1^* - jA_2^*$

$$A^* = -A_1 - jA_2$$

$$= -(A_1 + jA_2)$$

$$= -(A)$$

$$= -A.$$

Hence A is skew-Hermitian.

Conversely, suppose A is skew-hermitian.

We have to show that $A_1^* = -A_1, A_2^* = A_2$

Since $A = A_1 + jA_2$

$$A^* = A_1^* - jA_2^*$$

$$\text{So } -A = A_1^* - jA_2^*$$

$$\text{Then } -(A_1 + jA_2) = (A_1^* - jA_2^*)$$

On comparing, we get $A_1^* = -A_1, A_2^* = A_2$. □

Proposition 2.2. *The bicomplex matrix A is Hermitian if and only if the complex matrices A_1 is Hermitian and A_2 is skew-Hermitian.*

Proof. First suppose A is hermitian, We have to show that A_1 is hermitian and A_2 is skew-hermitian.

Since $A = A_1 + jA_2, A_1, A_2 \in M_n(\mathbb{C}(i))$

Also $A^* = A$

$$\text{implies } A_1^* - jA_2^* = A_1 + jA_2$$

On comparing, we get $A_1^* = A_1, A_2^* = -A_2$

Hence A_1 is hermitian and A_2 is skew-hermitian.

Conversely, $A_1^* = A_1, A_2^* = -A_2$

We have to prove that A is hermitian. Since $A = A_1 + jA_2$

$$A^* = A_1^* - jA_2^*$$

$$A^* = A_1 + jA_2 = A.$$

Hence A is hermitian □

A bicomplex matrix can also be expressed as

$$A = A_1 e + A_2 e^\dagger, \tag{2.2}$$

where $A_1, A_2 \in M_n(\mathbb{C}(i))$ such that $A_1 = A_1 - iA_2$ and $A_2 = A_1 + iA_2$.

Taking any \mathcal{BC} -norm on \mathcal{BC}^n and the corresponding matrix norm on $M_n(\mathcal{BC})$, one can show that the partial sums of the series (2.1) form a Cauchy sequence as the corresponding sums of the series of the form

$$e^{p(A_1 - iA_2)} = \sum_{m=0}^{\infty} \frac{p^m (A_1 - iA_2)^m}{m!} \tag{2.3}$$

and

$$e^{p(A_1 + iA_2)} = \sum_{m=0}^{\infty} \frac{p^m (A_1 + iA_2)^m}{m!} \tag{2.4}$$

are Cauchy sequences and hence both the series converges and satisfy the following

$$\|e^{pA_1}\| = \|e^{p(A_1 - iA_2)}\| \leq e^{p\|A_1\|} = e^{p\|A_1 - iA_2\|}$$

and

$$\|e^{pA_2}\| = \|e^{p(A_1 + iA_2)}\| \leq e^{p\|A_2\|} = e^{p\|A_1 + iA_2\|}$$

for all $p \succeq' 0$.

$$\text{We have } \|e^{pA}\| = \|e^{pA_1} e + e^{pA_2} e^\dagger\| = \frac{\sqrt{\|e^{pA_1}\|^2 + \|e^{pA_2}\|^2}}{\sqrt{2}}$$

Remark 2.3. *For all $p \succeq' 0$. we have,*

$$\begin{aligned} \| \|e^{pA}\|_{\mathcal{D}} \| &= \| \|e^{pA_1} e + e^{pA_2} e^\dagger\| \| \\ &= \| \|e^{pA_1}\|_1 e + \|e^{pA_2}\|_2 e^\dagger \| \\ &= \frac{\sqrt{\|e^{pA_1}\|_1^2 + \|e^{pA_2}\|_2^2}}{\sqrt{2}} \\ &= \|e^{pA}\|. \end{aligned}$$



Example. The following example is based on [5, Page 9]. Here we take the entries as bicomplex numbers.

