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Topology conservation of vorticity field in inviscid and viscous fluid flows

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Abstract
The equation governing topology conservation of a vector fields is considered under a generating vector field. Assuming the generating vector field as velocity field of a fluid flow, topology conservation of vorticity vector field is discussed in this paper. Usually, the topology conservation of vorticity field holds in the case of barotropic flows of inviscid flows. But such topology conservation of vorticity field lines are not only true for inviscid flows but also several examples of such topology conserving vorticity fields can be obtained for Newtonian and non-Newtonian fluid flows. We derive certain exact solutions for topology conserving vorticity fields both in the case of viscous flows and couple stress fluid flows.

Keywords
Topology conservation, Inviscid flows, Newtonian flows, Couple stress flows.

AMS Subject Classification
76A02, 76A05, 76D17, 53A55.

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Contents

1. Introduction

Topological properties are playing an increasing role in the field of hydrodynamics in the recent years[1, 3, 4, 7, 11, 12, 15, 16]. Topological concepts play an important role in the study of hydrodynamic and magnetohydrodynamics flows and equilibrium. Ideal hydrodynamic and magnetohydrodynamics flows conserve not only the vorticity flux and magnetic flux but also any kind of linkage and knottedness of vortex field lines and magnetic field lines respectively. Topological properties can be used to study the different ways in which ideal plasmas can minimize the magnetic energy and the corresponding constraints. Topological considerations apply mainly to inviscid flows. In the case of inviscid magnetohydrodnamics flows also they play an important role[5, 6, 8, 11, 15]. The different local invariants that appear in the filed of fluid mechanics can be classified in to four using the concept of differential forms and Lie derivative[15]. If we use four dimensional Euclidean space-time manifold, we get one more invariant for fluid flow[12]. In the case of inviscid incompressible flows, we can easily find all these invariants in connection with the vorticity field. But In the case of viscous flows these quantities associated with the vorticity field need not be invariants of flow.

In this paper we are investigating the conservation of field line topology in the case of both inviscid and viscous flows. We are concentrating on the conservation of vorticity vector filed lines in various types of flows. We construct several examples of viscous fluid flows where vorticity field lines are topology conserving. Some examples in the case of couple stress fluid flows are also obtained.

2. Fundamental equations

A necessary and sufficient condition that the flux of an arbitrary vector field **S** through an arbitrary material surface is constant as the motion proceeds is[8, 14]

$$\partial_t \mathbf{S} + (\mathbf{w}.\nabla)\mathbf{S} - (\mathbf{S}.\nabla)\ \mathbf{w} + \mathbf{S}(\nabla.\mathbf{w}) = 0, \tag{2.1}$$

and a necessary and sufficient condition for the vector tubes of **S** to be material tubes is

$$\mathbf{S} \times (\partial_t \mathbf{S} + (\mathbf{w}.\nabla)\mathbf{S} - (\mathbf{S}.\nabla)\mathbf{w}) = 0. \tag{2.2}$$

The generating vector field in the equations (2.1) and (2.2) is **w** . Thus, flux of the field **S** is conserved if it satisfies the equation (2.1). It can be easily conclude that if a field is flux conserving, then its vector tubes are material tubes, which follows from (2.1) and (2.2).

The condition for topology conservation of an arbitrary vector field **S** is given by[5, 14]

$$\partial_t \mathbf{S} + (\mathbf{w}.\nabla)\mathbf{S} - (\mathbf{S}.\nabla)\mathbf{w} = \lambda \mathbf{S}. \tag{2.3}$$

where $\mathbf{w}(\mathbf{x},t)$ is the generating vector field and $\lambda(\mathbf{x},t)$ is a scalar function. From this equation it can be shown that the null points of the field lines and the orientation of field lines are preserved. But, in general, (2.2) need not preserve such properties[5].

