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On the solutions of a higher order recursive sequence

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Abstract

In this paper, we introduce an explicit formula and discuss the global behavior of solutions of the recursive sequence

$$x_{n+1} = \frac{ax_{n-2k-1}}{b+c\prod_{l=0}^{k} x_{n-2l-1}}, \quad n = 0,1,\ldots,$$

where a,b,c are positive real numbers, the initial conditions $x_{-2k-1}, x_{-2k}, \ldots, x_{-1}, x_0$ are real numbers and k is a nonnegative integer. We show that every admissible solution with $\prod_{l=0}^{k} x_{-2(l+1)+i} = \frac{a-b}{c}$, $i = 1,2$ is periodic with prime period $2k+2$. Otherwise, the solution converges to zero if $a < b$ or converges to a period-$(2k+2)$ solution if $a > b$. We finally study some special cases and give illustrative examples.

Keywords

Difference equation, periodic solution, convergence.

AMS Subject Classification

39A20.

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Contents

1. Introduction

Difference equations have played an important role in analysis of mathematical models of biology, physics and engineering. Recently, there has been a great interest in studying properties of nonlinear and rational difference equations. One can see [2–7, 9, 10, 13–16, 18–22, 25, 26] and the references therein.

We have discussed in [8] the global behavior of the solutions of the difference equation

$$x_{n+1} = \frac{Bx_{n-2r-1}}{C+D\prod_{i=l}^{k} x_{n-2i}}, \quad n = 0,1,\ldots,$$

where A,B,C are nonnegative real numbers and the initial conditions are nonnegative real numbers and l,r,k are nonnegative integers such that $l \leq k$ and $r \leq k$.

In [23], D. Simsek et al. introduced the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}}, \quad n = 0,1,\ldots,$$

where $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0,\infty)$.

Also in [24], D. Simsek et al. introduced the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1+x_{n-1}x_{n-3}}, \quad n = 0,1,\ldots,$$

with positive initial conditions.

R. Karatas et al. [17] discussed the positive solutions and the attractivity of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}x_{n-5}}, \quad n = 0,1,\ldots,$$

where the initial conditions are nonnegative real numbers.
In [11], E.M. Elsayed discussed the solutions of the difference equation

$$x_{n+1}=\frac{x_{n-5}}{-1+x_{n-2}x_{n-5}},\quad n=0,1,...,$$

where the initial conditions are nonzero real numbers with $x_{-5}x_{-2}\neq 1$, $x_{-4}x_{-1}\neq 1$ and $x_{-3}x_0\neq 1$.
Also in [12], E.M. Elsayed determined the solutions to some difference equations. He obtained the solution to the difference equation

$$x_{n+1}=\frac{x_{n-3}}{1+x_{n-1}x_{n-3}},\quad n=0,1,...,$$

where the initial conditions are nonzero positive real numbers.
In [1], the authors obtained the expressions of solutions of the following difference equations

$$x_{n+1}=\frac{x_{n-2k+1}}{\pm 1+\pm\prod_{l=0}^{k}x_{n-2l+1}},\quad n=0,1,\ldots,$$

with initial conditions $x_{-j},\ j=0,1,...,2k-1,$ where $k\in\{1,2,...\}$.

In this paper, we introduce an explicit formula and investigate the global behavior of solutions of the recursive sequence

$$x_{n+1}=\frac{ax_{n-2k-1}}{b+c\prod_{l=0}^{k}x_{n-2l-1}},\quad n=0,1,\ldots,\tag{1.1}$$

where a,b,c are positive real numbers, the initial conditions $x_{-2k-1},x_{-2k},...,x_{-1},x_0$ are real numbers and k is a nonnegative integer.

2. Solution of Equation (1.1)

In this section, we give an explicit formula for the solution of Equation (1.1) with $a\neq b$.
Let $\theta_i=\frac{a-b-c\alpha_i}{\alpha_i}$, where $\alpha_i=\prod_{l=0}^{k}x_{-2(l+1)+i}$, $i=1,2$.
We need the following lemma to prove the main result in this section.

Lemma 2.1. *Let $x_{-2k-1},x_{-2k},...,x_{-1}$ and x_0 be real numbers such that for any $i\in\{1,2\}$, $\alpha_i\neq-\frac{b}{c\sum_{r=0}^{n}(\frac{a}{b})^r}$ for all $n\in\mathbb{N}$. Then*

$$\prod_{l=0}^{k}x_{-2(l+1)+2t+i}=\frac{a-b}{(\frac{b}{a})^t\theta_i+c},\quad 0\leq t\leq k\quad \textit{and}\quad 1\leq i\leq 2.$$

Proof. Let $\mu(t,i)=\prod_{l=0}^{k}x_{-2(l+1)+2t+i}$, where $0\leq t\leq k$ and $1\leq i\leq 2$. It is required to show that

$$\mu(t,i)=\frac{a-b}{(\frac{b}{a})^t\theta_i+c}\quad \text{for}\quad 0\leq t\leq k\quad\text{and}\quad 1\leq i\leq 2.\tag{2.1}$$

