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Several exact solutions for three dimensional Schrodinger equation involving inverse square and power law potentials

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Abstract
Several exact solutions for steady state Schrodinger equation in three dimensional space are derived in this paper. The potentials are taken to be sum of an inverse square potential and a power law potential. Different new exact solutions of Schrodinger equation are derived for this potential with zero energy. The solutions are derived in cartesian coordinates without separation of variables. Certain exact solutions for non-zero energy are also derived for Schrodinger equation with inverse square potential.

Keywords
Schrodinger equation, Exact solution, Zero and non-zero energy, Inverse Square Potential, Power Law Potential.

AMS Subject Classification
35A09, 35A24, 35Q40.

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Contents

1. Introduction

Schrodinger equation is having many applications not only in modern physics but also in several other fields such as quantum information and econophysics[1, 3, 5, 27, 30]. So exact solutions to Schrodinger equations are also having applications in these and several other fields. Availability of exact solutions for any partial differential equation which represent a physical system or phenomena are inevitable for a better understanding of behavior of the system or phenomena. They can also be utilized to check the correctness of approximate methods developed for obtaining specific solutions.

There are several exact solutions available in the literature for Schrodinger equation corresponding to the well known potentials such as Coulomb, harmonic oscillator inverse square, Mie-type, Eckart, Poschl-Teller, Morse , Rosen-Morse, Woods-Saxen, Manning-Rosen, Scarf and Gendenshtein potentials [7, 9, 17, 18, 23, 24, 29]. In the case of such equations one can convert them in to ordinary differential equations by suitable transformations. These equations can be solved using well known special functions such as Bessel, Hermite, Legendre, Heun, Whittaker and confluent hypergeometric functions. Some of the recent works involving exact solutions of Schrodinger equation are [12, 13, 20, 21, 25, 26] and involving approximate solutions are [8, 11]. Exact solutions of Schrodinger equation in three dimensional cases are obtained mainly by solving the corresponding radial part of the equation after assuming the solution in the separation of variables form. The motion of particles in inverse square potential has been discussed in [10, 16, 28].

There are several situations where Schrodinger equation with zero energy can be applied [2, 4, 6, 14, 15, 19, 22]. Schrodinger equation can be simplified by assuming zero energy when discussing the case of scattering of ultracold particles. The zero energy eigenstates in the parabolic potential barrier play important role in statistical mechanics in Gelfand triplet[14]. Zero energy exact solutions are also motivated by studies in super symmetric quantum mechanics and they are also used in zero energy limit calculations in the study of loosely bound systems, scattering length and coupling parameter calculations[2].

Plan of the paper is as follows. In the next section certain exact solutions of Schrodinger equation with zero energy are derived. The potential is taken to be the sum of an inverse square potential and a power law potential. The required three dimensional solutions are obtained without separation of variables. In the third section certain exact solution of Schrodinger equation are derived with non-zero energy and in the case of inverse square potential. The exact solutions that we have derived in both cases are always bound state solutions.

2. Zero energy exact solutions

Most of the exact solutions available for Schrodinger equation in the literature are usually derived by the method of separation of variables in the spherical coordinates. This will lead to an ordinary differential equation corresponding to radial part. This equation is solved to find various exact solutions in the case of potentials which are functions of the radial vector $r=\sqrt{x^2+y^2+z^2}$ only. But, in this paper we will find out certain exact solutions to the three dimensional Schrodinger partial differential equations at zero energy level in cartesian coordinates without applying the method of separation of variables. The potential that we consider here is the sum of inverse square potential and a power law potential. The corresponding equation is given by

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=\left(\frac{a}{x^2+y^2+z^2}-b\left(x^2+y^2+z^2\right)^k\right)f(x,y,z) \tag{2.1}$$

where a, b and k are arbitrary parameters. Assuming particular forms for the required exact solutions we will solve the Schrodinger equation (2.1). The exact solutions that we are interested to derive are the bound state solutions $f(x,y,z)$ to Schrodinger equation which are zero at origin and also tends to zero as x or y or z tends to $\pm\infty$.