- (i) A diagonal matrix $A = \text{diag}(a_1, a_2, \dots, a_n)$ generated the (semi)group which is given by $e^{pA} = \text{diag}(e^{pa_1}, e^{pa_2}, \dots, e^{pa_n})$, for $a_1, a_2, \dots, a_n \in \mathcal{BC}$.
- (ii) We also have a case of $m \times m$ Jordan block.

$$A = \begin{bmatrix} \mu & 1 & 0 & \dots & \dots & 0 \\ 0 & \mu & 1 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 & \mu \end{bmatrix}_{m \times m}$$

where, $\mu = \mu_1 + j\mu_2 \in \mathcal{BC}$, is an eigen value of A . Also $\mu = e\mu_1 + e^\dagger\mu_2$ as its idempotent decomposition. Now we write A as

$$A = e \begin{bmatrix} \mu_1 & 1 & 0 & \dots & \dots & 0 \\ 0 & \mu_1 & 1 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 & \mu_1 \end{bmatrix}_{m \times m} + e^\dagger \begin{bmatrix} \mu_2 & 1 & 0 & \dots & \dots & 0 \\ 0 & \mu_2 & 1 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 & \mu_2 \end{bmatrix}_{m \times m}$$

Using idempotent decomposition of A we can decompose A into $A = D + N$, where,

$$D = \mu I = (\mu_1 e + \mu_2 e^\dagger)(I_1 e + I_2 e^\dagger).$$

Then the m th power of N is zero i.e N is a nilpotent matrix and the power series 2.1 with A replaced by N , becomes as

$$e^{pA} = e \begin{bmatrix} 1 & p & \frac{p^2}{2!} & \dots & \dots & \frac{p^{m-1}}{(m-1)!} \\ 0 & 1 & p & \dots & \dots & \frac{p^{m-2}}{(m-2)!} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \frac{p^2}{2!} \\ \vdots & \ddots & \ddots & \ddots & \ddots & p \\ 0 & \dots & \dots & \dots & 0 & 1 \end{bmatrix}_{m \times m}$$

$$+ e^\dagger \begin{bmatrix} 1 & p & \frac{p^2}{2!} & \dots & \dots & \frac{p^{m-1}}{(m-1)!} \\ 0 & 1 & p & \dots & \dots & \frac{p^{m-2}}{(m-2)!} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \frac{p^2}{2!} \\ \vdots & \ddots & \ddots & \ddots & \ddots & p \\ 0 & \dots & \dots & \dots & 0 & 1 \end{bmatrix}_{m \times m}$$

Since $DN = ND$ we have,

$$e^{pA} = e^{p(D+N)} = e^{p\mu I} e^{pN} = e^{p\mu} e^{pN}.$$

Thus $e^{pA} = e^{p\mu} e^{pN}$.

Proposition 2.4. Let $A \in M_n(\mathcal{BC})$. Then the map $\mathcal{D}_+ \ni p \mapsto e^{pA} \in M_n(\mathcal{BC})$ is continuous and satisfies

$$e^{(p+q)A} = e^{pA} e^{qA}, \text{ for } p, q \succeq' 0, e^{0A} = I. \tag{2.5}$$

Proof. We know that the series $\sum_{m=0}^{\infty} \frac{p^m \|A\|^m}{m!}$ converges so that

$$\sum_{m=0}^{\infty} \frac{p^m A^m}{m!} \cdot \sum_{m=0}^{\infty} \frac{q^m A^m}{m!} = \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{p^{n-m} A^{n-m}}{(n-m)!} \cdot \frac{q^m A^m}{m!} = \sum_{n=0}^{\infty} \frac{(p+q)^n A^n}{n!}.$$

Thus we have F.E 2.5. To show that $p \mapsto e^{pA}$ is continuous, we first show that by F.E one has $e^{(p+r)A} - e^{pA} = e^{pA}(e^{rA} - I)$, $\forall p, r \in \mathcal{D}_+$. Thus it suffices to show that

$$\lim_{r \rightarrow 0} e^{rA} = I.$$

It follows from the estimate that □

$$\begin{aligned} \|e^{rA} - I\|_{\mathcal{D}} &= \|(e^{rA_1} e + e^{rA_2} e^\dagger) - (I_1 e + I_2 e^\dagger)\|_{\mathbb{D}} \\ &= \|(e^{rA_1} - I_1)e + (e^{rA_2} - I_2)e^\dagger\|_{\mathbb{D}} \\ &= \|e^{rA_1} - I_1\|_1 e + \|e^{rA_2} - I_2\|_2 e^\dagger \\ &\leq (e^{|r|\|A_1\|} - I_1)e + (e^{|r|\|A_2\|} - I_2)e^\dagger \end{aligned}$$

$$\Rightarrow \lim_{r \rightarrow 0} \|e^{rA} - I\|_{\mathcal{D}} \leq 0.$$

We have the mapping $p \mapsto \Phi(p)$ is a homomorphism from the additive semigroup $(\mathcal{BC}, +)$ into the multiplicative semigroup $(M_n(\mathcal{BC}), \cdot)$.

