Malaya Journal of Matematik, Vol. 8, No. 2, 674-682, 2020

https://doi.org/10.26637/MJM0802/0060

Imprecise vector: Membership Surface and its arithmetic

Dhruba Das[1]

Abstract

In this article, a method has been developed to perform the arithmetic operations of imprecise vector and its membership surface based on Randomness – Impreciseness Consistency Principle which states that Dubois-Prade left reference function is a distribution function and the right reference function is a complementary distribution function. Using the proposed method one can easily perform the arithmetic operations of imprecise vector and can construct the membership surface. Here, we have shown how to perform the arithmetic operations of imprecise vectors using the proposed method, explained its application in real life situation and compared with existing method. In numerical examples discussed here, we have performed the arithmetic operations only for two dimensional vectors, but there is no restriction on dimensions to perform the arithmetic operations and to obtain the membership surface by the proposed method.

Keywords

Imprecise vector, Membership surface, Distribution function, Randomness-Impreciseness Consistency Principle.

AMS Subject Classification

62E86, 03E72, 03E20, 28A05.

[1] *Department of Statistics, Dibrugarh University, Dibrugarh-786004, India.*
Email: dhrubadas16@gmail.com

Article History: Received **26** February **2020**; Accepted **12** May **2020**

Contents

1. Introduction

Baruah ([1], [2], [3], [4], [5], [6]) has successfully shown the construction of a normal imprecise number based on the Randomness- Impreciseness Consistency Principle. The Randomness– Impreciseness Consistency Principle states that two laws of randomness are necessary and sufficient to define a normal imprecise number. Based on this principle Das and Baruah ([7], [8], [9], [10], [11]) have already shown the construction of the membership surface or presence level indicator surface of a normal imprecise vector. An imprecise vector (X,Y), where X and Y are imprecise numbers represented by $X=[a,b,c]$ and $Y=[p,q,r]$ respectively and if the membership function of X and Y be

$$\mu_{X|Y}(x,y)=\begin{cases} L(x) & a\le x\le b, p\le y\le r\\ R(x) & b\le x\le c, p\le y\le r\\ 0 & \text{otherwise.}\end{cases}$$

$$\mu_{Y|X}(x,y)=\begin{cases} L(y) & a\le x\le c, p\le y\le q\\ R(y) & a\le x\le c, q\le y\le r,\\ 0 & \text{otherwise.}\end{cases}$$

where $L(x)$, $a\le x\le b, p\le y\le r$ and $L(y)$, $a\le x\le c, p\le y\le q$ are the left reference functions and $R(x)$, $b\le x\le c, p\le y\le r$, $R(y)$, $a\le x\le c, q\le y\le r$ are the right reference functions respectively. According to Randomness-Impreciseness Consistency Principle $L(x)$, $a\le x\le b, p\le y\le r$ and $L(y)$, $a\le x\le c, p\le y\le q$ are distribution functions and $R(x)$, $b\le x\le c, p\le y\le r$ and $R(y)$, $a\le x\le c, q\le y\le r$ are complementary distribution functions. Then Das and Baruah ([7],[8]) have established that the membership surface of the imprecise vector (X,Y) can be obtained as follows

$$\mu_{X,Y}(x,y)=\begin{cases} L(x)L(y) & a\le x\le b, p\le y\le q\\ L(x)R(y) & a\le x\le b, q\le y\le r\\ R(x)L(y) & b\le x\le c, p\le y\le q\\ R(x)R(y) & b\le x\le c, q\le y\le r\\ 0 & \text{otherwise.}\end{cases}$$

Here it is assumed that X and Y are independently distributed. In this article, subnormal imprecise vector has been considered. The normal imprecise vector is a special case of a subnormal imprecise vector in the sense that a subnormal imprecise vector is nothing but a generalized imprecise vector. So, if the proposed method is applicable to subnormal imprecise vectors then it will be automatically applicable to normal imprecise vectors.

Consider an imprecise vector (X,Y), where X and Y are subnormal imprecise number represented by $X=[a,c,e]$ and $Y=[f,h,j]$ respectively. Assume that X and Y are independently distributed. The membership function of X and Y be

$$\mu_{X|Y}(x,y)=\begin{cases} L(x) & a\le x\le c, f\le y\le j\\ R(x) & c\le x\le e, f\le y\le j\\ 0 & otherwise\end{cases}$$

such that $L(a)=R(e)=0, L(c)=R(c)=T(x), 0\le T(x)<1$, where $L(x)$ is the distribution function of a random variable defined in the interval $[a,d]$, and $R(x)$ is the complementary distribution function of another random variable defined in the interval $[b,e]$ for $b<c\le e$.

$$\mu_{Y|X}(x,y)=\begin{cases} L(y) & f\le y\le h, a\le x\le e\\ R(y) & h\le y\le j, a\le x\le e\\ 0 & otherwise\end{cases}$$

such that $L(f)=R(j)=0, L(h)=R(h)=T(y), 0\le T(y)<1$, where $L(y)$ is the distribution function of a random variable defined in the interval $[f,i]$, and $R(y)$ is the complementary distribution function of another random variable defined in the interval $[g,j]$ for $g<h\le i$. Then the function defining the membership surface would be

$$\mu_{X,Y}(x,y)=\begin{cases} L(x)L(y) & a\le x\le c, f\le y\le h\\ L(x)R(y) & a\le x\le c, h\le y\le j\\ R(x)L(y) & c\le x\le e, f\le y\le h\\ R(x)R(y) & c\le x\le e, h\le y\le j\\ 0 & otherwise\end{cases}$$

Now we are going to propose a method how to perform the arithmetic operations on subnormal imprecise vectors. Mahanta et. al ([12], [13]) has shown an alternative method for dealing with the arithmetic of imprecise numbers. The proposed method is an extension of that method.