2.1 Case I

The equation (2.1) is converted into an ordinary differential equation by a suitable substitution. Putting $u=x^2+y^2+z^2$ in equation (2.1) we get an ordinary differential equation in terms of $g(u)=f(x,y,z)$ given by

$$4\left(ug''(u)+6g'(u)\right)+\left(bu^k-\frac{a}{u}\right)g(u)=0 \tag{2.2}$$

Now we use the following change of variables to the independent and dependent variables

$$z=\frac{\sqrt{bu^{k+1}}}{k+1} \tag{2.3}$$

and $g(u)=u^{\frac{-1}{4}}w(z)$. On substitution and simplification the differential equation (2.2) becomes

$$z^2w''(z)+zw'(z)+w(z)\left(z^2-\frac{4a+1}{4(k+1)^2}\right)=0 \tag{2.4}$$

This is a Bessel's equation whose solutions in terms of Bessel functions of first and second kinds are given by

$$c_1J_{\frac{\sqrt{4a+1}}{2(k+1)}}(z)+c_2Y_{\frac{\sqrt{4a+1}}{2(k+1)}}(z) \tag{2.5}$$

where c_1 and c_2 are arbitrary constants. Hence particular solutions to Schrodinger equation (2.1) are given by

$$\left(x^2+y^2+z^2\right)^{-\frac{1}{4}}\left(c_1J_{\frac{\sqrt{4a+1}}{2(k+1)}}\left(\frac{\sqrt{b}\left(x^2+y^2+z^2\right)^{\frac{k+1}{2}}}{k+1}\right)+c_2Y_{\frac{\sqrt{4a+1}}{2(k+1)}}\left(\frac{\sqrt{b}\left(x^2+y^2+z^2\right)^{\frac{k+1}{2}}}{k+1}\right)\right) \tag{2.6}$$

For real numbers $a\geq -1/4$ and $b>0$, the solution given by Bessel function of first kind satisfies the conditions for a bound state wave function, that is, they are going to zero as x, y or z tends to $\pm\infty$. Also it becomes zero if all x, y and z are zeroes. Graphical representation of a particular wave function is given in figure 1 for fixed values for the variable z.

2.2 Case II

Now we seek a solution in the form in the form $f(x,y,z)=xg(x^2+y^2+z^2)$. We convert the corresponding equation (2.1) into an ordinary differential equation by putting $u=x^2+y^2+z^2$ .Then we get an ordinary differential equation in terms of $g(u)$ given by

$$4ug''(u)+10g'(u)+\left(bu^k-\frac{a}{u}\right)g(u)=0 \tag{2.7}$$

Here we use the change of variables to the independent and dependent variables given by equation (2.3) and $g(u)=u^{-3/4}w(z)$. On substitution and simplification the differential equation (2.7) becomes

$$z^2w''(z)+zw'(z)+w(z)\left(z^2-\frac{4a+9}{4(1+k)^2}\right)=0 \tag{2.8}$$

This is again Bessel's equation whose solutions in terms of Bessel functions of first and second kinds are given by

$$c_1J_{\frac{\sqrt{4a+9}}{2(k+1)}}(z)+c_2Y_{\frac{\sqrt{4a+9}}{2(k+1)}}(z) \tag{2.9}$$

