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The Geodetic Vertex Covering Number of a Graph

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Abstract
A subset S of vertices in a connected graph G of order at least two is called a *geodetic vertex cover* if S is both a geodetic set and a vertex covering set. The minimum cardinality of a geodetic vertex cover is the *geodetic vertex covering number* of G denoted by $g_\alpha(G)$. Any geodetic vertex cover of cardinality $g_\alpha(G)$ is a g_α - set of G. Some general properties satisfied by geodetic vertex covering number of a graph are studied. The geodetic vertex covering number of several classes of graphs are determined. Some bounds for $g_\alpha(G)$ are obtained and the graphs attaining these bounds are characterized. A few realization results are given for the parameter $g_\alpha(G)$.

Keywords
Geodesic, geodetic set, vertex covering set, geodetic vertex cover, geodetic vertex covering number.

AMS Subject Classification
05C12

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Contents

1. Introduction

For basic graph theoretic terminology and basic definitions not given here we refer to Harary [5]. We consider finite, undirected, connected graphs without loops and multiple edges. Denote the number of vertices and edges of a graph G as $n = |V(G)|$ and $m = |E(G)|$ respectively. A vertex v is a *simplicial vertex or an extreme vertex* of G if the subgraph induced by its neighbors is complete.

Let $I[u,v]$ denote the set consisting of u, v, and all the vertices lying on a $u-v$ geodesic and for $S \subseteq V(G)$, $I[S]$ denote the union of all $I[u,v]$ for $u, v \in S$. The *geodetic number* $g(G)$ of G is the minimum cardinality of its geodetic sets and any geodetic set of cardinality $g(G)$ is a *minimum geodetic set* or a g - set of G. The geodetic number of a graph was introduced in [1,6] and further studied in [2 - 4]. A subset $S \subseteq V(G)$ is called a *vertex covering set* of G if every edge has at least one end point in S. A vertex covering set with minimum cardinality is a *minimum vertex covering set* of G. The *vertex covering number* of G is the cardinality of any minimum vertex covering set of G denoted as $\alpha(G)$. The vertex covering number of a graph was studied in [7].

A set of vertices (edges) in a graph G is *independent* if no two of the vertices (edges) are adjacent. The *independence number* $\beta(G)$ of G is the maximum number of vertices in an independent set of vertices of G. By a *matching* in a graph G, we mean an independent set of edges in G. A *caterpillar* is a tree of order 3 or more, the removal of whose end vertices produces a path called the spine of the caterpillar. A graph G is called *triangle free* if it does not contain cycles of length 3. A subset $S \subseteq V(G)$ is a *dominating set* if every vertex in $V-S$ is adjacent to at least one vertex in S. A *geodetic dominating set* of G is a subset S of vertices which is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set of a graph G is its *geodetic domination number* denoted by $\gamma_g(G)$.

In this paper we define geodetic vertex covering number $g_\alpha(G)$ of a graph and initiate a study of this parameter. We investigate about some general properties satisfied and some bounds attained by this parameter. Also few realization results are given for this parameter. We need the following theorems.

Theorem 1.1 ([3]) Every extreme vertex of a connected graph G belongs to every geodetic set of G.

Theorem 1.2 ([6]) For any tree T with k end vertices, $g(T) = k$.

Throughout this paper, G is considered as a connected graph of order at least two.

2. The Geodetic Vertex Cover of a Graph

Definition 2.1. *Let G be a connected graph of order at least 2. A set S of vertices of G is a geodetic vertex cover of G if S is both a geodetic set and a vertex covering set of G. The minimum cardinality of a geodetic vertex cover of G is defined as the geodetic vertex covering number of G and is denoted by $g_\alpha(G)$. Any geodetic vertex cover of cardinality $g_\alpha(G)$ is a g_α - set of G.*

Example 2.2. *Consider the graph G of Figure 2.1. Observe that $S_1 = \{v_2, v_4, v_5\}$ is a minimum vertex covering set of G so that $\alpha(G) = 3$, $S_2 = \{v_1, v_3, v_6\}$ is a minimum geodetic set of G so that $g(G) = 3$ and $S_3 = \{v_1, v_2, v_3, v_4, v_6\}$ is a g_α -set of G so that $g_\alpha(G) = 5$. Thus the geodetic vertex covering number of G is different from its vertex covering number and its geodetic number.*