Now, consider the case of barotropic inviscid fluid flow. The Euler's equation of motion under conservative body force is

$$\partial_t \mathbf{u} + \frac{1}{2}\nabla \mathbf{u}^2 - \mathbf{u} \wedge \omega = -\nabla \psi. \tag{2.4}$$

Here ω is the vorticity. ψ is defined as follows. Let φ be the potential of the body force and $\nabla W = \nabla P/\rho$, where P is the pressure and ρ is the density of the fluid. Then $\psi = \varphi - W$.

Taking the curl of this equation we get the vorticity equation

$$\partial_t \omega + (\mathbf{u}.\nabla)\omega - (\omega.\nabla)\mathbf{u} = -(\nabla.\mathbf{u})\omega. \tag{2.5}$$

From this it is clear that the evolution of the vorticity field is topology conserving. This is the classical result that in the case of a barotropic inviscid fluid flow with conservative body forces the topology of vorticity field is conserved.

In the case of incompressible viscous flow under conservative body forces the Navier-Stokes equation is given by

$$\nabla \cdot u = 0 \tag{2.6}$$

and

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nabla \phi + \nu \nabla^2 u, \tag{2.7}$$

where u is the velocity field P is the pressure, $\nabla \phi$ is the conservative body force and ν is the coefficient of kinematic viscosity. Then the corresponding vorticity equation is given by

$$\frac{\partial \omega}{\partial t} + \nabla \times (\omega \times u) - \nu \nabla^2 \omega = 0 \tag{2.8}$$

where $\omega = \nabla \times u$ is the vorticity vector field.

The equation of motion of an incompressible couple stress fluid flow under conservative body forces[13] are given by (2.6) and

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nabla \phi + \nu \nabla^2 u - \mu \nabla^4 u, \tag{2.9}$$

where μ is the parameter due to couple stress. Then the corresponding vorticity equation is given by

$$\frac{\partial \omega}{\partial t} + \nabla \times (\omega \times u) - \nu \nabla^2 \omega + \mu \nabla^4 \omega = 0 \tag{2.10}$$

As we mentioned earlier, in the case of inviscid incompressible flows the topology conservation of vorticity field lines are evident. Now the question is whether there exist topology conserving incompressible viscous fluid flows or not. If there exist such a flow then vorticity vector field must satisfy the equations (2.3) and (2.8) simultaneously. In the next section we derive certain exact solutions for Newtonian fluid flows and couple stress fluid flows for which the vorticity fields are topology conserving and which solves the equations (2.8) and (2.10) respectively.

3. Viscous flows

To find exact solutions to Navier-Stoke equation itself is very difficult, since it involves nonlinear partial differential equations in higher orders. So the available solutions for viscous flows are very rare in the literature. Existing solutions are mainly in the case of one dimensional flows, two dimensional flows or axisymmetric flows[2, 10]. Our problem is much more difficult as we have to solve the Navier-Stoke equation and the equation for topology conservation simultaneously. So, to obtain a solution for topology conserving vorticity field in the case of viscous flows we use the following ansatz form for the vector potential for the corresponding velocity field

$$\begin{aligned} V = e^{-A\nu t} & (a_1 \sin(-ax+by+bz) + a_2 \cos(-ax+by+bz), \\ & a_3 \sin(-ay+bx+bz) - a_4 \cos(-ay+bx+bz), \\ & a_5 \sin(-az+bx+by) + a_6 \cos(-az+bx+by)) \end{aligned} \tag{3.1}$$

Taking the curl of this equation we get the velocity field as

$$\begin{aligned} u = be^{-A\nu t} & (a_4 \sin(ay - b(x+z)) - a_6 \sin(b(x+y) - az) \\ & - a_3 \cos(ay - b(x+z)) + a_5 \cos(b(x+y) - az), \\ & a_2 \sin(ax - b(y+z)) + a_6 \sin(b(x+y) - az) \\ & + a_1 \cos(ax - b(y+z)) - a_5 \cos(b(x+y) - az), \\ & -(a_2 \sin(ax - b(y+z)) + a_4 \sin(ay - b(x+z)) \\ & + a_1 \cos(ax - b(y+z)) - a_3 \cos(ay - b(x+z)))) \end{aligned} \tag{3.2}$$