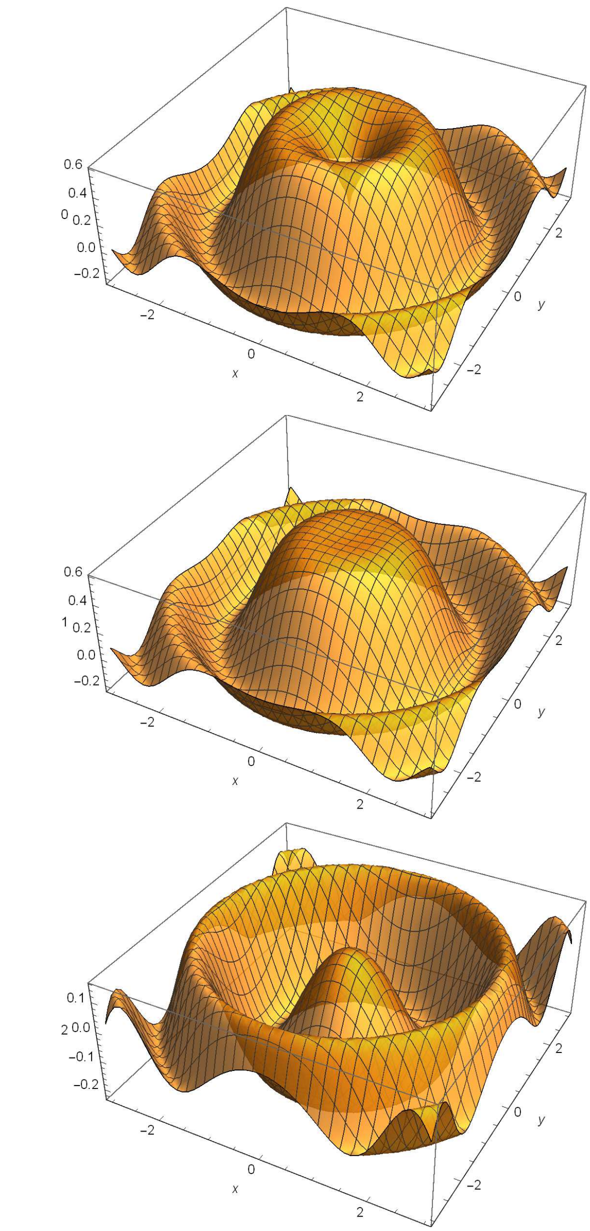

Figure 1. Graphical representation of wave function given by equation (2.6) with $c_1 = 1, c_2 = 0, a = 1, b = 2, k = 1$ and for $z = 0, z = 1$ and $z = 2$ respectively (Case I)

where c_1 and c_2 are arbitrary constants. Hence particular solutions to Schrodinger equation (2.1) are given by

$$x\left(x^2+y^2+z^2\right)^{-\frac{3}{4}}\left(c_1 J_{\frac{\sqrt{4a+9}}{2(k+1)}}\left(\frac{\sqrt{b(x^2+y^2+z^2)^{k+1}}}{k+1}\right)+ c_2 Y_{\frac{\sqrt{4a+9}}{2(k+1)}}\left(\frac{\sqrt{b(x^2+y^2+z^2)^{k+1}}}{k+1}\right)\right) \tag{2.10}$$

For any real numbers $a \geq -4/9$ and $b \geq 0$, Bessel function of first kind gives a bound sate solution of wave equation. Its graphical representation is given in figure 2

2.3 Case III

Here we assume that a solution can be written in the form $f(x,y,z) = xyg(x^2+y^2+z^2)$. We convert the corresponding equation (2.1) into an ordinary differential equation by putting $u = x^2+y^2+z^2$. Then we get an ordinary differential equation in terms of $g(u)$ given by

$$4ug''(u)+14g'(u)+\left(bu^k-\frac{a}{u}\right)g(u)=0 \tag{2.11}$$

Here we use the change of variables (2.3) and $g(u) = u^{-\frac{5}{4}}w(z)$. On substitution and simplification the differential equation (2.11) becomes

$$z^2w''(z)+zw'(z)+w(z)\left(z^2-\frac{25+4a}{4(k+1)^2}\right)=0 \tag{2.12}$$

This is a Bessel's equation whose solutions in terms of Bessel functions of first and second kinds are given by

$$c_1 J_{\frac{\sqrt{25+4a}}{2(k+1)}}(z)+c_2 Y_{\frac{\sqrt{25+4a}}{2(k+1)}}(z) \tag{2.13}$$

where c_1 and c_2 are arbitrary constants. Hence particular solutions to Schrodinger equation (2.1) are given by

$$(xy)\left(x^2+y^2+z^2\right)^{-\frac{5}{4}}\left(c_1 J_{\frac{\sqrt{25+4a}}{2(k+1)}}\left(\frac{\sqrt{b(x^2+y^2+z^2)^{k+1}}}{k+1}\right)+ c_2 Y_{\frac{\sqrt{25+4a}}{2(k+1)}}\left(\frac{\sqrt{b(x^2+y^2+z^2)^{k+1}}}{k+1}\right)\right) \tag{2.14}$$

For any real numbers $a \geq -25/4$ and $b \geq 0$, Bessel function of first kind gives a bound sate solution of wave equation. Its graphical representation is given in figure 3

