



Determining equations for infinitesimal transformation of second and third-order ODE using algorithm in open-source SageMath

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Abstract

To find exact solutions of nonlinear Ode using Lie symmetry technique it is required to find infinitesimal generator of the group admitted by differential equation, which becomes cumbersome if done manually.

The purpose of this paper is to develop algorithm in open-source SageMath to find the determining equations for infinitesimal transformation of Second and Third-order ODE which on solving gives value of infinitesimal. The algorithm developed in the paper is prepared in python language. The codes given in algorithm can be used by typing or by downloading the .odt file by using link https://drive.google.com/open?id=19T5FHV89Z4lum7_LbvsNIgnyF0_hlfJT. The codes given in .odt file can then copied and pasted in Sage Cell, SageMath cloud (CoCalc - Collaborative Calculation and Data Science) or in SageMath - Open-Source Mathematical Software System and run it.

By giving input of differential equation in interactive window the user can get the output as determining equations for infinitesimal transformation.

Keywords

Infinitesimal transformation, Lie symmetry, Ode of second and third order, SageMath software.

AMS Subject Classification

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1. Introduction

The differential equation is used in many mathematical modelling in science and other areas of research in which exact solution of differential equations is sought to give accurate

result.

The Lie symmetry method or infinitesimal group method starts with a general infinitesimal group of transformations. Using the invariance under the infinitesimal group the “determining equations” are derived. The determining equations are set of linear differential equations, the solution of which gives the transformation function or the infinitesimal of the dependent and independent variables [10]. To solve the differential equation using Lie symmetry the most essential thing is to find infinitesimal of the Lie group associated with the differential equations.

In the book “Symmetry Analysis of Differential Equations An Introduction” [2], Hydon suggested some algebra packages to find Lie symmetry various Mathematica software [7–9] but no such package available on open source SageMaths software. We used the related mathematical concepts of infinitesimal transformations and developed an algorithm in open-source SageMath software using python language which gives the determining equations as output by providing

the differential equations as input in the program.

2. Preliminaries

Following theorem [11] is used to develop algorithm

Theorem 2.1. *Let*

$$T = X(x, y) \frac{\partial}{\partial x} + Y(x, y) \frac{\partial}{\partial y} \quad (2.1)$$

be the infinitesimal generator of one parameter Lie group of transformations

$$x^* = f(x, y, \epsilon), \quad y^* = g(x, y, \epsilon) \quad (2.2)$$

associated with differential equation written in solved form

$$y^{(n)} = H(x, y, y', y'', \dots, y^{(n)}) \quad (2.3)$$

where $y^{(n)} = \frac{dy^n}{dx^n}$.

Let

$$T^{(n)} = X(x, y) \frac{\partial}{\partial x} + Y(x, y) \frac{\partial}{\partial y} + Y_{[x]}(x, y, y') \frac{\partial}{\partial y'} + \dots + Y_{[(n)x]}(x, y, y', \dots, y^{(n)}) \frac{\partial}{\partial y^{(n)}} \quad (2.4)$$

be the n^{th} extended infinitesimal generator of (2.1) where for $k = 1, 2, \dots, n$, $Y_{[(k)x]}$ is given by

$$Y_{[(k)x]} = D_x (Y_{[(k-1)x]}) - y^{(k)} D_x (X) \quad (2.5)$$

where $D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + y''' \frac{\partial}{\partial y''} + \dots$

Then (2.2) is admitted by (2.3) if and only if

$$T^{(n)}(y^{(n)} - H(x, y, y', y'', \dots, y^{(n)})) = 0 \quad (2.6)$$

when $y^{(n)} = H(x, y, y', y'', \dots, y^{(n)})$

Proof. For proof See [11]

□

For second-order ODE

$$y'' = H(x, y, y') \quad (2.7)$$

the equation (2.6) becomes

$$T^{(2)}(y'' - H(x, y, y')) = 0 \text{ when } y'' = H(x, y, y') \quad (2.8)$$

where

$$T^{(2)} = X \partial_x + Y \partial_y + Y_{[x]} \partial_{y'} + Y_{[xx]} \partial_{y''} \quad (2.9)$$

and

$$Y_{[x]} = Y_x + (Y_y - X_x)y' - X_y y'^2 \quad (2.10)$$

$$Y_{[xx]} = Y_{xx} + (2Y_{xx} - X_{xx})y' + (Y_{yy} - 2X_{xy})y'^2 - X_{yy}y'^3 + (Y_y - 2X_x)y'' - 3X_y y' y'' \quad (2.11)$$

Equating coefficients of various powers y' to zero, we obtain the determining equations for finding infinitesimal X and Y .