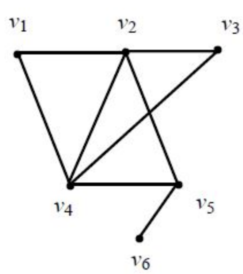

Figure 2.1 : G

Theorem 2.3. *Let G be any connected graph. Then $2 \leq max\{\alpha(G), g(G)\} \leq g_\alpha(G) \leq n$.*

Proof. Any geodetic set of G needs at least two vertices and so $2 \leq max\{\alpha(G), g(G)\}$. From the definition of geodetic vertex covering number of G, we have $max\{\alpha(G), g(G)\} \leq g_\alpha(G)$. Clearly $V(G)$ is a geodetic vertex cover of G. Hence $g_\alpha(G) \leq n$. Thus $2 \leq max\{\alpha(G), g(G)\} \leq g_\alpha(G) \leq n$. □

Remark 2.4. *The bounds in Theorem 2.3 are sharp. For C_4, $\alpha(C_4) = 2$, $g(C_4) = 2$ and $g_\alpha(C_4) = 2$. For K_n $(n \geq 2)$, $g_\alpha(K_n) = n$.*

Remark 2.5. *Clearly, union of a vertex covering set and a geodetic set is a geodetic vertex cover of G. Thus $2 \leq max\{\alpha(G), g(G)\} \leq g_\alpha(G) \leq min\{\alpha(G)+g(G), n\}$. Take the graph G in Figure 2.2. Observe that $S_1 = \{v_2, v_4\}$ is a minimum vertex covering set of G and hence $\alpha(G) = 2$, $S_2 = \{v_1, v_5\}$ is a g - set of G so that $g(G) = 2$ and $S_3 = \{v_1, v_2, v_4, v_5\} = S_1 \cup S_2$ is a g_α - set of G and so $g_\alpha(G) = 4 = \alpha(G) + g(G) < n = 6$.*

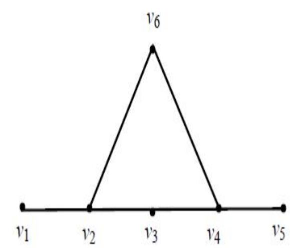

Figure 2.2 : G

Theorem 2.6. *Every simplicial vertex of a connected graph G belongs to every geodetic vertex cover of G.*

Proof. From the definition of g_α - set, every g_α - set of G is a g - set of G. Hence the result follows from Theorem 1.1. □

Corollary 2.7. *Let $K_{1,n-1}$ $(n \geq 3)$ be a star. Then $g_\alpha(K_{1,n-1}) = n-1$.*

Proof. The result follows from Theorem 2.6. □

Corollary 2.8. *For the complete graph $K_n (n \geq 2)$, $g_\alpha(K_n) = n$.*

Theorem 2.9. *If G is a connected graph of order $n \geq 2$, then*

(i) $g_\alpha(G) = 2$ if and only if G is either K_2 or $K_{2,n-2}$ $(n \geq 3)$.

(ii) $g_\alpha(G) = n$ if and only if $G = K_n$ $(n \geq 2)$.

Proof. (i) Let $g_\alpha(G) = 2$. Let $S = \{u, v\}$ be a minimum geodetic vertex cover of G. We claim that $G = K_2$ or $G = K_{2,n-2}$ $(n \geq 3)$. Suppose that $G = K_2$, then there is nothing to prove. If not, then $n \geq 3$ and since $S = \{u, v\}$ is a g_α - set of G, u and v cannot be adjacent in G. Let $W = V - S$. We claim that every vertex of W is adjacent to both u and v and no two vertices of W are adjacent.
Claim 1. Every vertex of W is adjacent to both u and v.
Suppose there is a vertex $w \in W$ such that w is adjacent to at most one vertex in S. Then w lies on a $u - v$ geodesic of length

at least 3. Let $P : u = v_0, v_1, ..., v_i = w, v_{i+1}, ..., v_m = v$ be a $u - v$ geodesic. Then the edges in $E(P) - \{v_0v_1, v_{m-1}v_m\}$ are not covered by any of the vertices u and v, which is a contradiction.