2.4 Case IV

Here we assume that a solution can be written in the form $f(x,y) = (xyz)g(x^2+y^2+z^2)$. We convert the corresponding equation (2.1) into an ordinary differential equation by putting $u = x^2+y^2+z^2$ .Then we get an ordinary differential equation in terms of $g(u)$ given by

$$4ug''(u)+18g'(u)+\left(bu^k-\frac{a}{u}\right)g(u)=0 \tag{2.15}$$

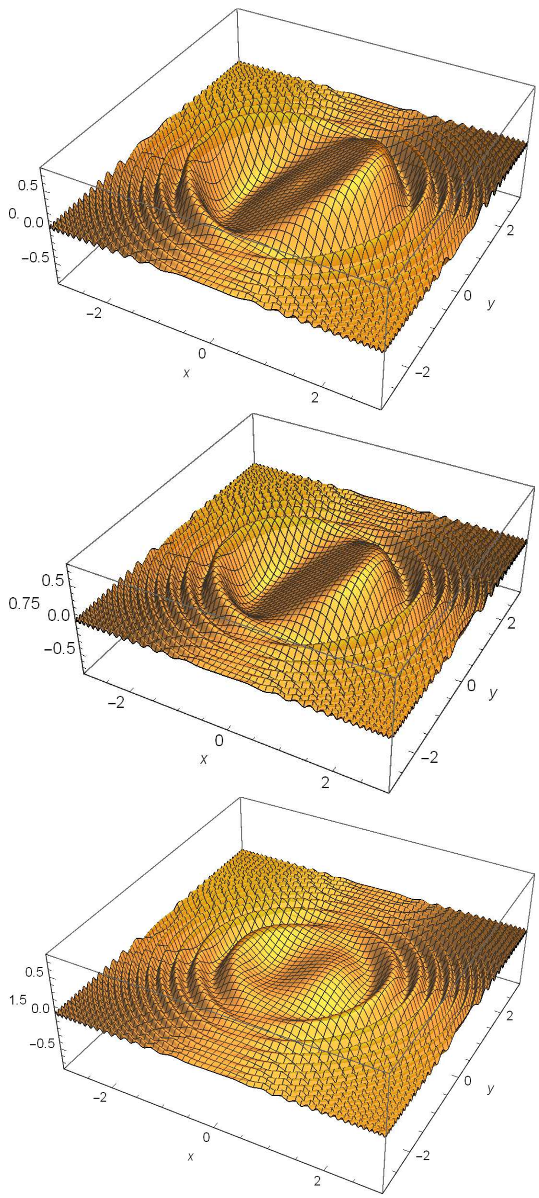

Figure 2. Representation of wave function given by equation (2.10) with $c_1 = 1, c_2 = 0, a = 1, b = 2, k = 3$ and and for $z = 0, z = .75$ and $z = 1.5$ respectively (Case II)

Here we use the change of variables given by (2.3) and $g(u) = u^{-\frac{7}{4}} w(z)$.

$$z^2 w''(z) + z w'(z) + w(z)\left(z^2 - \frac{49+4a}{4(k+1)^2}\right) = 0 \quad (2.16)$$

This is a Bessel's equation whose solutions in terms of Bessel functions of first and second kinds are given by

$$c_1 J_{\frac{\sqrt{49+4a}}{2(k+1)}}(z) + c_2 Y_{\frac{\sqrt{49+4a}}{2(k+1)}}(z) \quad (2.17)$$

where c_1 and c_2 are arbitrary constants. Hence particular solutions to Schrodinger equation (2.1) are given by

$$(xyz)\left(x^2+y^2+z^2\right)^{-\frac{7}{4}}\left(c_1 J_{\frac{\sqrt{25+4a}}{2(k+1)}}\left(\frac{\sqrt{b(x^2+y^2+z^2)^{k+1}}}{k+1}\right) + c_2 Y_{\frac{\sqrt{25+4a}}{2(k+1)}}\left(\frac{\sqrt{b(x^2+y^2+z^2)^{k+1}}}{k+1}\right)\right) \quad (2.18)$$

For any real numbers $a \geq -49/4$ and $b \geq 0$, Bessel function of first kind gives a bound sate solution of wave equation. Its graphical representation is given in figure 4

3. Non-zero energy solutions

Now we will derive certain exact solutions for three dimensional Schrodinger equation with non-zero energy. The potential that we consider here is the inverse square potential. The corresponding equation is given by

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{a}{x^2+y^2+z^2} - E\right) f(x,y) \quad (3.1)$$

Assuming particular forms for the solution we will find out certain exact solution of the equation (3.1). We are interested to derive only the bound state solutions $f(x,y)$ to Schrodinger equation which are zero at origin and also tends to zero as x or y tends to $\pm\infty$. Since the procedure is same as in the previous section, the exact solutions are summarized below.

