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Minimum irregularity of totally segregated bicyclic graphs

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Abstract

A connected graph G is totally segregated if every edge in G joins vertices of different degrees. In this paper we focus on special class of graphs called totally segregated ∞^+- bicyclic graphs and Θ- bicyclic graphs. Here we make an attempt to find the minimum irregularity of totally segregated ∞^+- bicyclic graphs and Θ- bicyclic graphs and present those extremal graphs.

Keywords

Totally segregated graph, irregularity, ∞^+- bicyclic graph, Θ- bicyclic graph.

AMS Subject Classification

05C07,05C38,05C75.

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Contents

1. Introduction

In many situations, it is of great importance to know the irregularity a given graph. In literature, several measures of such 'irregularity' were proposed [3] [6] [5] [4]. Among them, the most investigated one is irregularity of a graph introduced by Albertson [2]. Albertson defines the imbalance of an edge $e = uv \in E(G)$ as $|deg_G u - deg_G v|$ and irregularity of G as

$$irr(G) = \sum_{uv \in E(G)} |deg_G u - deg_G v| \tag{1.1}$$

The imbalance of an edge e in G is denoted by $imb_G(e)$ or $imb(e)$ if the graph under discussion is clear. Edges of a graph with different end vertex degrees are called imbalanced edges and edges with same end vertex degrees are called balanced edges. Note that graphs with same degree sequence may have different irregularity (Figure 1). Two non-isomorphic graphs G_1 and G_2 have the same degree sequence 3,3,2,1,1,1,1. They have different irregularities $(irr(G_1) = 10$ and $irr(G_2) = 8)$.

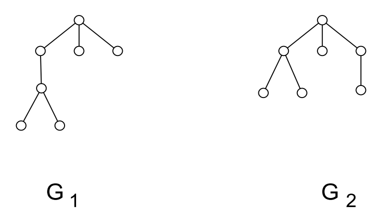

Figure 1. Two non-isomorphic graphs with same degree sequence

Another measure of irregularity is the total irregularity of a graph introduced by H. Abdo and D. Dimitrov in [1], which is defined as:

$$irr_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} |deg_G u - deg_G v| \tag{1.2}$$

A connected graph G is totally segregated if $deg_G u \neq deg_G v$, for every edge $uv \in E(G)$. The class of totally segregated graphs was studied by Jackson and Entringer [7]. For convenience, we abbreviate totally segregated bicyclic graph to TSB graph. There exists no connected totally segregated graph of order 2. A connected graph G is *uniformly segregated* if for every $uv \in E(G)$, $|deg_G u - deg_G v|$ is a constant. When this constant is zero the graph is regular. When this constant is $k \neq 0$, the graph is called k- segregated graph[8]. In [10] L.You et al. introduced three types of bicyclic graphs and discussed the total irregularity of those bicyclic graphs.

In [9] minimum irregularity of ∞- TSB graphs is determined. In this paper we make an attempt to find minimum irregularity of ∞^+- TSB graphs and Θ- TSB graphs and to present those extremal graphs.

2. Minimum Irregularity of Totally Segregated ∞^+- Bicyclic Graphs and Θ- Bicyclic Graphs

To make this work self content, we present ∞^+- bicyclic graph and Θ- bicyclic graphs which is introduced in [10]. A bicyclic graph is a simple connected graph in which the number of edges will exceed the number of vertices by one. Here we see two bicycles ∞^+- bicycle and Θ- bicycle. The ∞^+- bicycle denoted by $\infty(a,b,c), a,b \geq 3, c \geq 2$ (Figure 2) is obtained from two vertex-disjoint cycles C_a and C_b by connecting C_a and C_b with a path P_c of length $c-1$ ($c \geq 2$,) where $a,b \geq 3$ and $c \geq 2$.

