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Exact solutions for Klein-Gordon equation with quadratic non linearity

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Abstract
Availability of exact solutions are important for any nonlinear partial differential equation which represent any physical phenomenon. A general Klein-Gordon equation with quadratic non linearity is considered in this paper and several new exact solutions are derived. These solutions are derived in terms of Jacobi elliptic functions. The periodic solutions and hyperbolic solutions can be derived from these solutions as limiting cases of Jacobi elliptic function solutions.

Keywords
Klein-Gordon equation, Exact solutions, Periodic solutions, Jacobi elliptic functions.

AMS Subject Classification
35C05, 35C07, 35C08, 35C09, 35B10, 35A24.

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Contents

1. Introduction

There are several types of Klein-Gordon equation studied in the literature. The Klein-Gordon equation with quadratic non linearity is given by

$$\frac{\partial^2 f}{\partial t^2}-\frac{\partial^2 f}{\partial x^2}+\alpha f-\beta f^2=0 \tag{1.1}$$

and the equation with cubic non linearity is given by

$$\frac{\partial^2 f}{\partial t^2}-\frac{\partial^2 f}{\partial x^2}+\alpha f-\beta f^3=0 \tag{1.2}$$

where $f=f(x,t)$. These equations have found applications in many field of science and engineering. They are used to study problems in solid state physics, optics , classical quantum and relativistic mechanics [5, 8]. Several methods are applied to obtain exact solutions to these equations and their variants [1–4, 6, 9, 10].

In this paper the generalized Klein-Gordon equation with quadratic non linearity given by

$$\frac{\partial^2 f}{\partial t^2}+c_1\frac{\partial^2 f}{\partial x^2}+c_2+c_3 f+c_4 f^2=0 \tag{1.3}$$

is considered. Certain new exact solutions of this equation are derived in terms of Jacobi elliptic function method. Some of the known solutions can be obtained from these new solutions as particular cases. To obtain the required solutions a traveling wave ansatz method is applied in the next section. The required computations are done with the help of computer algebra system.

2. Exact solutions

The traveling wave transformation is used to obtain the exact solutions of the Klein-Gordon equation (1.3). Assume that the required solution is in the form

$$v=v(ax+bt) \tag{2.1}$$

Substituting this in the equation (1.3) we get an ordinary differential equation

$$\left(a^2c_1+b^2\right)v''+c_4v^2+c_3v+c_2=0 \tag{2.2}$$

where $v=v(\zeta)$ with $\zeta=ax+bt$. Any solution to this ordinary differential equation will lead to a solution to the Klein-Gordon equation. To obtain the required solutions several ansatz forms are assumed for the function v and the solutions are derived with the help of computer algebra system. The solutions are assumed in the form of the ansatz given by

$$v=\sum_{i=-N}^{N} A_i u^i \tag{2.3}$$

where u is some function of ζ. Comparing the highest order derivative and the highest degree of the differential equation, it can be easily seen that the $N=2$. Here $u(\zeta)$ is taken to be the different Jacobi elliptic functions with modulus k [7]. We take $e=k^2$ in all the calculations in this paper.

3. Jacobi elliptic function solutions

Assume the solutions in the form

$$v(\zeta)=a_0+a_1\operatorname{sn}(\zeta)+a_2\operatorname{sn}(\zeta)^2+a_{-1}\operatorname{ns}(\zeta)+a_{-2}\operatorname{ns}(\zeta)^2 \tag{3.1}$$

Substituting this ansatz in the equation (2.2), this will be a solution if the following algebraic equations are satisfied.