3.1 Case I

We convert equation (3.1) into an ordinary differential equation by a suitable substitution. Putting $u = x^2+y^2+z^2$ in the corresponding equation we get an ordinary differential equation in terms of $g(u) = f(x,y,z)$ given by

$$4ug''(u) + 6g'(u) - \left(\frac{a}{u} - E\right) g(u) = 0 \quad (3.2)$$

Solving this, particular solutions to Schrodinger equation (3.1) are given by

$$\left(x^2+y^2+z^2\right)^{-\frac{1}{4}}\left(c_1 J_{\frac{1}{2}\sqrt{4a+1}}\left(\sqrt{E\left(x^2+y^2+z^2\right)}\right) + c_2 Y_{\frac{1}{2}\sqrt{4a+1}}\left(\sqrt{E\left(x^2+y^2+z^2\right)}\right)\right)$$

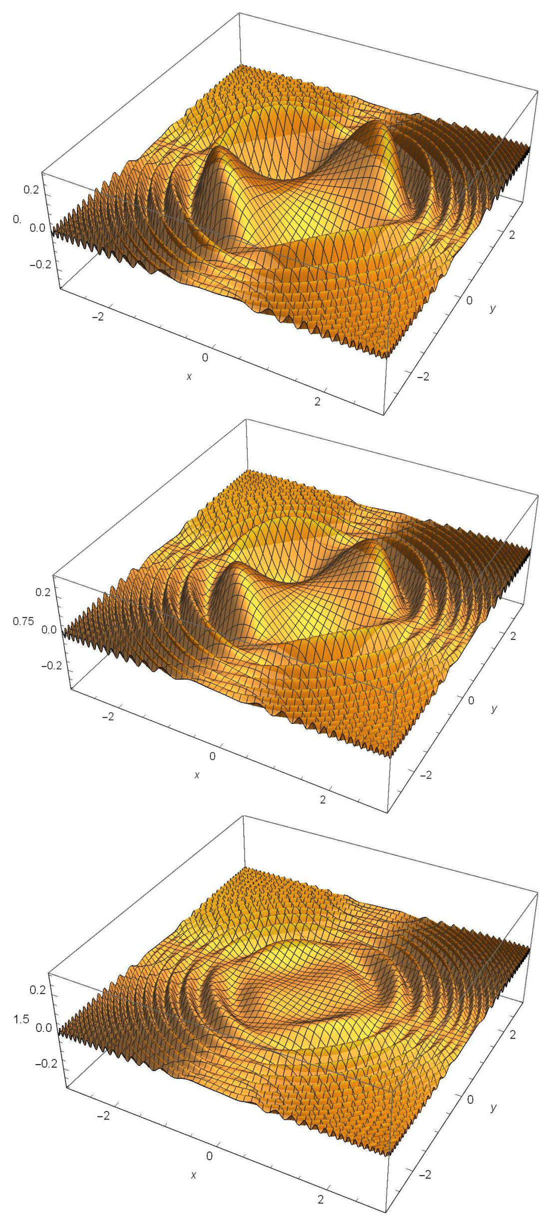

Figure 3. Representation of wave function given by equation (2.14) with $c_1 = 1, c_2 = 0, a = 1, b = 2, k = 3$ and and for $z = 0, z = 0.75$ and $z = 1.5$ respectively (Case III)