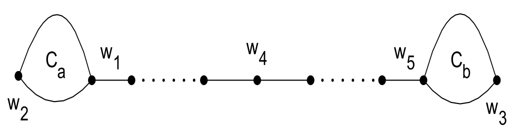

Figure 2. The graph $\infty(a,b,c)$ with $a \geq 3$, $b \geq 3$ and $c \geq 2$

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The Θ- bicycle , denoted by $\theta(a,b,c)$ (Figure 3), is a graph on $a+b-c$ vertices with the two cycles C_a and C_b having c common vertices, where $a,b \geq 3$ and $c \geq 2$.

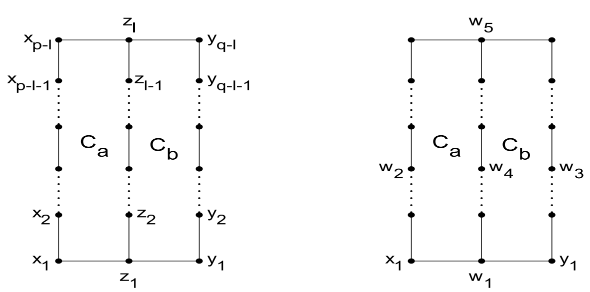

Figure 3. The graph $\theta(a,b,c)$ with $a \geq 3$, $b \geq 3$ and $c \geq 2$

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Set of graphs each of which is a bicyclic graph with ∞^+ bicycle, $\infty(a,b,c)$,($a \geq 3$, $b \geq 3$, $c \geq 2$) is called ∞^+- bicyclic graph and with Θ- bicycle, $\Theta(a,b,c)$ ($a \geq 3$, $b \geq 3$, $c \geq 2$) is called Θ- bicyclic graph.
Observe that any ∞^+- bicyclic gaph G (Θ- bicyclic graph) is obtained from a ∞^+ bicycle, $\infty(a,b,c), a,b \geq 3, c \geq 2$ (Θ- bicycle $\Theta(a,b,c)$ ($a \geq 3$, $b \geq 3$, $c \geq 2$)) (possibly) by attaching trees to some of its vertices. If G is obtained from $\infty(a,b,c)$ ($\Theta(a,b,c)$) by attaching trees to some of its vertices, then we call G as ∞^+- bicyclic graph (Θ- bicyclic graph).

Remark 2.1. For $n \leq 9$, a ∞^+- TSB graph of order n does not exist.

In the following theorem we find ∞^+- TSB graphs on n vertices with minimum irregularity.

Theorem 2.2. *If $\mathscr{B}_n^+$ is the set of all ∞^+- TSB graphs on n vertices, $(n \geq 10)$ and $\mathscr{B}_n^+ = \mathrm{B}_1^+ \cup \mathrm{B}_2^+ \cup \mathrm{B}_3^+ \cup \mathrm{B}_4^+ \cup \mathrm{B}^+$, where*

- $\mathrm{B}_1^+ = \{G \in \mathscr{B}_n^+ : n = 4k, k = 3,4,\cdots\}$
- $\mathrm{B}_2^+ = \{G \in \mathscr{B}_n^+ : n = 4k+1, k = 3,4,5,\cdots\}$
- $\mathrm{B}_3^+ = \{G \in \mathscr{B}_n^+ : n = 4k+2, k = 4,5,\cdots\}$
- $\mathrm{B}_4^+ = \{G \in \mathscr{B}_n^+ : n = 4k+3, k = 4,5,\cdots\}$
- $\mathrm{B}^+ = \{G \in \mathscr{B}_n^+ : n = 10,11,14,15\}$.

then,

1. $\min\{irr(G)\}_{G \in \mathrm{B}_1^+} = n+2$
2. $\min\{irr(G)\}_{G \in \mathrm{B}_2^+} = n+1$
3. $\min\{irr(G)\}_{G \in \mathrm{B}_3^+} = n+2$
4. $\min\{irr(G)\}_{G \in \mathrm{B}_4^+} = n+1$
5. $\min\{irr(G)\}_{G \in \mathrm{B}^+} = \begin{cases} 20 & if\, G \in \mathscr{B}_{10}^+ \\ 14 & if\, G \in \mathscr{B}_{11}^+ \\ 18 & if\, G \in \mathscr{B}_{14}^+ \cup \mathscr{B}_{15}^+ \end{cases}$