Clearly this velocity field satisfy the equation (2.6). Now the vorticity field is obtained as

$$\begin{aligned}\omega =& be^{-A\nu t}\left(-2a_1 b\sin(ax-b(y+z))\right.\\ &+2a_2 b\cos(ax-b(y+z))+a\left(a_3\sin(-ay+bx+bz)\right.\\ &+a_5\sin(b(x+y)-az)-a_4\cos(ay-b(x+z))\\ &\left.+a_6\cos(b(x+y)-az)\right), a\left(a_1(-\sin(ax-b(y+z)))\right.\\ &+a_5\sin(b(x+y)-az)+a_2\cos(ax-b(y+z))\\ &\left.+a_6\cos(b(x+y)-az)\right)-2b\left(a_3\sin(ay-b(x+z))\right.\\ &\left.+a_4\cos(ay-b(x+z))\right), a\left(a_1(-\sin(ax-b(y+z)))\right.\\ &+a_3\sin(-ay+bx+bz)+a_2\cos(ax-b(y+z))\\ &\left.-a_4\cos(ay-b(x+z))\right)+2b\left(a_5\sin(b(x+y)-az)\right.\\ &\left.\left.+a_6\cos(b(x+y)-az)\right)\right)\end{aligned} \tag{3.3}$$

This is a solution to the Navier-Stokes equation if the above fields satisfy the vorticity equation (2.8). Substituting these values in the vorticity equation and simplifying, it is not hard to show that equation (2.8) is satisfied if the following nonlinear algebraic equations are simultaneously satisfied.

$$\begin{aligned}&a^2-A+2b^2=0\\ &(a_2a_3-a_1a_4)\left(-a^2+ab+2b^2\right)=0\\ &(a_1a_3+a_2a_4)\left(-a^2+ab+2b^2\right)=0\\ &(a_2a_5+a_1a_6)\left(-a^2+ab+2b^2\right)=0\\ &(a_1a_5-a_2a_6)\left(-a^2+ab+2b^2\right)=0\\ &(a_4a_5-a_3a_6)\left(-a^2+ab+2b^2\right)=0\\ &(a_3a_5+a_4a_6)\left(-a^2+ab+2b^2\right)=0\end{aligned} \tag{3.4}$$

Solving this system of equations by computer algebra system we get three distinct solutions. Corresponding to these solutions we get exact solutions to the Navier-Stoke equation.

3.0.1 First solution

A solution to the above system of nonlinear equation is given by $A=3a^2$ and $b=-a$. Corresponding to this solution the exact solution for Navier-Stokes equation satisfying equations (2.6) and (2.8) is given by

$$\begin{aligned}u =& e^{-3a^2\nu t}a\left((a_3-a_5)\cos(a(x+y+z))\right.\\ &-(a_4+a_6)\sin(a(x+y+z)),(a_6-a_2)\sin(a(x+y+z))\\ &+(a_5-a_1)\cos(a(x+y+z)),(a_2+a_4)\sin(a(x+y+z))\\ &\left.+(a_1-a_3)\cos(a(x+y+z))\right)\end{aligned} \tag{3.5}$$

But for vorticity conservation this flow should also satisfy the equation (2.3). Substituting the velocity field and corresponding vorticity field in this equation, this equation is satisfied if $3a^2\nu+\lambda=0$. So choosing $\lambda=-3a^2\nu$, the topology of vorticity filed is conserved. Hence, we derived the required exact solution for viscous incompressible fluid flow for which the vorticity fields are topology conserving, given by (3.5).