2. Arithmetic operations of imprecise vectors

Consider two imprecise vectors (X_1,Y_1) and (X_2,Y_2) with membership surfaces

$$\mu_{X_1,Y_1}(x_1,y_1)=\begin{cases} L(x_1)L(y_1) & a\le x_1\le c, p\le y_1\le r\\ L(x_1)R(y_1) & a\le x_1\le c, r\le y_1\le t\\ R(x_1)L(y_1) & c\le x_1\le e, p\le y_1\le r\\ R(x_1)R(y_1) & c\le x_1\le e, r\le y_1\le t\\ 0 & \text{otherwise}\end{cases}$$

$$\mu_{X_2,Y_2}(x_2,y_2)=\begin{cases} L(x_2)L(y_2) & f\le x_2\le h, v\le y_2\le x\\ L(x_2)R(y_2) & f\le x_2\le h, x\le y_2\le z\\ R(x_2)L(y_2) & h\le x_2\le j, v\le y_2\le x\\ R(x_2)R(y_2) & h\le x_2\le j, x\le y_2\le z\\ 0 & \text{otherwise}\end{cases}$$

where X_1, X_2, Y_1, Y_2 are subnormal triangular imprecise numbers represented by X_1= [a, b, c], X_2= [d, e, f], Y_1= [p, q, r] and Y_2= [s, t, u] respectively with membership functions

$$\mu_{X_1|Y_1}(x_1,y_1)=\begin{cases} L(x_1) & a\le x_1\le c, p\le y_1\le t\\ R(x_1) & c\le x_1\le e, p\le y_1\le t\\ 0 & \text{otherwise}\end{cases}$$

$$\mu_{X_2|Y_2}(x_2,y_2)=\begin{cases} L(x_2) & f\le x_2\le h, v\le y_2\le z\\ R(x_2) & h\le x_2\le j, v\le y_2\le z\\ 0 & \text{otherwise}\end{cases}$$

$$\mu_{Y_1|X_1}(x_1,y_1)=\begin{cases} L(y_1) & a\le x_1\le e, p\le y_1\le r\\ R(y_1) & a\le x_1\le e, r\le y_1\le t\\ 0 & \text{otherwise}\end{cases}$$

$$\mu_{Y_2|X_2}(x_2,y_2)=\begin{cases} L(y_2) & f\le x_2\le j, v\le y_2\le x\\ R(y_2) & f\le x_2\le j, x\le y_2\le z\\ 0 & \text{otherwise}\end{cases}$$

where $L(a)=R(e)=0, L(c)=R(c)=T(x_1), 0\le T(x_1)<1; L(p)=R(t)=0, L(r)=R(r)=T(y_1), 0\le T(y_1)<1; L(f)$ $=R(j)$=0, $L(h)=R(h)=T(x_2), 0\le T(x_2)<1; L(v)=R(z)=0$ and $L(x)=R(x)=T(y_2), 0\le T(y_2)<1$. Here $L(x_1)$, $L(x_2)$, $L(y_1)$, $L(y_2)$ are distribution functions defined in the intervals $a\le x_1\le d$, $f\le x_2\le i$, $p\le y_1\le s$, $u\le y_2\le y$, where $c<d, h<i, r<s, x<y$ and $R(x_1), R(x_2), R(y_1), R(y_2)$ are complementary distribution functions defined in the intervals $b\le x_1\le e$, $g\le x_2\le j$, $q\le y_1\le t$ and $w\le y_2\le z$, where $b<c, g<h, q<r, w<x$, respectively. For $T(x_1)=1, T(x_2)=1, T(y_1)=1$ and $T(y_2)=1$, all are normal imprecise numbers.

Since $L(x_1), L(x_2), L(y_1), L(y_2)$ are distribution functions in $[a,d], [f,i], [p,s]$ and $[u,y]$ respectively and $R(x_1)$, $R(x_2)$, $R(y_1)$, $R(y_2)$ are complementary distribution functions in [b, e], $[g,j]$, $[q,t]$ and $[w,z]$ respectively, so there would exist some density functions for the distribution functions $L(x_1)$, (1-$R(x_1)$), $L(x_2)$, $(1-R(x_2)), L(y_1)$, $(1-R(y_1)), L(y_2)$ and $(1-R(y_2))$ in their respective ranges. Now let,

$$f(x_1) = \frac{d}{dx_1}(L(x_1)), a \leq x_1 \leq c, p \leq y_1 \leq t$$

$$g(x_1) = \frac{d}{dx_1}(T(x_1) - R(x_1)), c \leq x_1 \leq e, p \leq y_1 \leq t$$

$$f(y_1) = \frac{d}{dy_1}(L(y_1)), p \leq y_1 \leq r, a \leq x_1 \leq e$$

$$g(y_1) = \frac{d}{dy_1}(T(y_1) - R(y_1)), r \leq y_1 \leq t, a \leq x_1 \leq e,$$

where $f(x_1)$ and $f(y_1)$ are density functions defined in the intervals $[a,d]$ and $[p,s]$ respectively. Similarly, $g(x_1)$=$\frac{d}{dx_1}$(1-$R(x_1)$) and $\frac{d}{dy_1}(1-R(y_1))$ are density functions defined in the intervals $[b,e]$ and $[q,t]$ respectively.

Now, we shall discuss about the arithmetic operations of two imprecise vectors (X_1,Y_1) and (X_2,Y_2).