We call $(e^{pA})_{p \succeq' 0}$, the one-parameter semigroup generated by the matrix $A \in M_n(\mathcal{BC})$. □



Theorem 2.5. Let $\Phi(p) = e^{pA}$ for some $A \in M_n(\mathcal{BC})$. Then the function $\Phi(\cdot) : \mathcal{D}_+ \rightarrow M_n(\mathcal{BC})$ is differentiable and satisfies the differential equation

$$\frac{d}{dp}\Phi(p) = A\Phi(p) \text{ for } p \succeq' 0, \Phi(0) = I.$$

Conversely, every differential function $\Phi(\cdot) : \mathcal{D}_+ \rightarrow M_n(\mathcal{BC})$ satisfying the above differential equation is already of the form $\Phi(p) = e^{pA}$ for some $A \in M_n(\mathcal{BC})$. Moreover, we have $A = \dot{\Phi}(0)$.

Proof. Suppose that $\Phi(\cdot)$ satisfies D.E. Since the F.E in Proposition 2.4 implies that

$$\frac{\Phi(p+r) - \Phi(p)}{r} = \frac{\Phi(r) - I}{r} \cdot \Phi(p), \forall p, r \in \mathcal{D}, r \notin \mathbb{N}\mathbb{C} \cup \{0\}$$

so it suffices to show that

$$\lim_{r \rightarrow 0} \frac{\Phi(r) - I}{r} = A.$$

Now

$$\begin{aligned} & \frac{\Phi(r) - I}{r} - A \\ = & \frac{e^{rA} - I}{r} - A \\ = & \frac{(e^{rA_1}e + e^{rA_2}e^\dagger) - (eI_1 + e^\dagger I_2)}{r} - (A_1e + A_2e^\dagger) \\ = & \left(\frac{e^{rA_1} - I_1}{r} - A_1\right)e + \left(\frac{e^{rA_2} - I_2}{r} - A_2\right)e^\dagger. \end{aligned}$$

Also

$$\lim_{r \rightarrow 0} \left\| \frac{e^{rA_i} - I_i}{r} \right\| \leq \lim_{r \rightarrow 0} \left(\frac{e^{r\|A_i\|} - 1}{r} - \|A_i\| \right) = 0, \text{ where } i = 1, 2$$

which further implies that

$$\lim_{r \rightarrow 0} \frac{\Phi(r) - I}{r} - A = 0.$$

Conversely, we prove this by uniqueness.

Let $U(\cdot) : \mathcal{D}_+ \rightarrow M_n(\mathcal{BC})$ be another function satisfying D.E. Then the function $V(\cdot) : [0, p] \rightarrow M_n(\mathcal{BC})$ defined by

$$V(s) = \Phi(q)U(p - q)$$

for $0 \preceq' q \preceq' p$ for some fixed $p \succ' 0$ is differentiable with derivative

$$\frac{d}{dq}V(q) = 0.$$

This shows that $\Phi(p) = V(p) = V(0) = U(p)$ for arbitrary $p \succ' 0$. \square

Theorem 2.6. Let $\Phi(\cdot) : \mathcal{D}_+ \rightarrow M_n(\mathcal{BC})$ be a continuous function satisfying the F.E. Then there exists $A \in M_n(\mathcal{BC})$ such that $\Phi(p) = e^{pA}$ for each $p \succeq' 0$.

Proof. Since $\Phi(\cdot) : \mathcal{D}_+ \rightarrow M_n(\mathcal{BC})$ be a continuous function. The function $I(\cdot)$ defined by

$$I(p) = \int_0^p \Phi(q) dq; p \succeq' 0$$

is differentiable with

$$\frac{d}{dq}(I(p)) = \Phi(p).$$

This implies that

$$\lim_{p \rightarrow 0} \frac{1}{p}I(p) = I'(0) = \Phi(0) = I, \quad p \notin \mathbb{N}\mathbb{C} \cup \{0\}.$$

Therefore, $I(p_0)$ is different from zero, hence invertible, for some $p_0 \succ' 0$.