3.0.2 Second solution

Another solution to the system of nonlinear equation (3.4) is given by

$$\begin{aligned}A&=\frac{3a^2}{2}\\ b&=\frac{a}{2}.\end{aligned} \tag{3.6}$$

Corresponding to this solution the exact solution for Navier-Stokes equation satisfying equations (2.6) and (2.8) is given by

$$\begin{aligned}u =& \frac{1}{2}ae^{-\frac{3}{2}a^2\nu t}\left(a_4\sin\left(ay-\frac{1}{2}a(x+z)\right)\right.\\ &-a_6\sin\left(\frac{1}{2}a(x+y-2z)\right)+a_3\left(-\cos\left(\frac{1}{2}a(x-2y+z)\right)\right)\\ &+a_5\cos\left(\frac{1}{2}a(x+y-2z)\right), a_2\sin\left(ax-\frac{1}{2}a(y+z)\right)\\ &+a_6\sin\left(\frac{1}{2}a(x+y-2z)\right)+a_1\cos\left(ax-\frac{1}{2}a(y+z)\right)\\ &-a_5\cos\left(\frac{1}{2}a(x+y-2z)\right), -\left(a_2\sin\left(ax-\frac{1}{2}a(y+z)\right)\right.\\ &+a_4\sin\left(ay-\frac{1}{2}a(x+z)\right)+a_1\cos\left(ax-\frac{1}{2}a(y+z)\right)\\ &\left.\left.-a_3\cos\left(\frac{1}{2}a(x-2y+z)\right)\right)\right)\end{aligned} \tag{3.7}$$

But for vorticity conservation this flow should also satisfy the equation (2.3). Substituting the velocity field and corresponding vorticity field in this equation, this equation is satisfied if

$$\lambda=-\frac{1}{2}\left(3a^2\nu\right)$$

So, the topology of vorticity filed is conserved. Hence, we derived the second family of required exact solutions for viscous incompressible fluid flow for which the vorticity fields are topology conserving given by (3.7).

3.0.3 Third solution

We can find another solution to the system of nonlinear equation (3.4) which is given by

$$\begin{aligned}&A=a^2+2b^2\\ &a_1=a_2=a_5=a_6=0\end{aligned} \tag{3.8}$$

Corresponding to this solution the exact solution for Navier-Stokes equation satisfying equations (2.6) and (2.8) is given by

$$\begin{aligned}u =& e^{-\nu t\left(a^2+2b^2\right)}\left(a_4 b\sin(ay-b(x+z))\right.\\ &-a_3b\cos(ay-b(x+z)),0,\\ &\left.a_4\sin(-ay+bx+bz)+a_3\cos(ay-b(x+z))\right)\end{aligned} \tag{3.9}$$

But for vorticity conservation this flow should also satisfy the equation (2.3). Substituting the velocity field and corresponding vorticity field in this equation, this equation is satisfied if

$$\lambda = -\left(a^2+2b^2\right)\nu$$

So, the topology of vorticity field is conserved in this case also. Hence, we derived the third family of required exact solutions for viscous incompressible fluid flow for which the vorticity fields are topology conserving given by (3.9).

4. Couple stress flows

Consider the system of nonlinear partial differential equations (2.10) for the couple stress fluid flows which are in higher orders than Navier-Stokes equation. In the case of Navier-Stoke equation itself it is very difficult to find exact solutions. So it is much more difficult to find exact solutions to couple stress fluid flows. Hence the available solutions for such flows are very rare in the literature and are mainly in the case of one dimensional or two dimensional flows[9]. Here, to obtain exact solution for topology conserving vorticity field in the case of couple stress flows, we use the following ansatz form for the vector potential for the corresponding velocity field

$$\begin{aligned} V =& e^{-(A\nu+B\mu)t}\left(a_1\sin(by-ax+bz)+a_2\cos(by-ax+bz),\right.\\ & a_3\sin(-ay+bx+bz)-a_4\cos(-ay+bx+bz),\\ & \left.a_5\sin(-az+bx+by)+a_6\cos(-az+bx+by)\right) \end{aligned} \tag{4.1}$$