$$\begin{aligned}
&a_{-2}\left(6\left(a^2c_1+b^2\right)+a_{-2}c_4\right)=0,\\
&2a_{-1}\left(a^2c_1+a_{-2}c_4+b^2\right)=0,\\
&a_{-2}\left(-4(e+1)\left(a^2c_1+b^2\right)+2a_0c_4+c_3\right)+a_{-1}^2c_4=0,\\
&2a_{-2}a_1c_4-a_{-1}\left((e+1)\left(a^2c_1+b^2\right)-2a_0c_4-c_3\right)=0,\\
&2a_{-2}e\left(a^2c_1+b^2\right)+2a_2\left(a^2c_1+a_{-2}c_4+b^2\right)\\
&\qquad+a_0\left(a_0c_4+c_3\right)+2a_{-1}a_1c_4+c_2=0,\\
&2a_{-1}a_2c_4-a_1\left((e+1)\left(a^2c_1+b^2\right)-2a_0c_4-c_3\right)=0,\\
&a_2\left(-4(e+1)\left(a^2c_1+b^2\right)+2a_0c_4+c_3\right)+a_1^2c_4=0,\\
&2a_1\left(e\left(a^2c_1+b^2\right)+a_2c_4\right)=0,\\
&a_2\left(6e\left(a^2c_1+b^2\right)+a_2c_4\right)=0.
\end{aligned}$$

Solving this algebraic system simultaneously,

$$\begin{aligned}
&a_{-1}=a_1=0,\qquad a_{-2}=\frac{3\phi}{2c_4\sqrt{\psi\phi}},\\
&a_0=-\frac{c_3+\frac{(e+1)\phi}{\sqrt{\psi\phi}}}{2c_4},\qquad a_2=\frac{3e\phi}{2c_4\sqrt{\psi\phi}},\\
&b=\frac{1}{2}\sqrt{-\frac{4a^2c_1\psi+\sqrt{\psi\phi}}{\psi}}.
\end{aligned} \tag{3.2}$$

where $\psi=e(e+14)+1$ and $\phi=c_3^2-4c_2c_4$. Then the original solutions for the equation (1.3) are given by

$$v_1=\frac{3e\phi\operatorname{sn}(ax+bt)^2}{2c_4\sqrt{\psi\phi}}+\frac{3\phi\operatorname{sn}(ax+bt)^2}{2c_4\sqrt{\psi\phi}}+\frac{c_3+\frac{(e+1)\phi}{\sqrt{\psi\phi}}}{2c_4} \tag{3.3}$$

where the value of b is given in (3.2). The second ansatz is taken to be

$$v(\zeta)=a_0+a_1\operatorname{nc}(\zeta)+a_2\operatorname{nc}(\zeta)^2+a_{-1}\operatorname{cn}(\zeta)+a_{-2}\operatorname{cn}(\zeta)^2 \tag{3.4}$$

Substituting this ansatz in the equation (2.2), this will be a solution if the following algebraic equations are satisfied.

$$\begin{aligned}
&a_{-2}\left(a_{-2}c_4-6(e-1)\left(a^2c_1+b^2\right)\right)=0,\\
&2a_{-1}\left(a_{-2}c_4-(e-1)\left(a^2c_1+b^2\right)\right)=0,\\
&a_{-2}\left(4(2e-1)\left(a^2c_1+b^2\right)+2a_0c_4+c_3\right)+a_{-1}^2c_4=0,\\
&a_{-1}\left((2e-1)\left(a^2c_1+b^2\right)+2a_0c_4+c_3\right)+2a_{-2}a_1c_4=0,\\
&-2a_2(e-1)\left(a^2c_1+b^2\right)-2a_{-2}\left(e\left(a^2c_1+b^2\right)-a_2c_4\right)\\
&\qquad+a_0\left(a_0c_4+c_3\right)+2a_{-1}a_1c_4+c_2=0,\\
&a_1\left((2e-1)\left(a^2c_1+b^2\right)+2a_0c_4+c_3\right)+2a_{-1}a_2c_4=0,\\
&a_2\left(4(2e-1)\left(a^2c_1+b^2\right)+2a_0c_4+c_3\right)+a_1^2c_4=0,\\
&-2a_1\left(e\left(a^2c_1+b^2\right)-a_2c_4\right)=0,\\
&a_2\left(a_2c_4-6e\left(a^2c_1+b^2\right)\right)=0.
\end{aligned}$$