For third-order ODE

$$y''' = H(x, y, y', y'') \quad (2.12)$$

the equation (2.6) becomes

$$T^{(3)}(y''' - H(x, y, y', y'')) = 0 \text{ when } y''' = H(x, y, y', y'') \quad (2.13)$$

where

$$T^{(3)} = X \partial_x + Y \partial_y + Y_{[x]} \partial_{y'} + Y_{[xx]} \partial_{y''} + Y_{[xxx]} \partial_{y'''} \quad (2.14)$$

and $Y_{[x]}$ and $Y_{[xx]}$ are given by (2.10) and (2.11) respectively and

$$Y_{[xxx]} = Y_{xxx} + (3Y_{xxy} - X_{xxx})y' + 3(Y_{xyy} - X_{xxy})y'^2 + (Y_{yyy} - 3X_{yy})y'^3 - X_{yyy}y'^4 + 3\{Y_{xy} - X_{xx} + (Y_{yy} - 3X_{xy})y' - 2X_{yy}y'^2\}y'' - 3X_y y'^2 + \{Y_y - 3X_x - 4X_y y'\}y''' \quad (2.15)$$

Equating coefficients of various powers y' to zero, we obtain the determining equations for finding infinitesimal X and Y .

3. Main Results

3.1 Notations and symbols

Symbols		Equivalent symbol in algorithm
$\frac{dy}{dx}$	derivative of y w.r.to x	y_x
$\frac{d^2y}{dx^2}$	second derivative of y w.r.to x	y_xx
$\frac{d^3y}{dx^3}$	Third derivative of y w.r.to x	y_xxx



3.2 Algorithm

SageMath algorithm for finding determining equation

```

print ("Program find determining equ
of the type y_xxx=H(x,y,y_x,y_xx) or
y_xx=H(x,y,y_x)
,write y_x for first and y_xx for
second derivative")
var('x,y,y_x,b,y_xx,y_xxx,c,d
,Y1,Y2,Y3,K,T')
function('X,Y,f') # Define function
import itertools
@interact
def deter_eq(A=input_box(default=y_xxx,
label='LHS of H'),w=input_box(default
=-y*y_xx,label='RHS of H')):
W=A-(w)
# define values of Y1,Y2,Y3
Y1=diff(Y(x,y),x)+(diff(Y(x,y),y)
-diff(X(x,y),x))*y_x-diff(X(x,y),y)*y_x^2
Y2=diff(Y(x,y),x,x)+(2*diff(Y(x,y),x,y)
-diff(X(x,y),x,x))*y_x
+(diff(Y(x,y),y,y)-2*diff(X(x,y),x,y))
*y_x^2-diff(X(x,y),y,y)*y_x^3
+(diff(Y(x,y),y)-2*diff(X(x,y),x)
-3*diff(X(x,y),y)*y_x)*y_xx
Y3=diff(Y(x,y),x,x,x)+
(3*diff(Y(x,y),x,x,y)-diff(X(x,y),x,x,x))
*y_x+3*(diff(Y(x,y),x,y,y)
-diff(X(x,y),x,x,y))*y_x^2+
(diff(Y(x,y),y,y,y)-3*diff(X(x,y),x,y,y))
*y_x^3-diff(X(x,y),y,y,y)*y_x^4
+3*(diff(Y(x,y),x,y)-diff(X(x,y),x,x)
+(diff(Y(x,y),y,y)-3*diff(X(x,y),x,y))
*y_x-2*diff(X(x,y),y,y)*y_x^2)*y_xx
-3*diff(X(x,y),y)*y_xx^2
+(diff(Y(x,y),y)-3*diff(X(x,y),x)
-4*diff(X(x,y),y)*y_x)*y_xxx
T=X(x,y)*diff(W,x)+Y(x,y)*diff(W,y)
+(Y1)*diff(W,y_x)+(Y2)*diff(W,y_xx)
+(Y3)*diff(W,y_xxx) #define T
if (A==y_xxx):
K=T(y_xxx=w).simplify_full()
elif (A==y_xx):
K=T(y_xx=w).simplify_full()
K=(numerator(K))
print("Determining equations
are given by :")
for i in range(0,3):
for k,s in itertools.product
(range(1,10),range(1,10)):
L=[(y_x^k)*(y_xx^s),(y_x)^k
,(y_xx)^k]
v=L[i]
if ((K.coefficient(v))!=0):
show(K.coefficient(v)==0)

```

```

K=(K-(K.coefficient(v)*v))
.simplify_full()

```

```
show(K==0)
```

4. Example

Example 1 We find the determining equations of second order ODE

$$y'' + 3yy' + y^3 = 0.$$

To find the determining equations we write the equation in solved form $y'' = -3yy' - y^3$.