The proof is by induction on t for each $1\leq i\leq 2$.
When $t=0$, we have for each $1\leq i\leq 2$,

$$\mu(0,i)=\prod_{l=0}^{k}x_{-2(l+1)+i}=\alpha_i.$$

On the other hand, the right hand side of Equality (2.1) when $t=0$ for each $1\leq i\leq 2$ is

$$\frac{a-b}{\theta_i+c}=\frac{a-b}{\frac{a-b-c\alpha_i}{\alpha_i}+c}=\alpha_i.$$

Suppose that for a certain $0\leq t\leq k-1$, we have

$$\mu(t,i)=\frac{a-b}{(\frac{b}{a})^t\theta_i+c}\quad\text{for each}\quad 1\leq i\leq 2.$$

Then for each $1\leq i\leq 2$,

$$\begin{aligned}\mu(t+1,i)&=\prod_{l=0}^{k}x_{-2(l+1)+2(t+1)+i}=x_{2t+i}\prod_{l=1}^{k}x_{-2l+2t+i}\\&=\frac{ax_{-2(k+1)+2t+i}}{b+c\prod_{l=0}^{k}x_{-2(l+1)+2t+i}}\prod_{l=0}^{k-1}x_{-2(l+1)+2t+i}\\&=\frac{a\prod_{l=0}^{k}x_{-2(l+1)+2t+i}}{b+c\prod_{l=0}^{k}x_{-2(l+1)+2t+i}}\,\frac{a\mu(t,i)}{b+c\mu(t,i)}\\&=\frac{a\frac{a-b}{(\frac{b}{a})^t\theta_i+c}}{b+c\frac{a-b}{(\frac{b}{a})^t\theta_i+c}}=\frac{a(a-b)}{b((\frac{b}{a})^t\theta_i+c)+c(a-b)}\\&=\frac{a-b}{(\frac{b}{a})^{t+1}\theta_i+c}.\end{aligned}$$

This completes the proof.

□

Theorem 2.2. *Let $x_{-2k-1},x_{-2k},...,x_{-1}$ and x_0 be real numbers such that for any $i\in\{1,2\}$, $\alpha_i\neq-\frac{b}{c\sum_{r=0}^{n}(\frac{a}{b})^r}$ for all $n\in\mathbb{N}$. Then the solution $\{x_n\}_{n=-2k-1}^{\infty}$ of Equation (1.1) can be written*

$$x_n=\begin{cases}x_{-2k-1}\prod_{j=0}^{\frac{n-1}{2(k+1)}}\frac{(\frac{b}{a})^{(k+1)j}\theta_1+c}{(\frac{b}{a})^{(k+1)j+1}\theta_1+c}, & n=1,(2k+3),...,\\ x_{-2k}\prod_{j=0}^{\frac{n-2}{2(k+1)}}\frac{(\frac{b}{a})^{(k+1)j}\theta_2+c}{(\frac{b}{a})^{(k+1)j+1}\theta_2+c}, & n=2,(2k+4),...,\\ x_{-2k+1}\prod_{j=0}^{\frac{n-3}{2(k+1)}}\frac{(\frac{b}{a})^{(k+1)j+1}\theta_1+c}{(\frac{b}{a})^{(k+1)j+2}\theta_1+c}, & n=3,(2k+5),...,\\ x_{-2k+2}\prod_{j=0}^{\frac{n-4}{2(k+1)}}\frac{(\frac{b}{a})^{(k+1)j+1}\theta_2+c}{(\frac{b}{a})^{(k+1)j+2}\theta_2+c}, & n=4,(2k+6),...,\\ \vdots & \\ x_{-1}\prod_{j=0}^{\frac{n-(2k+1)}{2(k+1)}}\frac{(\frac{b}{a})^{(k+1)j+k}\theta_1+c}{(\frac{b}{a})^{(k+1)j+k+1}\theta_1+c}, & n=2k+1,(4k+3),...,\\ x_0\prod_{j=0}^{\frac{n-2(k+1)}{2(k+1)}}\frac{(\frac{b}{a})^{(k+1)j+k}\theta_2+c}{(\frac{b}{a})^{(k+1)j+k+1}\theta_2+c}, & n=2(k+1),4(k+1),...,\end{cases}\tag{2.2}$$

where $\theta_i=\frac{a-b-c\alpha_i}{\alpha_i}$, $\alpha_i=\prod_{l=0}^{k}x_{-2(l+1)+i}$, $i=1,2$.

Proof. We can write the given solution (2.2) in the form

$$x_{2(k+1)m+2t+i}=x_{-2(k+1)+2t+i}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{(k+1)j+t}\theta_i+c}{(\frac{b}{a})^{(k+1)j+t+1}\theta_i+c},\quad m=-1,0,1,..., \tag{2.3}$$

where $0\le t\le k$ and $1\le i\le 2$.