Taking the curl of this equation we get the velocity field as

$$\begin{aligned} u =& be^{-(A\nu+B\mu)t}\left(a_4\sin(ay-b(x+z))-a_6\sin(b(x+y)-az)\right.\\ & -a_3\cos(ay-b(x+z))+a_5\cos(b(x+y)-az),\\ & a_2\sin(ax-b(y+z))+a_6\sin(b(x+y)-az)\\ & +a_1\cos(ax-b(y+z))-a_5\cos(b(x+y)-az),\\ & -(a_2\sin(ax-b(y+z))+a_4\sin(ay-b(x+z))\\ & \left.+a_1\cos(ax-b(y+z))-a_3\cos(ay-b(x+z)))\right) \end{aligned} \tag{4.2}$$

Clearly this velocity field satisfy the equation (2.6). Taking curl of this equation we get the vorticity vector field ω.

These are solutions to the couple stress flows if the above fields satisfy the vorticity equation (2.10). Substituting these values in the vorticity equation and simplifying, it is again not too hard to show that equation (2.10) is satisfied if the following nonlinear algebraic equations are simultaneously satisfied.

$$\begin{aligned} & a^4\mu+a^2\left(4b^2\mu+\nu\right)-A\nu+4b^4\mu+2b^2\nu-B\mu=0\\ & (a_2a_3-a_1a_4)\left(-a^2+ab+2b^2\right)=0\\ & (a_1a_3+a_2a_4)\left(-a^2+ab+2b^2\right)=0\\ & (a_2a_5+a_1a_6)\left(-a^2+ab+2b^2\right)=0\\ & (a_1a_5-a_2a_6)\left(-a^2+ab+2b^2\right)=0\\ & (a_4a_5-a_3a_6)\left(-a^2+ab+2b^2\right)=0\\ & (a_3a_5+a_4a_6)\left(-a^2+ab+2b^2\right)=0 \end{aligned} \tag{4.3}$$

Solving this system of equations by computer algebra system we get three distinct solutions. Corresponding to these solutions we get exact solutions to the couple stress flows.

4.0.1 First solution

A solution to the above system of nonlinear equation is given by

$$\begin{aligned} B &= \frac{9a^4\mu+3a^2\nu-A\nu}{\mu}\\ b &= -a \end{aligned} \tag{4.4}$$

Corresponding to this solution the exact solution for couple stress flows satisfying equations (2.6) and (2.10) is given by

$$\begin{aligned} u =& e^{e^{-3a^2t\left(3a^2\mu+\nu\right)}}\left(a\left(a_3-a_5\right)\cos(a(x+y+z))\right.\\ & -a\left(a_4+a_6\right)\sin(a(x+y+z)),\\ & a\left(a_6-a_2\right)\sin(a(x+y+z))\\ & +a\left(a_5-a_1\right)\cos(a(x+y+z)),\\ & a\left(a_2+a_4\right)\sin(a(x+y+z))\\ & \left.+a\left(a_1-a_3\right)\cos(a(x+y+z))\right) \end{aligned} \tag{4.5}$$

But for vorticity conservation this flow should also satisfy the equation (2.3). Substituting the velocity field and corresponding vorticity field in this equation, this equation is satisfied if $9a^4\mu+3a^2\nu+\lambda=0$. So choosing $\lambda=-(9a^4\mu+a^2\nu)$, the topology of vorticity filed is conserved. Hence, we derived the required exact solution for incompressible couple stress fluid flow for which the vorticity fields are topology conserving, given by (4.5).