Proof. Let $G \in \mathrm{B}_1^+$. Since G is a bicyclic graph, G has odd number $(n+1)$ of edges. But irregularity of any graph is always even [2]. Hence at least one edge has an imbalance greater than one. Hence $irr(G) \geq n+2$.
TSB graphs $G \in \mathrm{B}_1^+$ with irregularity $n+2$ are depicted in the Figure 4.

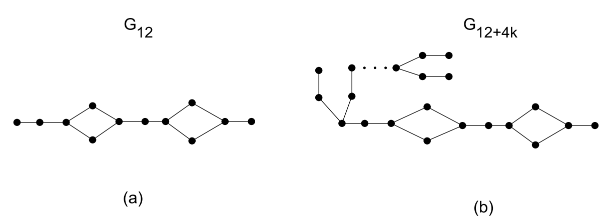

Figure 4. ∞^+- TSB graph with minimum irregularity on $4k$ vertices ,$k \geq 3$

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Let $G \in \mathrm{B}_2^+$. Since G is a TSB graph, it has $n+1$ edges. Hence $irr(G) \geq n+1$. TSB graphs $G \in \mathrm{B}_2^+$ with irregularity $n+1$ is presented in Figure 5. G_{13}, the 1-segregated ∞^+- bicyclic graph with 13 vertices, is given in Figure 5 (a). 1-segregated graph of this type on $4k+1$ vertices, $k \geq 3$, is constructed from G_{13} (Figure 5 (b)).

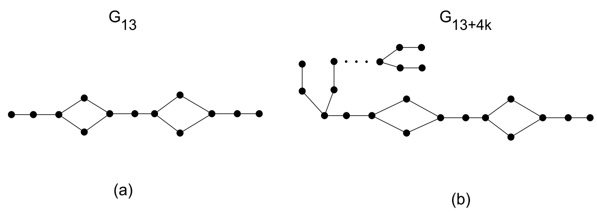

Figure 5. ∞^+- TSB graph with minimum irregularity on $4k+1$ vertices, $k \geq 3$

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Let $G \in B_3^+$. In this case, order of the graph G is even and hence it has odd number $(n+1)$ of edges. But irregularity of a graph is even [2]. Thus $irr(G) \geq n+2$. TSB graph $G \in B_3^+$ with irregularity $n+2$ is presented in Figure 6.

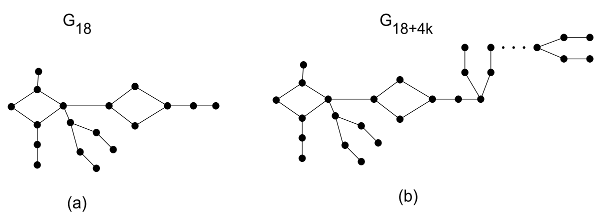

Figure 6. ∞^+- TSB graph with minimum irregularity on $4k+2$ vertices, $k \geq 4$

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Let $G \in B_4^+$. Since G is TSB graph, it has $n+1$ edges. Hence $irr(G) \geq n+1$. TSB graphs $G \in B_4^+$ with irregularity $n+1$ is given in Figure 7. G_{19}, the 1-segregated ∞^+ bicyclic graph with 19 vertices, is depicted in Figure 7 (a). 1-segregated ∞^+- bicyclic graph on $4k+3$ vertices, $k \geq 4$, is constructed from G_{19} (Figure 7 (b)).

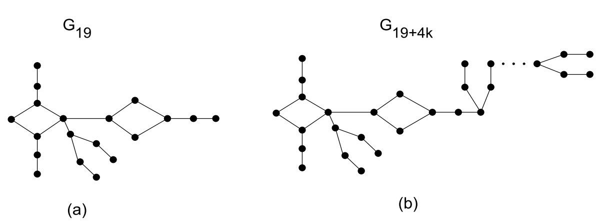

Figure 7. ∞^+- TSB graph with minimum irregularity on $4k+3$ vertices, $k \geq 4$.