Addition of imprecise vectors

We are going to discuss about the addition of the subnormal imprecise vectors (X_1,Y_1) and (X_2,Y_2). Let $Z = X_1 + X_2 = [a+f, c+h, e+j]$ be the imprecise number of X_1+X_2 and $U = Y_1+Y_2 = [p+v, r+x, t+z]$ be the imprecise number of Y_1+Y_2. We start with equating $L(x_1)$ with $L(x_2)$, $R(x_1)$ with $R(x_2)$,$L(y_1)$ with $L(y_2)$ and $R(y_1)$ with $R(y_2)$. And so, we obtain $x_2 = \phi_1(x_1), x_2 = \phi_2(x_1), y_2 = \phi_1(y_1)$ and $y_2 = \phi_2(y_1)$ respectively. Let $z = x_1 + x_2$, so we have $z = x_1 + \phi_1(x_1), z = x_1 + \phi_2(x_1)$ so that $x_1 = \psi_1(z)$ and $x_1 = \psi_2(z)$, say. Similarly, let $u = y_1 + y_2$, so we have $u = y_1 + \phi_1(y_1), u = y_1 + \phi_2(y_1)$ so that $y_1 = \psi_1(t)$ and $y_1 = \psi_2(t)$, say. Replacing, x_1 by $\psi_1(z)$ in $f(x_1)$ and by $\psi_2(z)$ in $g(x_1)$, we obtain $f(x_1) = \eta_1(z)$ and $g(x_1) = \eta_2(z)$, say.

Now let,

$$\frac{dx_1}{dz} = \frac{d}{dz}(\psi_1(z)) = m_1(z)$$

and

$$\frac{dx_1}{dz} = \frac{d}{dz}(\psi_2(z)) = m_2(z)$$

The distribution function for X_1+X_2, would now be given by

$$\int_{a+f}^{z} \eta_1(z)m_1(z)dz, a+f \leq z \leq c+h \tag{2.1}$$

and the complementary distribution function would be given by

$$T(z) - \int_{c+h}^{z} \eta_2(z)m_2(z)dz, c+h \leq z \leq e+j$$

where $T(z)$ should be obtained from equation (2.1). This distribution function and the complementary distribution function constitute the imprecise membership function of X_1+X_2 as

$$\mu_Z(z) = \begin{cases} \int_{a+f}^{z} \eta_1(z)m_1(z)dz & a+f \leq z \leq c+h \\ T(z) - \int_{c+h}^{z} \eta_2(z)m_2(z)dz & c+h \leq z \leq e+j \\ 0 & otherwise \end{cases}$$

In the same way, replacing y_1 by $\psi_1(u)$ in $f(y_1)$ and by $\psi_2(u)$ in $g(y_1)$, we obtain $f(y_1) = \eta_1(u)$ and $g(y_1) = \eta_2(u)$, say.

Now let,

$$\frac{dy_1}{dz} = \frac{d}{du}(\psi_1(u)) = m_1(u)$$

and

$$\frac{dy_1}{du} = \frac{d}{du}(\psi_2(u)) = m_2(u)$$

The distribution function for Y_1+Y_2, would now be given by

$$\int_{p+v}^{u} \eta_1(u)m_1(u)du, p+v \leq u \leq r+x$$

and the complementary distribution function would be given by

$$T(u) - \int_{r+x}^{u} \eta_2(u)m_2(u)du, r+x \leq u \leq t+z, \tag{2.2}$$

where $T(u)$ should be obtained from equation (2.2).

This distribution function and the complementary distribution function constitute the imprecise membership function of Y_1+Y_2 as

$$\mu_U(u) = \begin{cases} \int_{p+v}^{u} \eta_1(u)m_1(u)du & p+v \leq u \leq r+x \\ T(u) - \int_{r+x}^{u} \eta_2(u)m_2(u)du & r+x \leq u \leq t+z \\ 0 & otherwise \end{cases}$$

Now, the membership surface of the imprecise vector (Z,U) will be as shown in equation (2.3).

$$\mu_{Z,U}(z,u)=\begin{cases}\int_{p+v}^{u}\int_{a+f}^{z}\eta_1(z)m_1(z)\eta_1(u)m_1(u)dzdu & (a+f)\le z\le(c+h),\\ & (p+v)\le u\le(r+x)\\ \int_{a+f}^{z}\{T(u)-\int_{r+x}^{u}\eta_2(u)m_2(u)du\}\eta_1(z)m_1(z)dz & (a+f)\le z\le(c+h),\\ & (r+x)\le u\le(t+z)\\ \int_{p+v}^{u}\{T(z)-\int_{c+h}^{z}\eta_2(z)m_2(z)dz\}\eta_1(u)m_1(u)du & (c+h)\le z\le(e+j),\\ & (p+v)\le u\le(r+x)\\ \{T(z)-\int_{c+h}^{z}\eta_2(z)m_2(z)dz\}\{T(u)-\int_{r+x}^{u}\eta_2(u)m_2(u)du\} & (c+h)\le z\le(e+j),\\ & (r+x)\le u\le(t+z)\\ 0 & otherwise\end{cases}\tag{2.3}$$

Subtraction of imprecise vectors

For subtraction, let $Z=X_1-X_2$. Then the imprecise membership function of $Z=X_1-X_2$ would be given by $Z=X_1+(-X_2)$. Suppose $-X_2=[-f,-h,-j]$ be the imprecise number of $(-X_2)$. Say,

$$f(x_2)=\frac{d}{dx_2}(L(x_2))=\eta_1(t), f\le x_2\le h, v\le y_2\le z$$
$$g(x_2)=\frac{d}{dx_2}\{T(x_2)-R(x_2)\}=\eta_1(t), h\le x_2\le j, v\le y_2\le z$$

Let $t=-x_2$ so that $\frac{dx_2}{dt}=-1=m(t)$, say. Replacing $x_2=-t$ in $f(x_2)$ and $g(x_2)$ we obtain $f(x_2)=\eta_1(t)$ and $g(x_2)=\eta_2(t)$, say. Then the membership function of $(-X_2)$ would be given by

$$\mu_{(-X_2)}(t)=\begin{cases}\int_{-j}^{t}\eta_2(t)m(t)dt & -j\le t\le -h\\ T(t)-\int_{-h}^{t}\eta_1(t)m(t)dt & -h\le t\le -f\\ 0 & otherwise\end{cases}$$

Then we can easily find the membership function of X_1-X_2 by addition of imprecise numbers X_1 and $-X_2$ as described in the earlier subsection.