Input: We write input as y_xx which is left hand-side and $-3*y*y_x-y^3$ which is right side of equation $y_xx = -3*y*y_x-y^3$

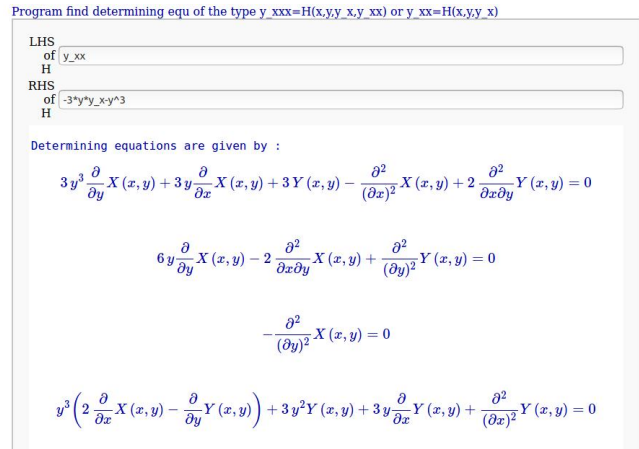


Figure 1. Determining equation for second order ODE

Output: We get the following determining equations which on solving gives values of infinitesimals X and Y

$$\begin{aligned}
 3y^3 \frac{\partial X}{\partial y} - 3y \frac{\partial X}{\partial x} + 3Y - \frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial x \partial y} &= 0 \\
 6y \frac{\partial X}{\partial y} - 2 \frac{\partial^2 X}{\partial x \partial y} + \frac{\partial^2 Y}{\partial y^2} &= 0 \\
 -\frac{\partial^2 Y}{\partial y^2} &= 0 \\
 y^3 \left(2 \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} \right) + 3y^2 Y + 3y \frac{\partial Y}{\partial x} + \frac{\partial^2 Y}{\partial x^2} &= 0
 \end{aligned} \tag{4.1}$$

For details of this example and solutions of determining equations refer [1] (page 49-54)



Example 2 We find the determining equations for symmetries of the Chazy equation which third order ODE **on solving gives values of infinitesimals X and Y**

$$y''' - 2yy'' + 3y'^2 = 0$$

To find the determining equations we write the equation in solved form $y''' = 2yy'' - 3y'^2$

Input: We write input as y_xxx which is left hand-side and $2*y*y_xx-3*y_x^2$ which is right side of equation $y_xxx = 2*y*y_xx-3*y_x^2$

```

Program find determining equ of the type y_xxx=H(x,y,y_x,y_xx) or y_xx=H(x,y,y_x)
LHS of H y_xxx
RHS of H 2*y*y_xx-3*y_x^2
Determining equations are given by :
-2*y*(d/dy)^2 X(x,y) - 9*(d/dx)(d/dy) X(x,y) + 3*(d/dy)^2 Y(x,y) = 0
-6*(d/dy)^2 X(x,y) = 0
2*y*(d/dx)^2 X(x,y) - 2*(d/dx)(d/dy) Y(x,y) - (d/dx)^3 X(x,y) + 6*(d/dx)(d/dy) Y(x,y) + 3*(d/dx)^2 (d/dy) Y(x,y) = 0
2*y*(2*(d/dx)(d/dy) X(x,y) - (d/dy)^2 Y(x,y)) + 3*(d/dx) X(x,y) - 3*(d/dx)^2 (d/dy) X(x,y) + 3*(d/dx)(d/dy)^2 Y(x,y) + 3*(d/dy) Y(x,y) = 0
    
```

Figure 2. Determining equation

$$\begin{aligned}
 -2y \frac{\partial X}{\partial y} - 9 \frac{\partial X}{\partial x} + 3 \frac{\partial^2 Y}{\partial y^2} &= 0 \\
 -6y \frac{\partial^2 Y}{\partial y^2} &= 0 \\
 2y \left(\frac{\partial^2 X}{\partial x^2} - 2 \frac{\partial^2 X}{\partial x \partial y} \right) - \frac{\partial^3 X}{\partial x^3} + 6 \frac{\partial Y}{\partial x} + 3 \frac{\partial^3 X}{\partial x^2 \partial y} &= 0 \\
 2y \left(2 \frac{\partial^2 X}{\partial x \partial y} - 2 \frac{\partial^2 Y}{\partial y^2} \right) + 3 \frac{\partial X}{\partial x} - 3 \frac{\partial^3 X}{\partial x^2 \partial y} + 3 \frac{\partial^3 Y}{\partial x \partial y^2} + 3 \frac{\partial Y}{\partial y} &= 0 \\
 2y \frac{\partial^2 X}{\partial y^2} - 3 \frac{\partial^3 X}{\partial x \partial y^2} + 6 \frac{\partial X}{\partial y} + \frac{\partial^3 Y}{\partial y^3} &= 0 \\
 -\frac{\partial^3 Y}{\partial y^3} &= 0 \\
 -2y \frac{\partial X}{\partial x} - 2Y - 3 \frac{\partial^2 Y}{\partial x^2} + 3 \frac{\partial^2 Y}{\partial x \partial y} &= 0 \\
 -3 \frac{\partial X}{\partial x \partial y} &= 0 \\
 -2y \frac{\partial^2 Y}{\partial x^2} + \frac{\partial^3 Y}{\partial x^3} &= 0
 \end{aligned}
 \tag{4.2}$$