With this theorem, we have characterized that all continuous one-parameter (semi-groups) on \mathcal{BC}^n as matrix-valued exponential function $(e^{pA})_{p \succeq' 0}$. \square

3. Uniformly Continuous Operator Semigroups

In this section, we study dynamical system (semigroup) on infinite-dimensional Banach modules. The work of this section is based on [5], Section 3 [Page 14].

Next let $L(\mathbf{B})$ denote the space of all bounded (continuous) linear operator $\Phi(\cdot) : \mathcal{D} \rightarrow \mathbf{B}$ with bicomplex scalars. We know that $L(\mathbf{B})$ is a \mathcal{BC} -Banach algebra with respect to operator norm

$$\begin{aligned} \|\Phi\| &= \sup_{hyp} \left\{ \|\Phi x\|_{hyp} : \|x\|_{hyp} \preceq' 1 \right\} \\ &= \sup_{hyp} \left\{ \|\Phi x\|_{hyp} : \|x\|_{hyp} = 1 \right\}. \end{aligned}$$

(cf. [1, Page 76]).

Again we consider the Cauchy's problem in this context.

Problem 3.1. Find all the maps $\Phi(\cdot) : \mathcal{D}_+ \rightarrow L(\mathbf{B})$ satisfying the F.E

$$\Phi(p+q) = \Phi(p)\Phi(q), \text{ for all } p, q \succeq' 0, \Phi(0) = I.$$

Definition 3.2. A family $(\Phi(p))_{p \succeq' 0}$ of bounded linear operators on a Banach \mathcal{BC} -module \mathbf{B} is called a (one-parameter) semigroup (or linear dynamical system) on \mathbf{B} if it satisfies the F.E. If the F.E holds even for each $p, q \in \mathcal{D}$, we call $(\Phi(p))_{p \in \mathcal{D}}$ a one-parameter group on \mathbf{B} .

Remark 3.3. A semigroup $(\Phi(p))_{p \succeq' 0}$ is a linear dynamical system such that

- (i) p is time,
- (ii) F.E is a law of determinism,



(iii) $\{\Phi(p)x : p \in \mathcal{D}_+\}$ as the orbit of the initial value x .
 For any operator $A \in L(\mathbf{B})$. As in matrix case, define a bicomplex operator-valued exponential function by

$$e^{pA} = \sum_{m=0}^{\infty} \frac{p^m A^m}{m!},$$

where, the convergence of this series takes place in the generalized \mathcal{BC} Banach-algebra $L(\mathbf{B})$.

Lemma 3.4. Let \mathbf{B} be a Banach module which is also a algebra and $l^p(\mathbf{B})$ be the space of all p -summable sequence $x = (x_1, x_2, x_3, \dots)$ of bicomplex numbers such that

$$\sum_{i=1}^{\infty} |x_i|_k^p = |x_1|_k^p + |x_2|_k^p + |x_3|_k^p + \dots \text{ converges in } \mathcal{BC}.$$

Then

$$\sum_{i=1}^{\infty} |x_i|_k^p < \infty$$

if and only if

$$\sum_{i=1}^{\infty} |\xi_1^i|^p < \infty$$

and

$$\sum_{i=1}^{\infty} |\xi_2^i|^p < \infty,$$

where $x_i = \xi_1^i e_1 + \xi_2^i e_2$.

Note 3.5. The \mathcal{D} -norm on $l^p(X)$ for $1 \leq p < \infty$ is defined as

$$\|x\|_{\mathcal{D}} = \left(\sum_{i=1}^{\infty} |x_i|_k^p \right)^{\frac{1}{p}}.$$

Proposition 3.6. Let \mathbf{B} be a Banach module which is an algebra and $l^2(\mathbf{B})$ be the space of all square summable sequence of bicomplex numbers. Then an element $x = (x_i)$ in $l^2(\mathbf{B})$ is in $\mathbb{N}\mathbb{C}$ of $l^2(\mathbf{B})$ if and only if x_i is either zero or $x_i \in \mathbb{N}\mathbb{C}$ of \mathbf{B} for some i , where $\mathbb{N}\mathbb{C}$ denotes the null cone.