In the same way, it can be obtained the membership function of $U=Y_1-Y_2$ and it is possible to obtain the membership surface of the imprecise vector (Z,U) as described in case of addition.

Scalar Product of imprecise vectors

Similarly for scalar product say $Z=X_1.X_2=[a.f,c.h,e.j]$ where $X_1=[a,c,e],(a,c,e>0), X_2=[f,h,j],(f,h,j>0)$ be the imprecise number of $X_1.X_2$ and $U=Y_1.Y_2=[p.v,r.x,t.z]$ where $Y_1=[p,r,t],(p,r,t>0)$, $Y_2=[v,x,z],(v,x,z>0)$ be the imprecise number of $Y_1.Y_2$. Equating $L(x_1)$ with $L(x_2)$, $R(x_1)$ with $R(x_2)$, $L(y_1)$ with $L(y_2)$ and $R(y_1)$ with $R(y_2)$. And so, we obtain $x_2=\phi_1(x_1), x_2=\phi_2(x_1), y_2=\phi_1(y_1)$ and $y_2=\phi_2(y_1)$ respectively. Let $z=x_1.x_2$, so we have $z=x_1.\phi_1(x_1)$, z=$x_1.\phi_2(x_1)$ so that $x_1=\psi_1(z)$ and $x_1=\psi_2(z)$, say. Similarly, let $u=y_1.y_2$, so we have $u=y_1.\phi_1(y_1), u=y_1.\phi_2(y_1)$ so that $y_1=\psi_1(t)$ and $y_1=\psi_2(t)$, say. Replacing, x_1 by $\psi_1(z)$ in $f(x_1)$ and by $\psi_2(z)$ in $g(x_1)$, we obtain $f(x_1)=\eta_1(z)$ and $g(x_1)=\eta_2(z)$, say.

Now let,

$$\frac{dx_1}{dz}=\frac{d}{dz}(\psi_1(z))=m_1(z)$$

and

$$\frac{dx_1}{dz}=\frac{d}{dz}(\psi_2(z))=m_2(z)$$

The distribution function for $X_1.X_2$, would now be given by

$$\int_{af}^{z}\eta_1(z)m_1(z)dz,\ \ af\le z\le ch$$

and the complementary distribution function would be given by

$$T(z)-\int_{ch}^{z}\eta_2(z)m_2(z)dz,\ \ ch\le z\le ej$$

This distribution function and the complementary distribution function constitute the imprecise membership function of $X_1.X_2$ as

$$\mu_Z(z)=\begin{cases}\int_{af}^{z}\eta_1(z)m_1(z)dz & af\le z\le ch\\ T(z)-\int_{ch}^{z}\eta_2(z)m_2(z)dz & ch\le z\le ej\\ 0 & otherwise\end{cases}$$

In the same way, replacing, y_1 by $\psi_1(u)$ in $f(y_1)$ and by $\psi_2(u)$ in $g(y_1)$, we obtain $f(y_1)=\eta_1(u)$ and $g(y_1)=\eta_2(u)$, say.

Now let,

$$\frac{dy_1}{du}=\frac{d}{du}(\psi_1(u))=m_1(u)$$

and

$$\frac{dy_1}{du}=\frac{d}{du}(\psi_2(u))=m_2(u)$$

The distribution function for $Y_1.Y_2$, would now be given by

$$\int_{pv}^{u}\eta_1(u)m_1(u)du,\ \ pv\le u\le rx$$

and the complementary distribution function would be given by

$$T(u) - \int_{rx}^{u} \eta_2(u)m_2(u)du, \quad rx \le u \le tz$$

This distribution function and the complementary distribution function constitute the imprecise membership function of $Y_1.Y_2$ as

$$\mu_U(u) = \begin{cases} \int_{pv}^{u} \eta_1(u)m_1(u)du & pv \le u \le rx \\ T(u) - \int_{rx}^{u} \eta_2(u)m_2(u)du & rx \le u \le tz \\ 0 & otherwise \end{cases}$$

Now by adding the imprecise numbers X_1X_2 and Y_1Y_2 we shall get the required result.

3. Numerical Example

We shall discuss here numerical examples. Consider two imprecise vectors (X_1,Y_1) and (X_2,Y_2) with membership surfaces

$$\mu_{X_1,Y_1}(x_1,y_1) = \begin{cases} \frac{(x_1-1)(y_1-3)}{8} & 1 \le x_1 \le 2, 3 \le y_1 \le 5 \\ \frac{(x_1-1)(6-y_1)}{4} & 1 \le x_1 \le 2, 5 \le y_1 \le 6 \\ \frac{(4-x_1)(y_1-3)}{16} & 2 \le x_1 \le 4, 3 \le y_1 \le 5 \\ \frac{(4-x_1)(6-y_1)}{8} & 2 \le x_1 \le 4, 5 \le y_1 \le 6 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu_{X_2,Y_2}(x_2,y_2) = \begin{cases} \frac{(x_2-2)(y_2-4)}{8} & 2 \le x_2 \le 3, 4 \le y_2 \le 6 \\ \frac{(x_2-2)(7-y_2)}{4} & 2 \le x_2 \le 3, 6 \le y_2 \le 7 \\ \frac{(5-x_2)(y_2-4)}{16} & 3 \le x_2 \le 5, 4 \le y_2 \le 6 \\ \frac{(5-x_2)(7-y_2)}{8} & 3 \le x_2 \le 5, 6 \le y_2 \le 7 \\ 0 & \text{otherwise.} \end{cases}$$