Claim 2. No two vertices of W are adjacent.

Suppose there exist vertices $w_i, w_j \in W$ such that w_i and w_j are adjacent. Since from Claim 1, every vertex of W is adjacent to both u and v and $S = \{u, v\}$ is a g_α - set of G, w_i and w_j lie on the $u - v$ geodesics, u, w_i, v and u, w_j, v, respectively. Then the edge w_iw_j is not covered by any of the vertices of S, which is a contradiction. Hence no two vertices of W are adjacent in G.

Thus G is the complete bipartite graph $K_{2,n-2} (n \geq 3)$ with partite sets S and W.

Conversely, let $G = K_2$ or $K_{2,n-2}$ $(n \geq 3)$. If $G = K_2$, then by Corollary 2.8, $g_\alpha(G) = 2$. If not, let $G = K_{2,n-2}$ $(n \geq 3)$. Let $U = \{u_1, u_2\}$ and $W = \{w_1, w_2, ..., w_{n-2}\}$ be the bipartition of G. Clearly every vertex w_i $(1 \leq i \leq n-2)$ lies on the geodesic u_1, w_i, u_2, and the vertices u_1 and u_2 cover all the edges of G. Hence U is a geodetic vertex cover of G and so $g_\alpha(G) = 2$.

(ii) Assume that $G = K_n$ $(n \geq 2)$. Then by Corollary 2.8, $g_\alpha(G) = n$. Conversely, let $g_\alpha(G) = n$. We prove that $G = K_n$ $(n \geq 2)$. For $n = 2$, the result holds from (i). Let $n \geq 3$. Contrarily assume that there exist two non-adjacent vertices u and v in G. Let a vertex x be adjacent to u lying on a $u - v$ geodesic. Then $V(G) - \{x\}$ is a geodetic vertex cover of G, giving a contradiction to $g_\alpha(G) = n$. Thus $G = K_n$. □

Theorem 2.10. *Let G be a connected graph with $n \geq 3$. Then $g_\alpha(G) = 3$ if and only if either $G = K_3$ or there exists a minimum geodetic set S on 3 vertices such that $V(G) - S$ is an independent set or there exists a minimum geodetic set S on 2 vertices such that $\langle V(G) - S \rangle$ is a star.*

Proof. Let $g_\alpha(G) = 3$. Let $S = \{u, v, w\}$ be a minimum geodetic vertex cover of G. Since $g(G) \leq g_\alpha(G)$, we have $g(G) = 2$ or 3.

Case 1. $g(G) = 3$. If $n = 3$, then by Theorem 2.9, $G = K_3$. If $n \geq 4$, then $V(G) - S \neq \Phi$. Since S is a g_α - set of G, every edge of G is incident with at least one vertex in S. Hence $V(G) - S$ is an independent set of vertices of G.

Case 2. $g(G) = 2$. Let $S' = \{u, v\} \subset S$ be a g - set of G. Also since $S = \{u, v, w\}$ is a g_α - set of G, the edges not covered by the vertices of S' should have exactly one end in w. Suppose the other ends of any two of these edges, say x and y are adjacent, then the edge xy will not be covered by any of the vertices of S, which is a contradiction. Hence $\langle V(G) - S \rangle$ must be a star.

Conversely, if $G = K_3$, by Corollary 2.8, $g_\alpha(G) = 3$. If G has a minimum geodetic set S on 3 vertices such that $V(G) - S$ is independent, then every edge of G has at least one end in S so that S is both a minimum geodetic set and a vertex cover of G. Hence S is a g_α - set of G and so $g_\alpha(G) = 3$. If G has a minimum geodetic set S on 2 vertices such that $\langle V(G) - S \rangle$ is a star, then S is not a vertex cover of G. Let w be the cut vertex of the star induced by $V(G) - S$. Then $S' = S \cup \{w\}$ will be a vertex cover of G so that S' is a geodetic vertex cover of G. □

Theorem 2.11. *Let G be a connected graph with $g(G) \geq n - 1$. Then $g_\alpha(G) = g(G)$.*