Proof. First suppose that $x = \{x_i\}$ belongs to the null cone of $l^2(\mathbf{B})$. Then there exists $0 \neq y = \{y_i\}$ in $l^2(\mathbf{B})$ such that $xy = 0$. Thus one has

$$\{x_1 y_1, x_2 y_2, x_3 y_3, \dots\} = \{0, 0, 0, \dots\}$$

Now since $y \neq 0$ implies that $y_i \neq 0$ for some i say $y_k \neq 0$. Therefore, $x_k y_k = 0$ and $y_k \neq 0$ implies that either $x_k = 0$ or x_k has to be an element of the null cone $\mathbb{N}\mathbb{C}$ of \mathbf{B} . This proves the direct part.

Conversely suppose that $x_i = 0$ or $x_i \in \mathbb{N}\mathbb{C}$ for some i , implies that $x_i y_i = 0$ for some $y_i \in \mathcal{BC}$ and so that $y = \{0, 0, 0, \dots, y_i, \dots, 0, 0, 0\} \in l^2(\mathbf{B})$ such that $xy = 0$ implies that x is in the null cone. \square

Definition 3.7. The point spectrum or the discrete spectrum denoted by $\sigma_{\mathcal{D}}(\Phi)$ is the set of all bicomplex numbers μ such that $(\Phi_{\mathcal{D}} - \mu I)^{-1}$ does not exist.

Example 3.8. Let $\phi : \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$ be given by $\phi(n) = n - 1$. Then composition operator $C_{\phi} : l^2(\mathbb{N} \cup \{0\}) \rightarrow l^2(\mathbb{N})$ defined as

$$C_{\phi}(x_n) = x_n \circ \phi = (x \circ \phi)(n) = x(n - 1).$$

for all $x_n \in l^2(\mathbb{N} \cup \{0\})$

has unbounded point spectrum. Infact it is a unilateral shift operator.

Now we state some results from [2], which will be used throughout this section.

Another notation of spectrum is defined in [2] and they called it the reduced spectrum and also proved that this reduced spectrum is a non-empty compact subset of \mathcal{BC} .

So in this case the reduced spectrum of unilateral shift is unit disc in \mathcal{BC} .

For $A \in L(\mathbf{B})$, the reduced spectrum of A is denoted by $\sigma_{red}(A)$ and its resolvent set $\rho_{red}(A) = \mathcal{BC} \setminus \sigma_{red}(A)$. Since $\sigma_{red}(A)$ is non-empty compact subset of \mathcal{BC} , $\rho_{red}(A)$ is open, and one can show that the resolvent map

$$R_{red}(\mu, A) = (\mu - A)^{-1} \in L(\mathbf{B})$$

constructs an analytic function from $\rho_{red}(A)$ into $L(\mathbf{B})$, (cf. [2]).

Definition 3.9. [2, Definition 3.13] The reduced spectrum of any operator $A \in L(\mathbf{B})$ is denoted by $\sigma_{red}(A)$ and defined as $\sigma_{red}(A) = (e\sigma(A_1) + e^{\dagger}\mathbb{C}(i)) \cap (e\mathbb{C}(i) + e^{\dagger}\sigma(A_2)) = e\sigma(A_1) + e^{\dagger}\sigma(A_2)$.

Definition 3.10. [2, Definition 3.5] Let $\Phi \in L(\mathbf{B})$. The spectrum of Φ denoted by $\sigma_{hyp}(\Phi)$ is defined as the complement of $\rho_{hyp}(\Phi)$.

$$i.e., \quad \sigma_{hyp}(\Phi) = \{\mu \in \mathcal{D} : (\Phi - \mu I) \text{ is not invertible}\}.$$

Further, we define an operator $R(\mu : \Phi) = (\Phi - \mu I)^{-1}$ for every $\mu \in \rho_{hyp}(\Phi)$ called the resolvent operator.

Now, we consider for each $p \succeq 0$, the function $\mu \rightarrow e^{p\mu}$, is analytic on all of \mathcal{BC} . We can define the exponent of A through the operator-valued version of Cauchy's integral formula.

Definition 3.11. [5, Definition 3.4] Let $A \in L(\mathbf{B})$ and choose an open neighbourhood O of $\sigma_{red}(A)$ with smooth, positively oriented boundary C such that

$$C_i = w_i : w = w_1 e + w_2 e^{\dagger} \in C,$$



for $i=1,2$.