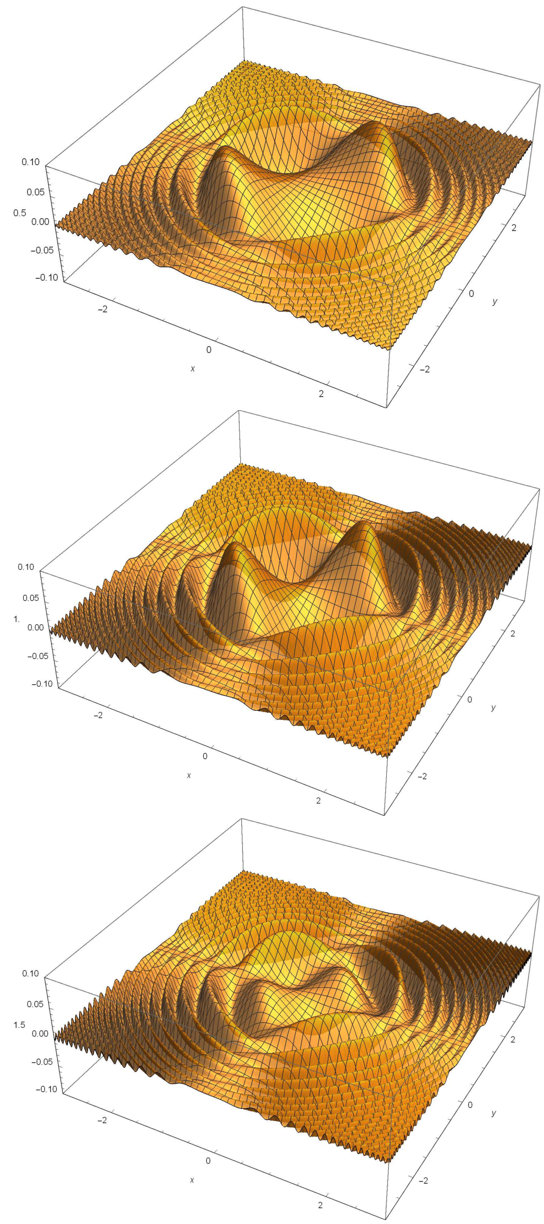

Figure 4. Representation of wave function given by equation (2.18) with $c_1 = 1, c_2 = 0, a = 1, b = 2, k = 3$ and and for $z = 0.5, z = 1$ and $z = 1.5$ respectively (Case IV)

(3.3)

For real numbers $a \geq -1/4$, the solution given by Bessel function of first kind satisfies the conditions for a bounded state wave function, that is, they are going to zero as x, y or z tends to $\pm\infty$. Also it becomes zero if all x, y and z become zero.—

3.2 Case II

Here we assume that a solution can be written in the form $f(x,y,z) = xg(x^2+y^2+z^2)$. Then we can derive particular solutions to Schrodinger equation (3.1) which are given by

$$x\left(x^2+y^2+z^2\right)^{-\frac{3}{4}}\left(c_1 J_{\frac{1}{2}\sqrt{4a+9}}\left(\sqrt{E\left(x^2+y^2+z^2\right)}\right) + c_2 Y_{\frac{1}{2}\sqrt{4a+9}}\left(\sqrt{E\left(x^2+y^2+z^2\right)}\right)\right) \tag{3.4}$$

For any real numbers $a \geq -9/4$, Bessel function of first kind gives a solution to wave equation which satisfies the conditions for a desirable wave function for all positive energy..

3.3 Case III

Next we assume that a solution can be written in the form $f(x,y,z) = xyg(x^2+y^2+z^2)$. Then we can derive particular solutions to Schrodinger equation (3.1) which are given by

$$xy\left(x^2+y^2+z^2\right)^{-\frac{5}{4}}\left(c_1 J_{\frac{1}{2}\sqrt{4a+25}}\left(\sqrt{E\left(x^2+y^2+z^2\right)}\right) + c_2 Y_{\frac{1}{2}\sqrt{4a+25}}\left(\sqrt{E\left(x^2+y^2+z^2\right)}\right)\right) \tag{3.5}$$

For any real numbers $a \geq -25/4$, here the Bessel function of first kind gives a solution to wave equation which satisfies the required conditions for a desirable wave function for all positive energy levels.

3.4 Case IV

Here we assume that a solution can be written in the form $f(x,y,z) = (xyz)g(x^2+y^2+z^2)$. Then we can derive particular solutions to Schrodinger equation (3.1) which are given by

$$(xyz)\left(x^2+y^2+z^2\right)^{-\frac{7}{4}}\left(c_1 J_{\frac{1}{2}\sqrt{4a+49}}\left(\sqrt{E\left(x^2+y^2+z^2\right)}\right) + c_2 Y_{\frac{1}{2}\sqrt{4a+49}}\left(\sqrt{E\left(x^2+y^2+z^2\right)}\right)\right) \tag{3.6}$$

Here also for any real numbers $a \geq -49/4$, the Bessel function of first kind gives a solution to wave equation which satisfies the required conditions for a desirable wave function for all positive energy levels.