Let $G \in B^+$. In this case ∞^+- TSB graph with minimum irregularity is given in Figure 8.

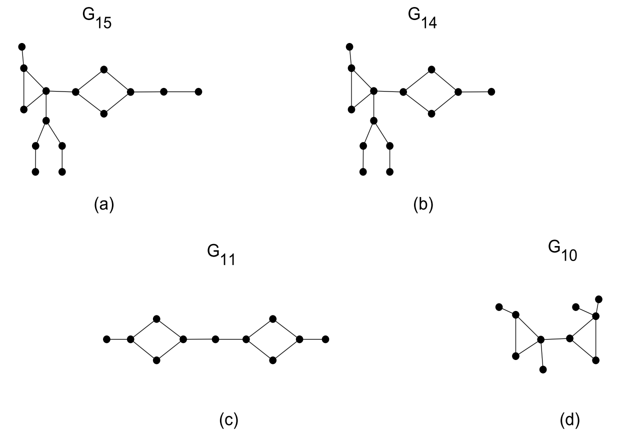

Figure 8. ∞^+- TSB graph with minimum irregularity on n vertices, $n = 10, 11, 14, 15$.

□

Remark 2.3. For $n \leq 4$, a Θ- TSB graph of order n does not exist.

In the following theorem we find Θ- TSB graphs with n vertices having minimum irregularity.

Theorem 2.4. *If $\mathscr{B}_n^{++}$ is the set of all Θ- TSB graphs on n vertices, $(n \geq 5)$ and $\mathscr{B}_n^{++} = B_1^{++} \cup B_2^{++} \cup B_3^{++} \cup B_4^{++} \cup B^{++}$, where*

- $B_1^{++} = \{G \in \mathscr{B}_n^{++} : n = 4k, k = 3, 4, \cdots\}$
- $B_2^{++} = \{G \in \mathscr{B}_n^{++} : n = 4k+1, k = 3, 4, 5, \cdots\}$
- $B_3^{++} = \{G \in \mathscr{B}_n^{++} : n = 4k+2, k = 3, 4, 5, \cdots\}$
- $B_4^{++} = \{G \in \mathscr{B}_n^{++} : n = 4k+3, k = 3, 4, 5, \cdots\}$
- $B^{++} = \{G \in \mathscr{B}_n^{++} : n = 5, 6, 7, 8, 9, 10, 11\}$.

then,

1. $\min\{irr(G)\}_{G \in B_1^{++}} = n+2$
2. $\min\{irr(G)\}_{G \in B_2^{++}} = n+1$
3. $\min\{irr(G)\}_{G \in B_3^{++}} = n+2$
4. $\min\{irr(G)\}_{G \in B_4^{++}} = n+1$
5. $$\min\{irr(G)\}_{G \in B^{++}} = \begin{cases} 6 & \text{if } G \in \mathscr{B}_5^{++} \\ 8 & \text{if } G \in \mathscr{B}_6^{++} \cup \mathscr{B}_7^{++} \\ 10 & \text{if } G \in \mathscr{B}_8^{++} \cup \mathscr{B}_9^{++} \\ 14 & \text{if } G \in \mathscr{B}_{10}^{++} \cup \mathscr{B}_{11}^{++} \end{cases}$$

Proof. Let $G \in B_1^{++}$. Since G is a bicyclic graph on n vertices, G has odd number $(n+1)$ of edges. But irregularity of any graph is always even [2]. Hence at least one edge has an imbalance greater than one. Hence $irr(G) \geq n+2$.
TSB graphs $G \in B_1^{++}$ with irregularity $n+2$ are depicted in Figure 9.