Proof. Let G be a connected graph with $g(G) \geq n - 1$. By Theorem 2.3, $g(G) \leq g_\alpha(G) \leq n$. If $g(G) = n$, then $g_\alpha(G) = n$ and so $g(G) = g_\alpha(G)$. If $g(G) = n - 1$, then let $S = \{x_1, x_2, ..., x_{n-1}\}$ be a g - set of G. Then there exists a vertex say, x not in S. Then x lies on a geodesic P joining any two vertices of S. Let x be adjacent to the vertices x_i and x_j on P for some $i \neq j$. Then all the edges of G including x_ix and xx_j are covered by the vertices of S. Hence S is a g_α - set of G. Thus $g(G) = g_\alpha(G)$. □

Remark 2.12. *The converse of Theorem 2.11 need not be true. For the graph G given in Figure 2.3, $S = \{v_1, v_2, v_5\}$ is both a minimum geodetic set and a minimum geodetic vertex cover of G so that $g_\alpha(G) = g(G) = 3$ but $g(G) < n - 1$.*

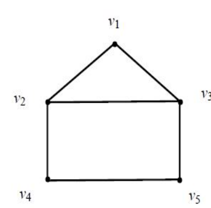

Figure 2.3 : G

Theorem 2.13. *Let G be a connected graph of order $n \geq 2$. Then $g_\alpha(G) = g(G)$ if and only if either $G = K_n$ or there exists a minimum geodetic set S such that $V(G) - S$ is independent.*

Proof. Let $g_\alpha(G) = g(G)$. If $G = K_n$, then clearly $g_\alpha(G) = g(G) = n$. If not, let S be a g - set of G. Since $g_\alpha(G) = g(G)$, S is a geodetic vertex cover of G. Hence every edge of G is incident with at least one vertex in S and so no edge of G has two ends in $V(G) - S$. Thus no pair of vertices of $V(G) - S$ are adjacent and hence $V(G) - S$ is an independent set.

Conversely, if $G = K_n$, then clearly $g_\alpha(G) = g(G) = n$. If S is a minimum geodetic set of G such that $V(G) - S$ is independent, then no edge of G has two ends in $V(G) - S$. Thus every edge of G has at least one end in S. Hence S is a minimum geodetic vertex cover of G so that $|S| = g(G) = g_\alpha(G)$. □

Theorem 2.14. *Let T be a tree of order $n \geq 2$. Then the following statements are equivalent.*

(i) $g_\alpha(T) = g(T)$.

(ii) *T is a star.*

(iii) $\alpha(T) = 1$.

(iv) *The set of all end vertices of T is a vertex cover of T.*

Proof. Let S consist of all end vertices of T. Since T is a tree, from Theorem 1.2, S is the unique g - set of T.

(i)⇒ (ii) Let $g_\alpha(T) = g(T)$. We claim that T is a star. If T is not a star, then $diam\ T \geq 3$ and so T has at least one edge other than the end edges. Let S' be the set of all edges of T which are not end edges. It is clear that S will not cover the edges in S'. Since, from Theorem 2.6, any geodetic vertex cover of T contains S, $g_\alpha(T) > |S| = g(T)$, which is a contradiction.

(ii) ⇒ (iii) Let T be a star. If $n = 2$, then an end vertex of T will cover the edge of T and if $n \geq 3$, then the cut vertex of T will cover all the edges in T. Hence $\alpha(T) = 1$.

(iii)⇒ (iv) Let $\alpha(T) = 1$. Then there exists a vertex, say x, in T such that x is an end vertex of all the edges in T. Hence all the edges in T are end edges in T and so S forms a vertex cover of T.

(iv) ⇒ (i) Suppose that S is a vertex cover of T. Then from Theorem 1.2, S is a g - set and by Theorem 2.6, S is a g_α - set of T. Hence $g_\alpha(T) = g(T)$.