4. Conclusion

Several exact solutions of Schrodinger equation in three dimensional space are derived in this paper. Usually the solutions of these equations with different potentials are derived in spherical coordinates, after assuming the solution in the variable separable form. Then the resulting radial equation is solved for different potential which are given as a function of only the radial variable r. The other parts of solution will be having the same solution always. The solutions that we have derived in this paper are derived without using separation of variables. There are several papers which discuss the application of zero energy potentials [2, 4, 6, 14, 15, 19, 22]. We derived zero energy solutions of Schrodinger equation in the case of the potential which is a sum of inverse square potential and power law potential and all these solutions are vanishing at origin and at infinity. Certain non-zero energy solutions for the inverse square potentials are derived in the third section. The assumed potentials have many applications and such problems are recently discussed in [10, 16, 28]. All the solutions are represented in terms of Bessel functions and all of them vanishes at the origin and tends to zero as $|x|$, $|y|$ or $|z|$ tends to infinity. The solutions given in terms of Bessel functions of first kind are all bound state solutions for the Schrodinger equation, for parameters in certain intervals. The solutions obtained in the third section are continuum wave functions for all positive values of energy E. All the derived solutions are new in the literature and they can be used to analyze the physical situations where the inverse square potentials and power law potentials play a major role. The obtained solutions can not only be applied to the areas of modern physics and chemistry but also in the emerging fields such as quantum information and econophysics [1, 3, 5, 27, 30], where Schrodinger equation is finding many new applications.

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References

[1] K. Ahn, M. Y. Choi, B. Dai, S. Sohn and B. Yang, Modeling stock return distributions with a quantum harmonic oscillator, *Europhysics Letters*, 120(3) (2018), https://doi.org/10.1209/0295-5075/120/38003

[2] A. D. Alhaidari, Exact solutions of Dirac and Schrödinger equations for a large class of power-law potentials at zero energy, *International Journal of Modern Physics A(Particles and Fields; Gravitation; Cosmology)*, 17(30)(2002), 4551–4566.

[3] B. E. Baaquie, *Quantum Finance*, Cambridge Univ. Press, Cambridge, 2004.

[4] B. Bagchi, C. Quesne , Zero-energy states for a class of quasi-exactly solvable rational potentials, *Physics Letters A*, 230(1–2) (1997), 1–6.

[5] G. Chen, D. A. Church, B.G. Englert, C. Henkel, B. Rohwedder, M. O. Scully, and M. S. Zubairy, *Quantum Computing Devices: Principles, Designs, and Analysis*, Chapman and Hall/CRC, New York, 2007.

[6] Jamil Daboul and Michael Martin Nieto, Exact, E=0, classical solutions for general power-law potentials, *Phys. Rev. E*, 52, (1995), https://doi.org/10.1103/PhysRevE.52.4430

[7] C. Eckart , The penetration of a potential barrier by electrons, *Phys. Rev.* 35(11)(1930), 1303–1309

[8] M. N. Farizky, A. Suparmi , C. Cari, M. Yunianto, Solution of three dimensional Schrodinger equation for Eckart and Manning-Rosen non-central potential using asymptotic iteration method, *Journal of Physics: Conference Series* 776(2016), 012085

[9] Frederick L. Scarf, New Soluble Energy Band Problem, *Phys. Rev.* 112 (1958), https://doi.org/10.1103/PhysRev.112.1137

[10] Elisa Guillaumín-España, H. N. Núñez-Yépez, and A. L. Salas-Brito, Classical and quantum dynamics in an inverse square potential, *Journal of Mathematical Physics*, 55(2014), 103509, https://doi.org/10.1063/1.4899083

[11] M. Hamzavi and S.M. Ikhdair, Approximate l-state solution of the trigonometric Pöschl–Teller potential, *Molecular Physics*, 110(24)(2012), https://doi.org/10.1080/00268976.2012.695029

[12] Felix Iacob, and Marina Lute, Exact solution to the Schrödinger's equation with pseudo-Gaussian potential, *Journal of Mathematical Physics*, 56(2015), 121501, https://doi.org/10.1063/1.4936309