Solving this algebraic system simultaneously,

$$\begin{aligned}
&a_{-1}=a_1=0,\qquad a_{-2}=-\frac{3(e-1)\phi}{2c_4\sqrt{\psi\phi}},\\
&a_0=\frac{(2e-1)\sqrt{\psi\phi}-c_3\psi}{2c_4\psi},\qquad a_2=-\frac{3e\phi}{2c_4\sqrt{\psi\phi}},\\
&b=-\frac{1}{2}\sqrt{-\frac{4a^2c_1\psi+\sqrt{\psi\phi}}{\psi}}.
\end{aligned} \tag{3.5}$$

where $\psi=16(e-1)e+1$ and $\phi=c_3^2-4c_2c_4$. Then the original solutions for the equation (1.3) are given by

$$\begin{aligned}
v_2=&-\frac{3e\phi\operatorname{cn}(ax+bt)^2}{2c_4\sqrt{\psi\phi}}-\frac{3(e-1)\phi\operatorname{nc}(ax+bt)^2}{2c_4\sqrt{\psi\phi}}\\
&+\frac{(2e-1)\sqrt{\psi\phi}-c_3\psi}{2c_4\psi}
\end{aligned} \tag{3.6}$$

where the value of b is given in (3.5). The next ansatz Jacobi elliptic function ansatz is taken to be

$$v(\zeta)=a_0+a_1\operatorname{cs}(\zeta)+a_2\operatorname{cs}(\zeta)^2+a_{-1}\operatorname{sc}(\zeta)+a_{-2}\operatorname{sc}(\zeta)^2 \tag{3.7}$$

Substituting this ansatz in the equation (2.2), this will be a solution if the following algebraic equations are satisfied.

$$\begin{aligned}
&a_{-2}\left(a_{-2}c_4-6(e-1)\left(a^2c_1+b^2\right)\right)=0,\\
&2a_{-1}\left(a_{-2}c_4-(e-1)\left(a^2c_1+b^2\right)\right)=0,\\
&a_{-2}\left(-4(e-2)\left(a^2c_1+b^2\right)+2a_0c_4+c_3\right)+a_{-1}^2c_4=0,\\
&a_{-1}\left(-(e-2)\left(a^2c_1+b^2\right)+2a_0c_4+c_3\right)+2a_{-2}a_1c_4=0,\\
&-2a_2(e-1)\left(a^2c_1+b^2\right)+2a_{-2}\left(a^2c_1+a_2c_4+b^2\right)\\
&\qquad+a_0\left(a_0c_4+c_3\right)+2a_{-1}a_1c_4+c_2=0,\\
&a_1\left(-(e-2)\left(a^2c_1+b^2\right)+2a_0c_4+c_3\right)+2a_{-1}a_2c_4=0,\\
&a_2\left(-4(e-2)\left(a^2c_1+b^2\right)+2a_0c_4+c_3\right)+a_1^2c_4=0,\\
&2a_1\left(a^2c_1+a_2c_4+b^2\right)=0,\\
&a_2\left(6\left(a^2c_1+b^2\right)+a_2c_4\right)=0.
\end{aligned}$$

(3.8)

Solving this algebraic system simultaneously,

$$
\begin{aligned}
&a_{-1}=a_1=0, \qquad a_{-2}=\frac{3(e-1)\phi}{2c_4\sqrt{\psi\phi}},\\
&a_0=\frac{\frac{(e-2)\phi}{\sqrt{\psi\phi}}-c_3}{2c_4}, \qquad a_2=-\frac{3\phi}{2c_4\sqrt{\psi\phi}},\\
&b=\frac{1}{2}\sqrt{\frac{\phi}{\sqrt{\psi\phi}}-4a^2c_1}.
\end{aligned}
\tag{3.9}
$$

where $\psi=(e-16)e+16$ and $\phi=c_3^2-4c_2c_4$. Then the original solutions for the equation (1.3) are given by

$$
v_3=-\frac{3\phi\,\mathrm{cs}(ax+bt)^2}{2c_4\sqrt{\psi\phi}}+\frac{3(e-1)\phi\,\mathrm{sc}(ax+bt^2}{2c_4\sqrt{\psi\phi}}+\frac{\frac{(e-2)\phi}{\sqrt{\psi\phi}}-c_3}{2c_4}
\tag{3.10}
$$

where the value of b is given in (3.9). The next ansatz function is taken to be

$$
v(\zeta)=a_0+a_1\mathrm{cd}(\zeta)+a_2\mathrm{cd}(\zeta)^2+a_{-1}\mathrm{dc}(\zeta)+a_{-2}\mathrm{dc}(\zeta)^2
\tag{3.11}
$$