□

Remark 2.15. *The results in Theorem 2.14 are not equivalent for any connected graph G of order $n \geq 2$. See graph G of Figure 2.4, $S = \{v_1, v_3, v_4\}$ is both a g - set and a g_α - set of G. So $g_\alpha(G) = g(G) = 3$. But $S' = \{v_2, v_3\}$ is a minimum vertex covering set and so $\alpha(G) = 2$. Moreover, G is not a star.*

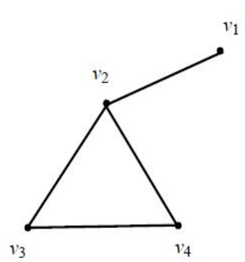

Figure 2.4 : G

Theorem 2.16. *Let T be a tree of order $n \geq 4$. If every cut vertex of T lies on a diametral path of length $2k+1$ or $2k+2$, then $g_\alpha(T) = g(T) + k$.*

Proof. Let T be a tree of order $n \geq 4$ with every cut vertex of T lies on a diametral path of length $2k+1$ or $2k+2$. Let $P : v_0, v_1, v_2, ..., v_m$ ($m = 2k+1$ or $2k+2$) be a diametral path of T. Let S be the set of all end vertices of T. Then by Theorem 1.2, S is the unique g - set of T and by Theorem 2.6, any geodetic vertex cover of T contains S. Let T' be a tree obtained from T by removing all the end vertices. Then $T' : v_1, v_2, ..., v_{m-1}$ is a path of length $2k-1$ or $2k$. It is clear that S will cover all the end edges of T and S will not cover the remaining edges of T. That is, S will not cover the edges of T'. So to cover the edges of T', we can include the vertices $v_2, v_4, v_6, ..., v_{2k}$ to a geodetic vertex cover of P. Clearly, $S' = S \cup \{v_2, v_4, v_6, ..., v_{2k}\}$ is a minimum geodetic vertex cover of T and so $g_\alpha(T) = |S| + k = g(T) + k$. □

Theorem 2.17. *Let T be a tree of order $n \geq 3$ with diameter d. Then $g_\alpha(T) = n-1$ if and only if T is either a star or a double star.*

Proof. Let $g_\alpha(T) = n-1$. Let $P : v_0, v_1, v_2, ..., v_d$ be a diametral path of T. Then $d \geq 2$. If $d \geq 4$, then $S = V(T) - \{v_1, v_3\}$ is a geodetic vertex cover of T and so $g_\alpha(T) \leq n-2$, giving a contradiction. Then $d = 2$ or 3 and hence T is either a star or a double star. Converse is clear. □

Theorem 2.18. *Let T be a caterpillar of order $n \geq 2$ with diameter d. Then $g_\alpha(T) = \lceil \frac{d}{2} \rceil + k - 1$, where k is the number of end vertices of T.*

Proof. Let T be a caterpillar. Let $P : v_0, v_1, v_2, ..., v_d$ be a diametral path and k denote number of end vertices of T. If d is even, let $S = \{v_0, v_2, v_4, ..., v_d\}$ and if d is odd, let $S = \{v_0, v_2, v_4, ..., v_{d-1}, v_d\}$. Then $|S| = \lceil \frac{d}{2} \rceil + 1$ and S covers all the edges of P. Since any vertex of P lies on the $v_0 - v_d$ geodesic, S is a geodetic vertex cover of the diametral path P. Since T is a caterpillar, $S' = (V(T) - V(P)) \cup \{v_0, v_d\}$ is the set of all end vertices of T. Then by Theorem 2.6, every geodetic vertex cover of T contains S'. Now, it is clear that $S'' = (S - \{v_0, v_d\}) \cup S'$ is a minimum geodetic vertex cover of T and so $g_\alpha(T) = \lceil \frac{d}{2} \rceil - 1 + k = \lceil \frac{d}{2} \rceil + k - 1$.

□

Theorem 2.19. (i) *For the cycle C_n ($n \geq 4$), $g_\alpha(C_n) = \lceil \frac{n}{2} \rceil$.*

(ii) *For the wheel $W_n = K_1 + C_{n-1}$ ($n \geq 5$), $g_\alpha(W_n) = \lceil \frac{n-1}{2} \rceil + 1$.*

(iii) *For the graph $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$, $g_\alpha(G) = n-1$.*

Proof. (i) Let $C_n : v_1, v_2, ..., v_n, v_1$ be a cycle of order n. It is clear that $S = \{v_1, v_3, v_5, ..., v_{2\lceil \frac{n}{2} \rceil - 1}\}$ is a g_α - set of C_n and so $g_\alpha(C_n) = \lceil \frac{n}{2} \rceil$.
(ii) Let $C_n : v_1, v_2, ..., v_{n-1}, v_1$ be the cycle of W_n and x, the

vertex of K_1 in W_n. Then $S = \{x, v_1, v_3, ..., v_{2\lceil \frac{n-1}{2} \rceil - 1}\}$ is a g_α - set of W_n. Hence $g_\alpha(W_n) = \lceil \frac{n-1}{2} \rceil + 1$.
(iii) Let $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$. Then $n \geq 3$ and G has exactly one cut vertex, say x, and all the remaining vertices are simplicial vertices of G. Then by Theorem 2.6, $S = V(G) - \{x\}$ is a subset of any geodetic vertex cover of G and so $g_\alpha(G) = n - 1$.