In this case, we may write it as $C = C_1e + C_2e^\dagger$, where C_i are corresponding curves. We define the exponential function as

$$\begin{aligned}
 e^{pA} &= \frac{1}{2\pi i} \int_C e^{p\lambda} R_{red}(\mu, A) d\mu & (3.1) \\
 &= \frac{1}{2\pi i} \int_C e^{p\mu} (\mu I - A)^{-1} d\mu \\
 &= \frac{1}{2\pi i} e \int_{C_1} e^{p\mu_1} (e\mu_1 I_1 - A_1)^{-1} d\mu_1 \\
 &\quad + \frac{1}{2\pi i} e^\dagger \int_{C_2} e^{p\mu_2} (e^\dagger \mu_2 I_2 - A_2)^{-1} d\mu_2 \\
 &= \frac{1}{2\pi i} e \int_{C_1} e^{p\mu_1} R_{red}(\mu_1, A_1) d\mu_1 \\
 &\quad + \frac{1}{2\pi i} e^\dagger \int_{C_2} e^{p\mu_2} R_{red}(\mu_2, A_2) d\mu_2 d\mu_2 \\
 &= e^{pA_1} e + e^{pA_2} e^\dagger.
 \end{aligned}$$

for each $p \succeq' 0$

We see from the general theory that e^{pA} is bounded \mathcal{BC} linear operator on \mathbf{B} as e^{pA_1} and e^{pA_2} are bounded operators on complex Banach space see, [5, Definition 3.4, page 15] and that its definition does not depend on the particular choice of O . As the Functional calculus give the homomorphism from the generalized algebra of bicomplex holomorphic functions into generalized algebra of bicomplex bounded linear operators $L(\mathbf{B})$. So we get, F.E for $(e^{pA})_{p \succeq' 0}$ from

$$e^{(p+q)\mu} = e^{p\mu} \cdot e^{q\mu},$$

for $p, q \succeq' 0$.

Also the continuity of $p \rightarrow e^{pA} \in L(\mathbf{B})$ follows from the continuity of $p \rightarrow e^{p(\cdot)}$ for the topology of uniform convergence on compact subsets of \mathcal{BC} , see, [[5] Page 15].

Definition 3.12. Let X and Y be a \mathcal{D} -normed \mathcal{BC} -module and $\Phi : X \rightarrow Y$ be a \mathcal{BC} -linear operator. Then the adjoint $\Phi^* : X^* \rightarrow Y^*$ defined as

$$\Phi^*(f)(x) = f(\Phi(x)) = (C_\Phi f) \quad \forall x \in X \text{ and } f \in Y^*.$$

Theorem 3.13. [2, Theorem 3.15] Let $\Phi \in L(\mathbf{B})$ and $\Omega \subset \mathcal{BC}$ be a domain containing the reduced spectrum of Φ . Let $w_1 + jw_2 \in \Omega$ and $c_1 \subset A_1$ and $c_2 \subset A_2$ be simple, positively oriented, closed curves which are unions of continuously differentiable Jordan curves and such that c_1 surrounds $w_1 - iw_2$, c_2 surrounds $w_1 + iw_2$, and $C = c_1 * c_2 \subset \Omega$.

Then

(1) Let $\Phi = e\Phi_1 + e^\dagger\Phi_2$. Then the resolvent operator series,

$$\sum_{n \geq 0} \lambda^{-n-1} \Phi^n$$

converges in the bi-disk $\mathbb{U}(0, r_1, r_2)$ centered in the origin for $r_1 > \|\Phi_1\|$ and $r_2 > \|\Phi_2\|$.

(2) The reduced spectrum $\sigma_{red}(\Phi)$ of Φ is non-empty and

compact.

(3) For every $m \in \mathbb{N} \cup \{0\}$, we have

$$\Phi^m = \frac{1}{2\pi i} \int_C (\lambda I - \Phi)^{-1} \mu^m d\mu.$$

Proposition 3.14. [2, Theorem 3.16], (Spectral radius) Let \mathbf{B} be a Banach module over \mathcal{BC} and let $\Phi : \mathbf{B} \rightarrow \mathbf{B}$ be a \mathcal{BC} -linear bounded operator. The spectral radius $r(\Phi)$ satisfies

$$r(\Phi) \leq \frac{\sqrt{2}}{2} (r(\Phi_1) + r(\Phi_2)).$$

Theorem 3.15. [2, Theorem 3.19] Let \mathbf{B} be a Banach module over \mathcal{BC} and let $\Phi \in L(\mathbf{B})$. Let $U(a_1 + ja_2; r_1, r_2)$ be a bi-disk whose closure is contained in an open set $V \supset \sigma_{red}(\Phi)$ on which f is bi-complex holomorphic. Let C_l be union of continuously differentiable Jordan curves for $l = 1, 2$ and such that $C = c_1 * c_2$ is contained in $U(a, r_1, r_2)$. Then the operator

$$\frac{1}{2\pi i} \int_C f(\mu) (\mu I - \Phi)^{-1} d\mu$$

does not depend on the choice of C .