For details of this example and solutions of determining equations (see Clarkson and Olver [12]) and [1] (page 61).

```

2*y*(d/dy)^2 X(x,y) - 3*(d/dx)(d/dy)^2 X(x,y) + 6*(d/dy) X(x,y) + (d/dy)^3 Y(x,y) = 0
- (d/dy)^3 X(x,y) = 0
-2*y*(d/dx) X(x,y) - 2Y(x,y) - 3*(d/dx)^2 X(x,y) + 3*(d/dx)(d/dy) Y(x,y) = 0
-3*(d/dy) X(x,y) = 0
-2*y*(d/dx)^2 Y(x,y) + (d/dx)^3 Y(x,y) = 0
    
```

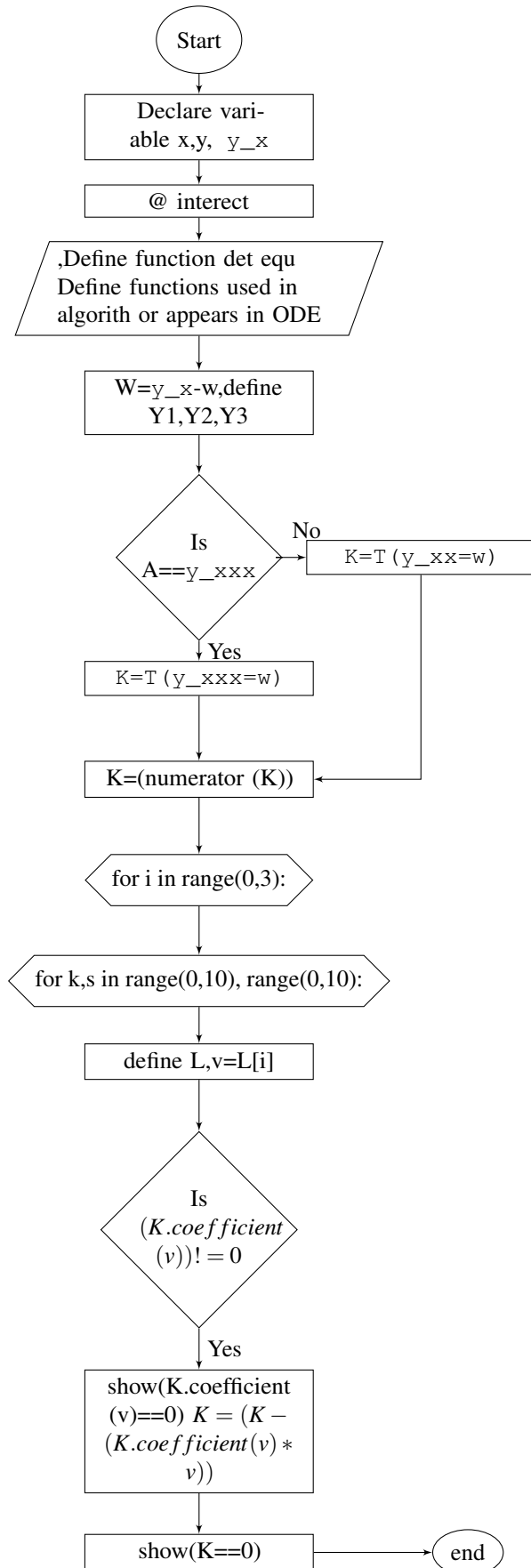
Figure 3. Determining equation for third order ODE

Example 3 Consider the second order ODE $y'' = \frac{F(x)}{y^2}$. In this case the given ODE involves a function $F(x)$ so the codes in algorithm can be modified by defining function F in algorithm and can be used.

Output: We get the following determining equations which



4.1 Flowchart of algorithm



Remark 4.1. The algorithm given in this paper is tested with SageMath Cloud. The algorithm given in the paper can be modified by user by defining new function if the differential equation involves any functions of x or y .

5. Conclusion

In this paper we have developed algorithm to find the determining equations for infinitesimal transformation of Second and Third-order ODE in open-source SageMath software. The algorithm can be used by researcher working in the field differential equations with use of lie symmetry. The algorithm can be developed further to find the determining equations with the values of infinitesimals directly.

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