□

Theorem 2.20. *For any connected graph G, $g_\alpha(G) \leq n - \lfloor \frac{diam\ G}{2} \rfloor$.*

Proof. Let $P : v_0, v_1, v_2, ..., v_d$ be a diametral path of G. If d is even, then $S = \{v_0, v_2, v_4, ..., v_d\}$ covers all the edges of P and each vertex of P lies on a $v_0 - v_d$ geodesic. Hence $(V(G) - V(P)) \cup S$ is a g_α - set. So $g_\alpha(G) \leq n - (d+1) + (\frac{d}{2} + 1) = n - \frac{d}{2}$. Similarly, if d is odd, then $S' = \{v_0, v_2, v_4, ..., v_{d-1}, v_d\}$ covers all the edges of P and every vertex of P lies on a $v_0 - v_d$ geodesic. Hence $(V(G) - V(P)) \cup S'$ is a g_α - set. So $g_\alpha(G) \leq n - (d+1) + (\lceil \frac{d}{2} \rceil + 1) = n - \lfloor \frac{d}{2} \rfloor$. Thus in both cases, we have $g_\alpha(G) \leq n - \lfloor \frac{diam\ G}{2} \rfloor$.

□

Remark 2.21. *The bound in Theorem 2.20 is sharp. For the path $P_7 : v_0, v_1, v_2, v_3, v_4, v_5, v_6, S = \{v_0, v_2, v_4, v_6\}$ is the unique minimum geodetic vertex cover of P_7 and so $g_\alpha(P_7) = 4$. Also since diam $P_7 = 6$, we have $n - \lfloor \frac{diam\ P_7}{2} \rfloor = 7 - 3 = 4$. Thus $g_\alpha(P_7) = n - \lfloor \frac{diam\ P_7}{2} \rfloor$.*

Theorem 2.22. *If G is a triangle free graph with $\delta(G) \geq 2$ and M is a maximal matching of G, then $g_\alpha(G) \leq 2|M|$.*

Proof. Let S consist of all end vertices of the edges of M. Since M is a maximal matching of G, no edge of G has its two ends in $V(G) - S$. Hence $V(G) - S$ is independent so that S is a vertex cover of G. Thus every edge of G has at least one end in S. Since $\delta(G) \geq 2$, there exist at least two neighbors x and y in S for every $v \in V(G) - S$. Since G has no triangles, the path x, v, y is an $x - y$ geodesic. Hence S is a g_α - set of G. Thus $g_\alpha(G) \leq 2|M|$. □

Theorem 2.23. *Let G be a triangle free graph with $\delta(G) \geq 2$. Then $g_\alpha(G) = n - \beta(G)$, where $\beta(G)$ is the independence number of G.*

Proof. Let S be a maximum independent set of vertices of G so that $|S| = \beta(G)$. Then $V(G) - S$ is a minimum vertex cover of G. Since G is triangle free and $\delta(G) \geq 2$, every vertex in S has at least two neighbors which are not adjacent in $V(G) - S$. Thus, every vertex $v \in S$ lies on an $x - y$ geodesic for some vertices $x, y \in V(G) - S$. Hence $V(G) - S$ is also a g - set of G. Thus $V(G) - S$ is a g_α - set and hence $g_\alpha(G) = n - \beta(G)$. □

Theorem 2.24. *Every geodetic vertex cover of a connected graph G is a geodetic dominating set of G.*

Proof. Let S be a geodetic vertex cover of G. Then S is both a geodetic set and a vertex cover of G. Since S is a vertex cover, every edge of G has at least one end in S and hence every vertex in $V(G) - S$ has at least one neighbour in S so that S is a dominating set of G. Hence S is a geodetic dominating set of G.