where the membership functions of X_1, Y_1, X_2 and Y_2 are as follows

$$\mu_{X_1|Y_1}(x_1,y_1) = \begin{cases} \frac{(x_1-1)}{2} & 1 \le x_1 \le 2, 3 \le y_1 \le 6 \\ \frac{(4-x_1)}{4} & 2 \le x_1 \le 4, 3 \le y_1 \le 6 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu_{Y_1|X_1}(x_1,y_1) = \begin{cases} \frac{(y_1-3)}{4} & 1 \le x_1 \le 4, 3 \le y_1 \le 5 \\ \frac{(6-y_1)}{2} & 1 \le x_1 \le 4, 5 \le y_1 \le 6 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{X_2|Y_2}(x_2,y_2) = \begin{cases} \frac{(x_2-2)}{2} & 2 \le x_2 \le 3, 4 \le y_2 \le 7 \\ \frac{(5-x_2)}{4} & 3 \le x_2 \le 5, 4 \le y_2 \le 7 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{Y_2|X_2}(x_2,y_2) = \begin{cases} \frac{(y_2-4)}{4} & 2 \le x_2 \le 5, 4 \le y_2 \le 6 \\ \frac{(7-y_2)}{2} & 2 \le x_2 \le 5, 6 \le y_2 \le 7, \\ 0 & \text{otherwise} \end{cases}$$

Here $L(x_1) = \frac{(x_1-1)}{2}$, $L(x_2) = \frac{(x_2-2)}{2}$, $L(y_1) = \frac{(y_1-3)}{4}$, $L(y_2) = \frac{(y_2-4)}{4}$ are distribution functions defined in the intervals $1 \le x_1 \le 3$, $2 \le x_2 \le 4$, $3 \le y_1 \le 7$, $4 \le y_2 \le 8$ and $R(x_1) = \frac{(4-x_1)}{4}$, $R(x_2) = \frac{(5-x_2)}{4}$, $R(y_1) = \frac{(6-y_1)}{2}$, $R(y_2)$ $=\frac{(7-y_2)}{2}$ are complementary distribution functions defined in the intervals $0 \le x_1 \le 4$, $1 \le x_2 \le 5$, $4 \le y_1 \le 6$ and $5 \le x_2 \le 7$, respectively. Now,we shall discuss here the arithmetic operations of subnormal imprecise vectors (X_1,Y_1) and (X_2,Y_2) without using the method of alpha cuts.

Addition of two imprecise vectors (X_1,Y_1) and (X_2, Y_2):

For addition, consider $X_1+X_2 = [3,5,9]$ and $Y_1 + Y_2 = [7, 11, 13]$. Equating the distribution functions and complementary distribution functions of X_1 and X_2 , we get $x_2 = \phi_1(x_1) = x_1+1$ and $x_2 = \phi_2(x_1) = x_1+1$. Let $x = x_1+x_2$, so we shall have $x = x_1+\phi_1(x_1) = 2x_1+1$ and $x = x_1+\phi_2(x_1) = 2x_1+1$ so that $x_1 = \psi_1(x) = \frac{x-1}{2}$ and $x_1 = \psi_2(x) = \frac{x-1}{2}$, respectively. Replacing x_1 by $\psi_1(x)$ and $\psi_2(x)$ in the density functions $f(x_1)$ and $g(x_1)$ respectively, we get $f(x_1) = \frac{1}{2} = \eta_1(x)$ and $g(x_1) = \frac{1}{4} = \eta_2(x)$. Again $m_1(x) = \frac{d}{dx}(\psi_1(x)) = \frac{1}{2}$ and $m_2(x) = \frac{d}{dx}(\psi_2(x)) = \frac{1}{2}$. Now the membership function of X_1+X_2 would be given by

$$\mu_X(x) = \begin{cases} \frac{x-3}{4} & 3 \le x \le 5 \\ \frac{9-x}{8} & 5 \le x \le 9 \\ 0 & otherwise \end{cases}$$

Similarly, equating the distribution functions and complementary distribution functions of Y_1 and Y_2 , we get $y_2 = \phi_1(y_1) = y_1+1$ and $y_2 = \phi_2(y_1) = y_1+1$. Let $y = y_1+y_2$, so we shall have $y = y_1+\phi_1(y_1) = 2y_1+1$ and $y = y_1+\phi_2(y_1) = 2y_1+1$ so that $y_1 = \psi_1(y) = \frac{y-1}{2}$ and $y_1 = \psi_2(y) = \frac{y-1}{2}$, respectively. Replacing y_1 by $\psi_1(y)$ and $\psi_2(y)$ in the density functions $f(y_1)$ and $g(y_1)$ respectively, we get $f(y_1) = \frac{1}{4} = \eta_1(y)$ and $g(y_1) = \frac{1}{2} = \eta_2(y)$. Again $m_1(y) = \frac{d}{dy}(\psi_1(y)) = \frac{1}{2}$ and $m_2(y) = \frac{d}{dy}(\psi_2(y)) = \frac{1}{2}$. Now the membership function of Y_1+Y_2 would be given by

$$\mu_Y(y) = \begin{cases} \frac{y-7}{8} & 7 \le y \le 11 \\ \frac{13-y}{4} & 11 \le y \le 13 \\ 0 & otherwise \end{cases}$$

Now the membership surface of the imprecise vector (X,Y)= (X_1,X_2)+ (Y_1,Y_2) would be given by (see Figure. 1)

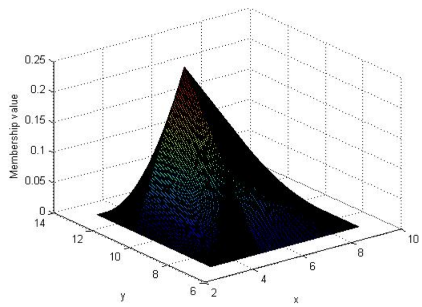

Figure 1. Membership surface of (X,Y)