Substituting this ansatz in the equation (2.2), this will be a solution if the following algebraic equations are satisfied.

$$
\begin{aligned}
&a_{-2}\left(6\left(a^2c_1+b^2\right)+a_{-2}c_4\right)=0,\\
&2a_{-1}\left(a^2c_1+a_{-2}c_4+b^2\right)=0,\\
&a_{-2}\left(-4(e+1)\left(a^2c_1+b^2\right)+2a_0c_4+c_3\right)+a_{-1}^2c_4=0,\\
&2a_{-2}a_1c_4-a_{-1}\left((e+1)\left(a^2c_1+b^2\right)-2a_0c_4-c_3\right)=0,\\
&2a_{-2}e\left(a^2c_1+b^2\right)+2a_2\left(a^2c_1+a_{-2}c_4+b^2\right)\\
&\qquad +a_0\left(a_0c_4+c_3\right)+2a_{-1}a_1c_4+c_2=0,\\
&a_1\left(-(e+1)\left(a^2c_1+b^2\right)+2a_0c_4+c_3\right)+2a_{-1}a_2c_4=0,\\
&a_2\left(-4(e+1)\left(a^2c_1+b^2\right)+2a_0c_4+c_3\right)+a_1^2c_4=0,\\
&2a_1\left(e\left(a^2c_1+b^2\right)+a_2c_4\right)=0,\\
&a_2\left(6e\left(a^2c_1+b^2\right)+a_2c_4\right)=0.
\end{aligned}
\tag{3.12}
$$

Solving this algebraic system simultaneously,

$$
\begin{aligned}
&a_{-1}=a_1=0, \qquad a_{-2}=\frac{3\phi}{2c_4\sqrt{\psi\phi}}, \qquad a_0=-\frac{c_3+\frac{(e+1)\phi}{\sqrt{\psi\phi}}}{2c_4},\\
&a_2=\frac{3e\phi}{2c_4\sqrt{\psi\phi}}, \qquad b=-\frac{1}{2}\sqrt{-\frac{4a^2c_1\psi+\sqrt{\psi\phi}}{\psi}}.
\end{aligned}
\tag{3.13}
$$

where $\psi=e(e+14)+1$ and $\phi\to c_3^2-4c_2c_4$. Then the original solutions for the equation (1.3) are given by

$$
v_4=\frac{3e\phi\,\mathrm{cd}((ax+bt)^2}{2c_4\sqrt{\psi\phi}}+\frac{3\phi\,\mathrm{dc}((ax+bt)^2}{2c_4\sqrt{\psi\phi}}-\frac{c_3+\frac{(e+1)\phi}{\sqrt{\psi\phi}}}{2c_4}
\tag{3.14}
$$

where the value of b is given in (3.13)

4. Discussion

Several new exact solutions for general Klein-Gordon equation with quadratic non linearity are derived in this paper. These solutions are derived using the Jacobi elliptic function ansatz forms. Since the computations are hard, the required calculations are performed using any of the computer algebra system such as Maple or Mathematica. Some of the trigonometric and hyperbolic solutions which are existing in the literature are obtained when the modulus of the Jacobi elliptic functions tends to zero or one. The trigonometric solutions are obtained by putting $e=0$ in the four solutions derived in this paper and are given by

$$
\begin{aligned}
v_5&=-\frac{-3\sqrt{\phi}\csc^2\left(ax+\frac{\sqrt{b_1}t}{2}\right)+c_3+\sqrt{\phi}}{2c_4},\\
v_6&=-\frac{-3\sqrt{\phi}\sec^2\left(ax-\frac{\sqrt{b_1}t}{2}\right)+c_3+\sqrt{\phi}}{2c_4},\\
v_7&=\frac{1}{8c_4}\left(-4c_3+\sqrt{\phi}-3\sqrt{\phi}\left(\cot^2\left(ax+\frac{\sqrt{b_2}t}{4}\right)\right.\right.\\
&\qquad\left.\left.+\sec^2\left(ax+\frac{\sqrt{b_2}t}{4}\right)\right)\right),\\
v_8&=-\frac{-3\sqrt{\phi}\sec^2\left(ax-\frac{\sqrt{b_1}t}{2}\right)+c_3+\sqrt{\phi}}{2c_4}
\end{aligned}
$$