□

Corollary 2.25. *If G is any connected graph, then $2 \leq \gamma_g(G) \leq g_\alpha(G) \leq n$.*

Remark 2.26. *Note that, $\gamma_g(K_2) = 2$. See graph G of Figure 2.5, $S_1 = \{v_1, v_2, v_3\}$ is a g_α - set and $S_2 = \{v_1, v_3\}$ is a γ_g - set of G. Thus $g_\alpha(G) = 3$ and $\gamma_g(G) = 2$ and so $\gamma_g(G) < g_\alpha(G)$. And $\gamma_g(K_n) = g_\alpha(K_n) = n$.*

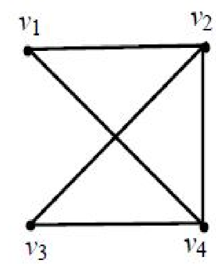

Figure 2.5 : G

Theorem 2.27. *Let S be a minimum geodetic dominating set of a connected graph G. Then S is a geodetic vertex cover of G if and only if $V(G) - S$ is an independent set.*

Proof. Let S be a minimum geodetic dominating set of G. If S is a geodetic vertex cover of G, then every edge of G has at least one end in S. Hence no two vertices in $V(G) - S$ are adjacent so that $V(G) - S$ is independent.

Conversely, let $V(G) - S$ be an independent set of G. Then every edge of G has at least one end in S so that S is also a vertex cover of G. Hence S is a geodetic vertex cover of G.

□

3. Realization Results

By Theorem 2.3, $2 \leq max\{\alpha(G), g(G)\} \leq g_\alpha(G) \leq n$. Also, we have $g_\alpha(G) \leq min\{\alpha(G) + g(G), n\}$. The following theorems give realization results for these parameters.

Theorem 3.1. *If a and n are positive integers such that $2 \leq a \leq n$, then there exists a connected graph G of order n with $g_\alpha(G) = a$.*

Proof. We prove this theorem by considering two cases.
Case (i) $2 \leq a = n$. Take $G = K_n$, then from Theorem 2.9 (ii),

$g_\alpha(G) = n = a$.
Case (ii) $2 \le a < n$. Take $H = K_{a-1}$, the complete graph on $a-1$ vertices $u_1, u_2, ..., u_{a-1}$. Add $n-a+1$ new vertices $v_1, v_2, ..., v_{n-a}, x$ to H and join the vertices $v_1, v_2, ..., v_{n-a}$ to both u_{a-1} and x. Thus we get the graph G of Figure 3.1.

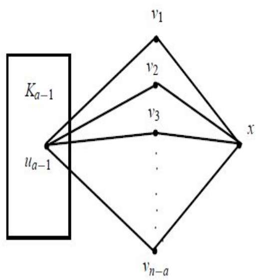

Figure 3.1 : G

Let $S = \{u_1, u_2, ..., u_{a-2}\}$ consist of all simplicial vertices of G so that by Theorem 1.1, they must belong to every geodetic set. Observe that $S' = S \cup \{x\}$ is a g - set and the edges of K_{a-1} and the edges $v_i x (1 \le i \le n-a)$ are covered by the vertices of S'. Now, to cover the edges $u_{a-1}v_i (1 \le i \le n-a)$, we must include at least the vertex u_{a-1} to S'. Hence a g_α - set of G is $S'' = \{u_1, u_2, ..., u_{a-2}, u_{a-1}, x\}$ with $|S''| = a < n$. □

Theorem 3.2. *Let $a, b \ge 2$ be any pair of integers. Then there is a connected graph G with $\alpha(G) = a$, $g(G) = b$ and $g_\alpha(G) = a + b$.*

Proof. Consider a cycle $C_4 : z_1, z_2, z_3, z_4, z_1$ of order 4 and a path $P : y_0, y_1, ..., y_{2(a-2)}$ of order $2(a-2)+1$. Obtain graph H from C_4 and P by joining the vertices z_3 in C_4 and y_0 in P. Add $b-1$ new vertices $x_1, x_2, ..., x_{b-1}$ to H and join each to the vertex z_1. The resultant graph G is shown in Figure 3.2.
Observe that the set $S = \{z_1, z_3, y_1, y_3, ..., y_{2(a-2)-1}\}$ is a minimum vertex cover for G with $|S| = a$.