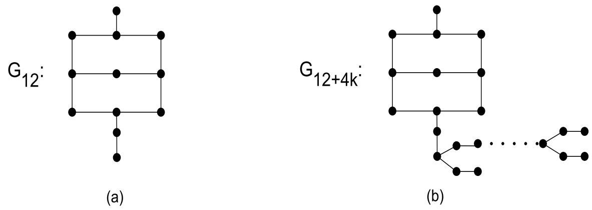

Figure 9. Θ- TSB graph with minimum irregularity on $4k$ vertices, $k \geq 3$.

Let $G \in \mathrm{B}_2^{++}$. Then $irr(G) \geq n+1$. TSB-graphs $G \in \mathrm{B}_2^{++}$ with minimum irregularity $n+1$ are presented in the Figure 10.

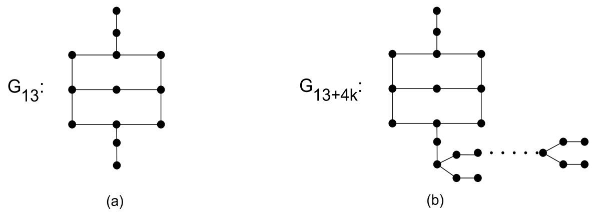

Figure 10. Θ- TSB graph with minimum irregularity on $4k+1$ vertices, $k \geq 3$.

Let $G \in \mathrm{B}_3^{++}$. Then $irr(G) \geq n+2$. TSB-graphs $G \in \mathrm{B}_3^{++}$ with minimum irregularity $n+2$ are presented in the Figure 11.

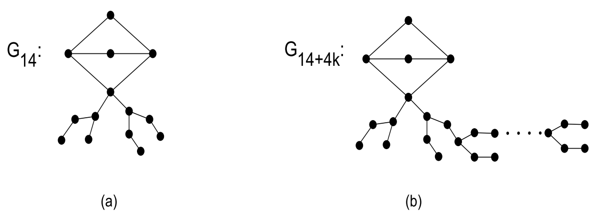

Figure 11. Θ- TSB graph with minimum irregularity on $4k+2$, $k \geq 3$.

Let $G \in \mathrm{B}_4^{++}$. Then $irr(G) \geq n+1$. TSB-graphs $G \in \mathrm{B}_4^{++}$ with minimum irregularity $n+1$ are given in the Figure 12.

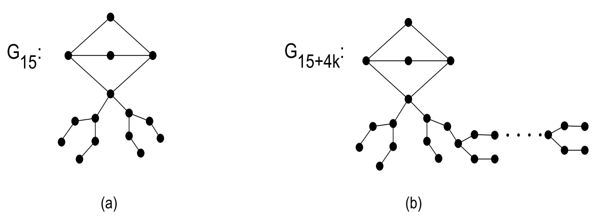

Figure 12. Θ- TSB graph with minimum irregularity on $4k+3$ vertices, $k \geq 3$.

Let $G \in \mathrm{B}^{++}$. Θ- TSB graphs with minimum irregularity are presented in Figure 13.

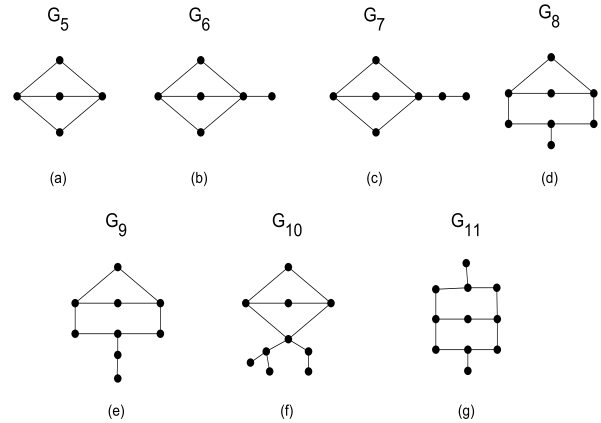

Figure 13. Θ- TSB graph with minimum irregularity on n vertices where $n = 5, 6, 7, 8, 9, 10, 11$.

$\square$

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