and, when $e=1$, the hyperbolic solutions are given by

$$
\begin{aligned}
v_9&=\frac{1}{8c_4}\left(-4c_3+\sqrt{\phi}+3\sqrt{\phi}\left(\coth^2\left(ax+\frac{\sqrt{b_3}t}{4}\right)\right.\right.\\
&\qquad\left.\left.-\mathrm{sech}^2\left(ax+\frac{\sqrt{b_3}t}{4}\right)\right)\right),\\
v_{10}&=\frac{-3\sqrt{\phi}\,\mathrm{sech}^2\left(ax-\frac{\sqrt{b_1}t}{2}\right)-c_3+\sqrt{\phi}}{2c_4},\\
v_{11}&=-\frac{3\sqrt{\phi}\,\mathrm{csch}^2\left(ax+\frac{\sqrt{b_4}t}{2}\right)+c_3+\sqrt{\phi}}{2c_4},\\
v_{12}&=\frac{\sqrt{\phi}-c_3}{2c_4}
\end{aligned}
$$

where $b_1=-4a^2c_1-\sqrt{\phi}, b_2=\sqrt{\phi}-16a^2c_1, b_3=-16a^2c_1-\sqrt{\phi}$ and $b_4=\sqrt{\phi}-4a^2c_1$. The ansatz method used in this paper are easier and powerful method in deriving the exact solutions of Klein-Gordon equations. Further exact solutions can also be derived for this equation by assuming similar ansatz forms. Such ansatz method can be applied to obtain exact solutions of other nonlinear partial differential equations also.

References

[1] M. N. Alam and X. Li, New soliton solutions to the nonlinear complex fractional Schrödinger equation and the conformable time-fractional Klein–Gordon equation with quadratic and cubic nonlinearity, *Physica Scripta*, 95(4)(2020), https://orcid.org/0000-0001-9352-7087

[2] R. Côte and Y. Martel, Multi-travelling waves for the nonlinear Klein-Gordon equation, *Trans. Amer. Math. Soc.* 370 (2018), 7461–7487

[3] Y. O. El-Dib, Periodic solution of the cubic nonlinear Klein–Gordon equation and the stability criteria via the He-multiple-scales method, *Pramana*, 92(7)(2019), https://doi.org/10.1007/s12043-018-1673-3

[4] A. Elgarayhi, New traveling wave solutions for the shallow water equations and the generalized Klein-Gordon equation, *Commun. Nonlinear Sci. Numer. Simul.* 13(5)(2008), 877–888

[5] M. A. Ghabshi, E.V. Krishnan and M. Alquran, Exact Solutions of a Klein-Gordon System by (G'/G)-Expansion Method and Weierstrass Elliptic Function Method, *Nonlinear Dynamics and Systems Theory*, 19 (3) (2019), 386–395,

[6] K. Hosseini, P. Mayeli and R. Ansari, Bright and singular soliton solutions of the conformable time-fractional Klein–Gordon equations with different nonlinearities, *Waves in Random and Complex Media*, 28(3)(2018), 426–434

[7] F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.

[8] A.M. Wazwaz, New travelling wave solutions to the Boussinesq and the Klein-Gordon equations, *Commun. Nonlinear Sci. Numer. Simul.* 13(2008), 889–901

[9] Z. Zhang, New Exact Traveling Wave Solutions for the Nonlinear Klein-Gordon Equation, *Turk. J. Phys.* 32 (2008), 235 – 240.

[10] A. I. Zhurov and A. D. Polyanin, Symmetry reductions and new functional separable solutions of nonlinear Klein–Gordon and telegraph type equations, *Journal of Nonlinear Mathematical Physics*, 27(2)(2020), 227–242

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