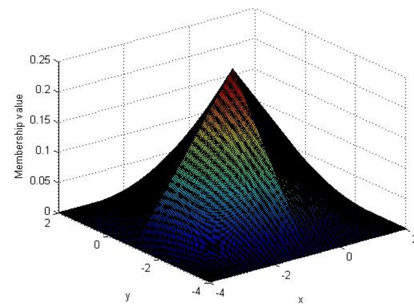

Figure 2. Membership surface of (X,Y)

$$\mu_{X,Y}(x,y)=\begin{cases}\frac{(x-3)(y-7)}{32} & 3\le x\le 5, 7\le y\le 11\\ \frac{(x-3)(13-y)}{16} & 3\le x\le 5, 11\le y\le 13\\ \frac{(9-x)(y-7)}{64} & 5\le x\le 9, 7\le y\le 11\\ \frac{(9-x)(13-y)}{32} & 5\le x\le 9, 11\le y\le 13\\ 0 & otherwise\end{cases}$$

Subtraction of two imprecise vectors (X_1, Y_1) and (X_2, Y_2):

For subtraction, $X_1-X_2=X_1+(-X_2)$. Now $-X_2$= [-5, -3, -2]. Let $t=-X_2$ so that $X_2=-t$ which implies $m(t)=-1$. Then the density function $f(x_2)$ and $g(x_2)$ would be, say,

$$f(x_2)=\frac{d}{dx_2}\frac{(x_2-2)}{2}=\frac{1}{2}=\eta_1(t), 2\le x_2\le 3, 4\le y_2\le 7$$
$$g(x_2)=\frac{d}{dx_2}\{\frac{1}{2}-\frac{(5-x_2)}{4}\}=\frac{1}{4}=\eta_2(t), 3\le x_2\le 5, 4\le y_2\le 7$$

Then the membership function of $(-X_2)$ would be given by

$$\mu_{(-X_2)}(x_2)=\begin{cases}\frac{(x_2+5)}{4} & -5\le x_2\le -3, 4\le y_2\le 7\\ \frac{-(x_2+2)}{2} & -3\le x_2\le -2, 4\le y_2\le 7\\ 0 & otherwise\end{cases}$$

Then by addition of imprecise numbers $X_1=[1,2,4]$ and $(-X_2)=[-5,-3,-2]$ the membership function of X_1-X_2 is given by

$$\mu_X(x)=\begin{cases}\frac{(x+4)}{6} & -4\le x\le -1\\ \frac{(2-x)}{6} & -1\le x\le 2\\ 0 & otherwise\end{cases}$$

Similarly, $Y_1-Y_2=Y_1+(-Y_2)$. Now $-Y_2=[-7,-6,-4]$. Let $t=-Y_2$ so that $Y_2=-t$ which implies $m(t)=-1$. Then the density function $f(x_2)$ and $g(x_2)$ would be, say,

$$f(y_2)=\frac{d}{dy_2}\frac{(y_2-4)}{4}=\frac{1}{4}=\eta_1(t), 2\le x_2\le 5, 4\le y_2\le 6$$
$$g(y_2)=\frac{d}{dy_2}\{1-\frac{(7-y_2)}{2}\}=\frac{1}{2}=\eta_1(t), 2\le x_2\le 5, 6\le y_2\le 7$$

Then the membership function of $(-Y_2)$ would be given by

$$\mu_{(-Y_2)}(y_2)=\begin{cases}\frac{(y_2+7)}{2} & -7\le x_2\le -6, 2\le y_2\le 5\\ \frac{-(y_2+4)}{4} & -6\le x_2\le -4, 2\le y_2\le 5\\ 0 & otherwise\end{cases}$$

Then by addition of imprecise numbers Y_1=[3, 5, 6] and (-Y_2) = [-7, -6, -4] the membership function of Y_1-Y_2 is given by

$$\mu_Y(y)=\begin{cases}\frac{(y+4)}{6} & -4\le y\le -1\\ \frac{(2-y)}{6} & -1\le y\le 2\\ 0 & otherwise\end{cases}$$

Now the membership surface of the imprecise vector (X,Y)= (X_1,X_2)- (Y_1,Y_2) would be given by (see Figure 2)

$$\mu_{X,Y}(x,y)=\begin{cases}\frac{(x+4)(y+4)}{36} & -4\le x\le -1, -4\le y\le -1\\ \frac{(x+4)(2-y)}{36} & -4\le x\le -1, -1\le y\le 2\\ \frac{(2-x)(y+4)}{36} & -1\le x\le 2, -4\le y\le -1\\ \frac{(2-x)(2-y)}{36} & -1\le x\le 2, -1\le y\le 2\\ 0 & otherwise\end{cases}$$

Scalar product of two imprecise vectors (X_1,Y_1) and (X_2,Y_2):

For scalar product, equating the distribution function and complementary distribution function of X_1 and X_2, we get $x_2=\phi_1(x_1)=x_1+1$ and $x_2=\phi_2(x_1)=x_1+1$. Let $x=x_1.x_2$, so we shall have $x=x_1.\phi_1(x_1)=x_1^2+x_1$ and $x=x_1.\phi_2(x_1)=x_1^2+x_1$, so that $x_1=\psi_1(x)=\frac{-1\pm\sqrt{1+4x}}{2}$ and $x_1=\psi_2(x)=\frac{-1\pm\sqrt{1+4x}}{2}$, respectively. Replacing x_1 by $\psi_1(x)$ and $\psi_2(x)$ in the density functions $f(x_1)$ and $g(x_1)$ respectively, we get $f(x_1)=\frac{1}{2}=\eta_1(x)$ and $g(x_1)=\frac{1}{4}=\eta_2(x)$. Again $m_1(x)=\frac{d}{dx}(\psi_1(x))$ and $m_2(x)=\frac{d}{dx}(\psi_2(x))$. Then the membership function of $X_1.X_2$ would be given by