4.0.2 Second solution

Another solution to the system of nonlinear equation (4.3) is given by

$$\begin{aligned} B &= \frac{9a^4\mu+6a^2\nu-4A\nu}{4\mu}\\ b &= \frac{a}{2} \end{aligned} \tag{4.6}$$

Corresponding to this solution the exact solution for couple stress fluid flow satisfying equations (2.6) and (2.10) is given by

$$u=\frac{1}{2}ae^{-\frac{3}{4}t\left(3a^4\mu+2a^2\nu\right)}\left(a_4\sin\left(ay-\frac{1}{2}a(x+z)\right)\right.$$
$$-a_6\sin\left(\frac{1}{2}a(x+y-2z)\right)+a_3\left(-\cos\left(\frac{1}{2}a(x-2y+z)\right)\right)$$
$$+a_5\cos\left(\frac{1}{2}a(x+y-2z)\right),a_2\sin\left(ax-\frac{1}{2}a(y+z)\right)$$
$$+a_6\sin\left(\frac{1}{2}a(x+y-2z)\right)+a_1\cos\left(ax-\frac{1}{2}a(y+z)\right)$$
$$-a_5\cos\left(\frac{1}{2}a(x+y-2z)\right),-\left(a_2\sin\left(ax-\frac{1}{2}a(y+z)\right)\right.$$
$$+a_4\sin\left(ay-\frac{1}{2}a(x+z)\right)+a_1\cos\left(ax-\frac{1}{2}a(y+z)\right)$$
$$\left.\left.-a_3\cos\left(\frac{1}{2}a(x-2y+z)\right)\right)\right) \tag{4.7}$$

But for vorticity conservation this flow should also satisfy the equation (2.3). Substituting the velocity field and corresponding vorticity field in this equation, this equation is satisfied if

$$\lambda=-\frac{3}{4}\left(3a^4\mu+2a^2\nu\right)$$

So, the topology of vorticity filed is conserved. Hence, we derived the second family of required exact solutions for couple stress fluid flow for which the vorticity fields are topology conserving given by (4.7).

4.0.3 Third solution

We can find another solution to the system of nonlinear equation (4.3) which is given by

$$\begin{aligned}B&=\frac{\nu\left(a^2-A+2b^2\right)}{\mu}+\left(a^2+2b^2\right)^2\\ a_1&=a_2=a_5=a_6=0\end{aligned} \tag{4.8}$$

Corresponding to this solution the exact solution for couple stress fluid flow satisfying equations (2.6) and (2.10) is given by

$$\begin{aligned}u=&e^{-\left(a^2+2b^2\right)\left(\mu\left(a^2+2b^2\right)+\nu\right)}\left(a_4b\sin(ay-b(x+z))\right.\\ &-a_3b\cos(ay-b(x+z)),0,\\ &\left.a_4\sin(-ay+bx+bz)+a_3\cos(ay-b(x+z))\right)\end{aligned} \tag{4.9}$$

But for vorticity conservation this flow should also satisfy the equation (2.3). Substituting the velocity field and corresponding vorticity field in this equation, this equation is satisfied if

$$\lambda=-\left(a^2+2b^2\right)\left(\mu\left(a^2+2b^2\right)+\nu\right)$$

So, the topology of vorticity field is conserved in this case also. Hence, we derived the third family of required exact solutions for couple stress fluid flow for which the vorticity fields are topology conserving given by (4.9).

5. Conclusion

In this paper we have discussed the topology conservation of vorticity field in different kinds of flows. It is known that the Euler's equation of motion for inviscid barotropic flows can be compared to the ideal form of Ohm's law. Also the induction equation in magnetohydrodynamics flows and vorticity equation for incompressible barotropic flows are having the same form. So vorticity field and magnetic fields are topology conserving vector fields in the case of inviscid hydrodynamics and magnetohydrodynamics respectively.

Topological properties can be used to analyze characteristics of different flow problems. Topological invariants are used in the study of turbulence. We derived several exact solutions for topology conserving vorticity fields in the case of incompressible viscous flows. Certain exact solutions for topology conserving vorticity fields are also derived for incompressible couple stress fluid flows.

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