[13] A. M. Ishkhanyan, Exact solution of the Schrödinger equation for the inverse square root potential $V_0/\sqrt{x}$, *Europhysics Letters*,112(1) (2015), https://doi.org/10.1209/0295-5075/112/10006

[14] T. Kobayashi, T. Shimbori, Zero-energy solutions and vortices in Schrödinger equations, *Phys. Rev. A*, 65(2002), https://doi.org/10.1103/PhysRevA.65.042108

[15] A. J.Makowski, Exact, zero-energy, square-integrable solutions of a model related to the Maxwell's fish-eye problem, *Annals of Physics*, 324(12)(2009), 2465–2472.

[16] R. P. Martínez-y-Romero, H. N. Núñez-Yépez, and A. L. Salas-Brito, The two dimensional motion of a particle in an inverse square potential: Classical and quantum aspects, *Journal of Mathematical Physics*, 54(2013), 053509, doi: 10.1063/1.4804356.

[17] F.Millard, Manning and Nathan Rosen A Potential Function for the Vibrations of Diatomic Molecules, *Phys. Rev.* 44(10)(1933), 953-960.

[18] P. M.Morse, Diatomic molecules according to the wave mechanics. II. Vibrational levels, *Phys. Rev.* 34(1929), 5764 ,https://doi.org/10.1103/PhysRev.34.57

[19] O. Mustafa, Auxiliary Quantization Constraints on The Von Roos Ordering-Ambiguity at Zero Binding Energies; Azimuthally Symmetrized Cylindrical Coordinates, *Modern Physics Letters A: Particles and Fields; Gravitation; Cosmology and Nuclear Physics*, 28(19)(2013), https://doi.org/10.1142/S021773231350082X

[20] I. H. Naeimi, J. Batle , S. Abdalla, Solving the two-dimensional Schrödinger equation using basis truncation: A hands-on review and a controversial case, *Pramana – J. Phys.* 89(2017), 70, https://doi.org/10.1007/s12043-017-1467-z

[21] T. Olsen, S. Latini, F. Rasmussen, K. S. Thygesen, Simple Screened Hydrogen Model of Excitons in Two-Dimensional Materials, *Phys. Rev. Lett.* 116(2016), https://doi.org/10.1103/PhysRevLett.116.056401

[22] J. Pade, Exact solutions of the Schrödinger equation for zero energy, *Eur. Phys. J. D*, 53(2009), https://doi.org/10.1140/epjd/e2009-00074-0

[23] G. P'oschl, E. Teller, Bemerkungen zur Quantenmechanik des anharmonischen Oszillators, *Zeitschrift f'ur Physik* 83(34)(1933) , 143–151.

[24] N. Rosen P. M. Morse On the Vibrations of Polyatomic Molecules, *Phys. Rev.* 42(1932), 210–217.

[25] M. S. Abdalla and H. Eleuch, Exact analytic solutions of the Schrödinger equations for some modified q-deformed potentials, *J. Appl. Phys.* ,115(2014), 234906, doi: 10.1063/1.4883296

[26] V. Tayari, N. Hemsworth, I. Fakih, A. Favron, E. Gaufrès, G. Gervais, R. Martel, T. Szkopek, Two-dimensional magnetotransport in a black phosphorus naked quantum well, *Nature Communications* 6 (2015), 7702, doi:10.1038/ncomms8702.

[27] T. Gao, Y. Chen, A quantum anharmonic oscillator model for the stock market, *Physica A: Statistical Mechanics and its Applications*, 468(2017), 307–314.

[28] V. M. Vasyutaa V. M. Tkachuk, Falling of a quantum particle in an inverse square attractive potential, *Eur. Phys. J. D*, 70(2016), 267, https://doi.org/10.1140/epjd/e2016-70463-3

[29] R. D. Woods, D. S.Saxon, Diffuse Surface Optical Model for Nucleon-Nuclei Scattering, *Phys. Rev* ., 95(2)(1954), https://doi.org/10.1103/PhysRev.95.577

[30] C.Zhang, L. Huang, A quantum model for the stock market, *Physica A: Statistical Mechanics and its Applications*, 389(24)(2010), 5769–5775.

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