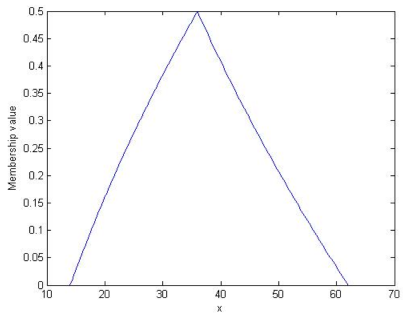

Figure 3. Membership curve of X

$$\mu_{X_1.X_2}(x)=\begin{cases}\frac{(\sqrt{1+4x}-3)}{4} & 2\le x\le 6\\ \frac{(9-\sqrt{1+4x})}{8} & 6\le x\le 20\\ 0 & otherwise\end{cases}$$

Similarly, equating the distribution functions and complementary distribution functions of Y_1 and Y_2, we get $y_2= \phi_1(y_1)$ $= y_1+1$ and $y_2=\phi_2(y_1)=y_1+1$. Let $y=y_1.y_2$, so we shall have $y=y_1.\phi_1(y_1)=y_1^2+y_1$ and $y=y_1.\phi_2(y_1)=y_1^2+y_1$ so that $y_1=\psi_1(y)=\frac{-1\pm\sqrt{1+4y}}{2}$ and $y_1=\psi_2(y)=\frac{-1\pm\sqrt{1+4y}}{2}$, respectively. Replacing y_1 by $\psi_1(y)$ and $\psi_2(y)$ in the density functions $f(y_1)$ and $g(y_1)$ respectively, we get $f(y_1)=\frac{1}{4}=\eta_1(y)$ and $g(y_1)=\frac{1}{2}=\eta_2(y)$. Again $m_1(y)=\frac{d}{dy}(\psi_1(y))$ and $m_2(y)=\frac{d}{dy}(\psi_2(y))$. Now the membership function of $Y_1.Y_2$ would be given by

$$\mu_{Y_1.Y_2}(y)=\begin{cases}\frac{(\sqrt{1+4y}-7)}{8} & 12\le y\le 30\\ \frac{(13-\sqrt{1+4y})}{4} & 30\le y\le 42\\ 0 & otherwise\end{cases}$$

Now by adding X_1X_2 and Y_1Y_2 we shall get the required result, i.e. the membership function of $X=X_1X_2+Y_1Y_2$ will be (see Figure 3)

$$\mu_X(x)=\begin{cases}\frac{-34+\sqrt{80x+36}}{40} & 14\le x\le 36\\ \frac{62-\sqrt{80x-1516}}{40} & 36\le x\le 62\\ 0 & otherwise\end{cases}$$

4. Real Life Application of the Proposed Method

Now we shall demonstrate the above theoretical results with the help of an real life example. The weather summary data for two weeks, 20 September - 26 September, 2015 and 27 September - 3 October, 2015 of Guwahati city in India, reported by CustomWeather[1], © 2015 have shown in Table 1 and Table 2. For simplicity and without any loss of generality we may assume here that temperature and humidity are triangular imprecise numbers of equal heights 0.5 respectively. The imprecise numbers were formed around the averages. Now, we can consider ([24 27.76 33] [51 88.46 100]) and ([25 30.01 35] [56 80.03 98]) be the realizations of the imprecise vector $(Temperature(T)\ Humidity(H))$ for the first and second week respectively.

[1] www.timeanddate.com/weather/india/guwahati/historic

Table 1. Weather summary data of Guwahati (20 Sep-26 Sep, 2015)

	Temperature	Humidity
High	33^0C	100%
Low	24^0C	51%
Average	27.76^0C	88.46%

Table 2. Weather summary data of Guwahati (27 Sep-03 Oct, 2015)

	Temperature	Humidity
High	35^0C	98 %
Low	25^0C	56%
Average	30.01^0C	80.03%

The membership surfaces of the imprecise vectors (T_1 H_1) =([24 27.76 33] [51 88.46 100]) and (T_2 H_2)= ([25 30.01 35] [56 80.03 98]) will be as follows

$$\mu_{(T_1,H_1)}(t,h)=\begin{cases}\frac{(t-24)(h-51)}{563.3984} & 24\le t\le 27.76, 51\le h\le 88.46\\ \frac{(t-24)(100-h)}{173.5616} & 24\le t\le 27.76, 88.46\le h\le 100\\ \frac{(33-t)(h-51)}{785.1616} & 27.76\le t\le 33, 51\le h\le 88.46\\ \frac{(33-t)(100-h)}{241.8784} & 27.76\le t\le 33, 88.46\le h\le 100\\ 0 & \text{otherwise.}\end{cases}$$

$$\mu_{(T_2,H_2)}(t,h)=\begin{cases}\frac{(t-25)(h-56)}{481.5612} & 25\le t\le 30.01, 56\le h\le 80.03\\ \frac{(t-25)(98-h)}{360.1188} & 25\le t\le 30.01, 80.03\le h\le 98\\ \frac{(35-t)(h-56)}{479.6388} & 30.01\le t\le 35, 56\le h\le 80.03\\ \frac{(35-t)(98-h)}{358.6812} & 30.01\le t\le 35, 80.03\le h\le 98\\ 0 & \text{otherwise.}\end{cases}$$

Now, if we would like to obtain the difference between the imprecise vectors $(T_1\ H_1)$ and $(T_2\ H_2)$ for the two weeks then using the proposed method the difference can be obtained along with its membership surface which is as shown below (See Figure 4)

$$\mu_{T,H}(t,h)=\begin{cases}\frac{(t+11)(h+47)}{1940.05} & -11\le t\le -2.25, -47\le h\le 8.43\\ \frac{(t+11)(44-h)}{1244.95} & -11\le t\le -2.25, 8.43\le h\le 44\\ \frac{(8-t)(h+47)}{2272.63} & -2.25\le t\le 8, -47\le h\le 8.43\\ \frac{(8-t)(44-h)}{1458.37} & -2.25\le t\le 8, 8.43\le h\le 44\\ 0 & \text{otherwise.}\end{cases}$$

The membership surface of an imprecise vector defines how the grade of membership of a vector in the space is determined. Suppose in the above example, the grade of membership of the vector $(-2.25, 8.43)$ is 0.25, which can be calculated easily from the above membership surface.