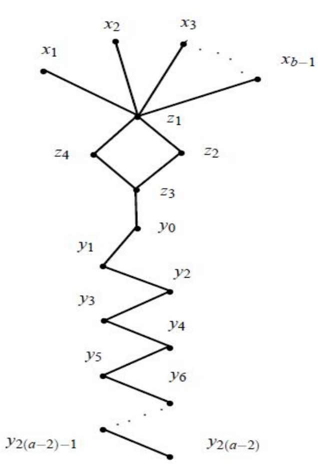

Figure 3.2 : G

Let $S' = \{x_1, x_2, ..., x_{b-1}, y_{2(a-2)}\}$ be the set of all simplicial vertices of G so that they must belong to every geodetic set of G. Moreover, S' itself is a geodetic set of G and hence S' is a minimum geodetic set of G with $|S'| = b$. Thus $\alpha(G) = a$ and $g(G) = b$. Clearly $S \cup S'$ is a g_α - set and so $g_\alpha(G) = a + b$.

□

Theorem 3.3. *Every pair of integers a,b with $1 \le a < b$ can be realized as the vertex covering number and geodetic vertex covering number, respectively, of some connected graph G.*

Proof. We prove this theorem by considering two cases.
Case(i) $1 = a < b$. Let $G = K_{1,b}$ $(b \ge 2)$ be a star. Then by Theorem 2.14 and Corollary 2.7, $\alpha(G) = 1$ and $g_\alpha(G) = b$.
Case(ii) $2 \le a < b$. Take $H = K_{a+1}$, the complete graph on $a+1$ vertices $u_1, u_2, ..., u_{a+1}$. Add $b-a$ new vertices $v_1, v_2, ..., v_{b-a}$ to H by joining each of them to u_{a+1} and get graph G of Figure 3.3.

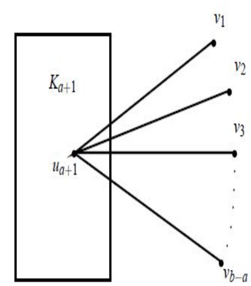

Figure 3.3 : G

Clearly the set $S = \{u_2, u_3, ..., u_{a+1}\}$ is a minimum vertex cover of G with $|S| = a$. Let $S' = \{u_1, u_2, ..., u_a, v_1, v_2, ..., v_{b-a}\}$ be the set of all simplicial vertices of G so that by Theorem 1.1, they must belong to every geodetic set of G. Moreover, S' itself is a geodetic set of G and hence S' is a minimum geodetic set of G. Also, S' covers all the edges of G. Hence S' is a minimum geodetic vertex cover of G with $|S'| = b$.

□

Theorem 3.4. *For any two positive integers a and b with $2 \le a \le b$, there exists a connected graph G with $g(G) = a$ and $g_\alpha(G) = b$.*

Proof. Let $P : u_1, u_2, ..., u_{2(b-a+1)}$ be a path of order $2(b-a+1)$. Add $a-1$ new vertices $v_1, v_2, ..., v_{a-1}$ to P and join these to $u_{2(b-a+1)}$. The resultant tree G is in Figure 3.4.

Clearly, $S = \{u_1, v_1, v_2, ..., v_{a-1}\}$ is the set of all simplicial vertices of G. Since G is a tree, by Theorem 1.2, $g(G) = a$. It is clear that the set S covers the edges $u_1u_2, v_iu_{2(b-a+1)}$ $(1 \le i \le a-1)$ and the remaining edges are covered by the vertices $u_3, u_5, ..., u_{2(b-a)+1}$. Hence the minimum geodetic vertex cover is $S' = \{u_3, u_5, ..., u_{2(b-a)+1}, u_1, v_1, v_2, ..., v_{a-1}\}$ with $|S'| = b$.

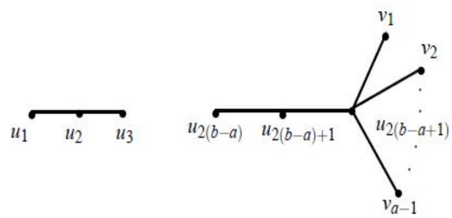

Figure 3.4 : G

□

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