Proposition 3.16. [2, Theorem 3.21] Let \mathbf{B} be a \mathcal{BC} -module and let $\mu \in L(\mathbf{B})$. Let f and g be two functions locally bicomplex holomorphic on the reduced spectrum of μ . Then

(i) $(f + g)(\Phi) = f(\Phi)g(\Phi);$

(ii) $(\alpha f)(\Phi) = \alpha f(\Phi),$ for all $\alpha \in \mathcal{BC};$

(iii) $(fg)(\Phi) = f(\Phi)g(\Phi) :$

(iv) If $f(Z) = \sum_{j=0}^{\infty} \alpha_j Z^j$ on a suitable bi-disk, then $f(\Phi) = \sum_{j=0}^{\infty} \alpha_j \Phi^j.$

Proposition 3.17. For $A \in L(\mathbf{B})$, define $(e^{pA})_{p \succeq' 0}$ as in equation (3.1). Then we have

(i) $(e^{pA})_{p \succeq' 0}$ is a semi-group on \mathbf{B} such that $p \rightarrow e^{pA}$ is a continuous map from $\mathcal{D}_+ \rightarrow L(\mathbf{B})$

(ii) The map $p \rightarrow e^{pA} \in L(\mathbf{B})$ is differentiable and satisfies the D.E

$$\frac{d}{dt}(\Phi(p)) = A\Phi(p)$$

for each $p \succeq' 0, \Phi(0) = I.$

Conversely, every differential function $\Phi(\cdot) : \mathcal{D}_+ \rightarrow L(\mathbf{B})$ satisfying the differential equation is of the form $\Phi(p) = e^{pA}$ for some $A \in L(\mathbf{B})$ and $A = \frac{d}{dp}(\Phi(0)) = I.$



Proof. The resolvent of A satisfies

$$\begin{aligned}
 \frac{d}{dp} e^{pA} &= \frac{d}{dp} \frac{1}{2\pi i} \int_C e^{p\mu} R(\mu, A) d\mu \\
 &= \frac{1}{2\pi i} \int_C \mu e^{p\mu} R(\mu, A) d\mu \\
 &= \frac{1}{2\pi i} e \int_{C_1} e^{p\mu_1} A_1 R(\mu_1, A_1) d\mu_1 + \frac{1}{2\pi i} e \int_{C_1} e^{p\mu_1} d\mu_1 \\
 &\quad + \frac{1}{2\pi i} e^\dagger \int_{C_2} e^{p\mu_2} A_2 R(\mu_2, A_2) d\mu_2 \\
 &\quad + \frac{1}{2\pi i} e^\dagger \int_{C_2} e^{p\mu_2} d\mu_2 \\
 &= \frac{1}{2\pi i} e \int_{C_1} e^{p\mu_1} A_1 R(\mu_1, A_1) d\mu_1 \\
 &\quad + \frac{1}{2\pi i} e^\dagger \int_{C_2} e^{p\mu_2} A_2 R(\mu_2, A_2) d\mu_2 \\
 &= e \frac{A_1}{2\pi i} \int_{C_1} e^{p\mu_1} R(\mu_1, A_1) d\mu_1 \\
 &\quad + e^\dagger \frac{A_2}{2\pi i} \int_{C_2} e^{p\mu_2} R(\mu_2, A_2) d\mu_2 \\
 &= e A_1 e^{pA_1} + e^\dagger A_2 e^{pA_2}
 \end{aligned}$$

for each $p \succeq' 0$.

The uniqueness is again proved as for Theorem 2.5. The above properties of $(e^{pA})_{p \succeq' 0}$ for $A \in L(\mathbf{B})$, proved by using power series or via, the functional calculus, will give us a simple and suitable answer to Problem (3.1). \square

Conclusion

In this paper, we have concluded that all continuous one parameter semigroup on $\mathbb{B}C^n$ can expressed in the term of exponential function e^{pA} , where p is greater than equal to 0.

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