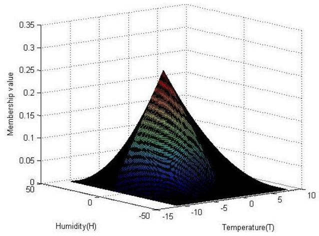

Figure 4. Membership surface of (T,H)

Moreover, in matrix algebra, the proposed method will be helpful to perform row and column operations to a matrix. Just as a normal imprecise number is defined with reference to a membership function, a normal imprecise vector too has to be defined with reference to a membership surface. The membership surface of an imprecise vector defines how the grade of membership of a vector in the space is determined. We have demonstrated here how to perform the arithmetic operations and to obtain the membership surface of imprecise vector.

5. Comparison with existing method and future scope

The standard method of alpha cut can be used to perform the arithmetic operations on imprecise vectors, but the main advantage of the proposed method is that besides arithmetic operations it gives the membership surface of the imprecise vector from which it is easily possible to find the grade of membership for any vector in the set of vectors.

In case of alpha cut, to obtain the membership grade of vectors in the space, one should have to go to the membership functions of the each imprecise elements of the imprecise vector and then has to multiply the membership values which is tedius and time consuming. But the main advantage of the proposed method is that if we construct the membership surface of the imprecise vector then the membership grade of any vector can be easily obtain from the membership surface.

Imprecise randomness defined in this way can be of use in sampling theory and therefore in demography. In sampling, for example, age, height, weight etc. of a person is usually taken as an integer. In fact, that should not be the case. A better way to deal with this would be to consider the data as intervals of length unity, and then assume that the data are in terms of triangular imprecise numbers. As soon as we assume without loss of generality that the data are triangular imprecise numbers, we would assume uniform laws of randomness to the left and to the right of the points of maximum membership. That way, the analysis would be more practical. Moreover, row and column transformations, rank of an imprecise matrix could be evaluated with the help of row and column membership surfaces which will be an extension of this work.

6. Conclusions

In this article, a method has been proposed to perform the arithmetic operations on imprecise vectors and to obtain its membership surface. This method is a simple way of dealing with imprecise vector arithmetic. This method is based on Randomness-Impreciseness Consistency Principle that states that Dubois-Prade left reference function is a distribution function and the right reference function is a complementary distribution function. Based on distribution theoretic approach it has been discussed here how to perform the arithmetic operations on imprecise vector and to obtain the membership surface. In order to define an imprecise vector, we would need to define an imprecise membership surface, and to define an imprecise membership surface we may take help of imprecise membership surface sections, which in the two dimensions is nothing but the membership function of a subnormal imprecise number. In this article, the proposed method is demonstrated with numerical examples, explained its applicability in real life situation, compared with existing method and discussed about the future scope of the method. Though the numerical examples have shown here only for two dimensional vectors, there is no restriction on dimensions for the proposed method to perform arithmetic operations and to obtain the membership surface.

Conflict of Interest

The author states that there is no conflict of interest.

References

[1] Baruah HK , Theory of Fuzzy sets: The Case of Subnormality, International Journal of Energy, *Information and Communications*, 2(3)(2011), 1-8.

[2] Baruah HK, An introduction to the theory of imprecise sets: The mathematics of partial presence, *J Math Comput Sci.*, 2(2)(2012), 110-124.

[3] Baruah HK, Fuzzy membership with respect to a reference function, *J Assam Sci Soc.*, 40(3) (1999), 65-73.

[4] Baruah HK, The Theory of Fuzzy Sets: Beliefs and Realities, International Journal of Energy, *Information and Communications*, 2(2)(2011), 1-21.

[5] Baruah HK,) Construction of the membership function of a fuzzy number, *ICIC Express Letters*, 5(2)(2011), 545-549.

[6] Baruah HK, In search of the root of fuzziness: the measure theoretic meaning of partial presence, *Annals of Fuzzy Mathematics and Informatics*, 2(1)(2011), 57-68.

[7] Das D., Baruah HK, *Construction of the Membership Surface of Imprecise Vector*, Springer Plus, 2014, 3:722.

[8] Das D., Baruah HK, Theory of Imprecise Sets: Imprecise Matrix, *National Academy Science Letters*, 39(4)(2016), 1–10.

[9] Baruah HK, Das D. *Imprecise Vector*, Lambert Academic Publishing, Germany, 2014.

[10] Das D., Dutta A., Mahanta S., Baruah HK, Construction of normal fuzzy numbers: a case study with earthquake waveform data, *Journal of Process Management - New Technologie*, 1(1)(2013), 1-6.

[11] Das, D., Mahanta, S., Chutia, R., Baruah, HK, Construction of normal fuzzy numbers: case studies with stock exchange data, *Annals of Fuzzy Mathematics and Informatics*, 5(3)(2013), 505-514.

[12] Mahanta S., Chutia R., Baruah HK, *Fuzzy Arithmetic Without Using the Method of Alpha Cuts*, *International Journal of Latest Trends in Computing*, 1(2)(2010), 73-80.

[13] Chutia, R., Mahanta, S., & Baruah, H. K, An alternative method of finding the membership of a fuzzy number, *International Journal of Latest Trends in Computing*, 1(2)(2010), 69-72.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