We prove the theorem by induction on m.
When $m=0$, using Lemma (2.1) we get

$$\begin{aligned}x_{2t+i}&=\frac{ax_{-2(k+1)+2t+i}}{b+c\prod_{l=0}^{k}x_{-2(l+1)+2t+i}}=\frac{ax_{-2(k+1)+2t+i}}{b+c\frac{a-b}{(\frac{b}{a})^t\theta_i+c}}\\&=\frac{ax_{-2(k+1)+2t+i}((\frac{b}{a})^t\theta_i+c)}{b((\frac{b}{a})^t\theta_i+c)+c(a-b)}\\&=x_{-2(k+1)+2t+i}\frac{(\frac{b}{a})^t\theta_i+c}{(\frac{b}{a})^{t+1}\theta_i+c}.\end{aligned}$$

On the other hand, using Formula (2.3), when $m=0$ we get

$$x_{2t+i}=x_{-2(k+1)+2t+i}\frac{(\frac{b}{a})^t\theta_i+c}{(\frac{b}{a})^{t+1}\theta_i+c},$$

as expected.
Now suppose that Formula (2.3) is true for a given $m\in\mathbb{N}$. Then

$$\begin{aligned}x_{2(k+1)(m+1)+2t+i}&=\frac{ax_{2(k+1)m+2t+i}}{b+c\prod_{l=0}^{k}x_{2(k+1)m+2(k+t-l)+i}}\\
&=\frac{ax_{-2(k+1)+2t+i}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{(k+1)j+t}\theta_i+c}{(\frac{b}{a})^{(k+1)j+t+1}\theta_i+c}}{b+c\prod_{l=0}^{k}(x_{-2(l+1)+2t+i}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{(k+1)j+k+t-l}\theta_i+c}{(\frac{b}{a})^{(k+1)j+k+t-l+1}\theta_i+c})}\\
&=\frac{ax_{-2(k+1)+2t+i}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{(k+1)j+t}\theta_i+c}{(\frac{b}{a})^{(k+1)j+t+1}\theta_i+c}}{b+c\prod_{l=0}^{k}x_{-2(l+1)+2t+i}\prod_{j=0}^{m}\prod_{l=0}^{k}\frac{(\frac{b}{a})^{(k+1)j+k+t-l}\theta_i+c}{(\frac{b}{a})^{(k+1)j+k+t-l+1}\theta_i+c}}\\
&=\frac{ax_{-2(k+1)+2t+i}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{(k+1)j+t}\theta_i+c}{(\frac{b}{a})^{(k+1)j+t+1}\theta_i+c}}{b+c\prod_{l=0}^{k}x_{-2(l+1)+2t+i}\prod_{j=0}^{m}\frac{\prod_{l=0}^{k}(\frac{b}{a})^{(k+1)j+k+t-l}\theta_i+c}{\prod_{l=0}^{k}(\frac{b}{a})^{(k+1)j+k+t-l+1}\theta_i+c}}\\
&=\frac{ax_{-2(k+1)+2t+i}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{(k+1)j+t}\theta_i+c}{(\frac{b}{a})^{(k+1)j+t+1}\theta_i+c}}{b+c\prod_{l=0}^{k}x_{-2(l+1)+2t+i}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{(k+1)j+t}\theta_i+c}{(\frac{b}{a})^{(k+1)(j+1)+t}\theta_i+c}}\\
&=\frac{ax_{-2(k+1)+2t+i}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{(k+1)j+t}\theta_i+c}{(\frac{b}{a})^{(k+1)j+t+1}\theta_i+c}}{b+c(\prod_{l=0}^{k}x_{-2(l+1)+2t+i})(\frac{(\frac{b}{a})^t\theta_i+c}{(\frac{b}{a})^{(k+1)(m+1)+t}\theta_i+c})}\\
&=\frac{ax_{-2(k+1)+2t+i}((\frac{b}{a})^{(k+1)(m+1)+t}\theta_i+c)\prod_{j=0}^{m}\frac{(\frac{b}{a})^{(k+1)j+t}\theta_i+c)}{(\frac{b}{a})^{(k+1)j+t+1}\theta_i+c}}{b((\frac{b}{a})^{(k+1)(m+1)+t}\theta_i+c)+c(\prod_{l=0}^{k}x_{-2(l+1)+2t+i})((\frac{b}{a})^t\theta_i+c)}.\end{aligned}$$

But from Lemma (2.1) we have $\prod_{l=0}^{k}x_{-2(l+1)+2t+i}=\frac{a-b}{(\frac{b}{a})^t\theta_i+c}$.
Therefore,

$$\begin{aligned}&x_{2(k+1)(m+1)+2t+i}\\
&=\frac{ax_{-2(k+1)+2t+i}((\frac{b}{a})^{(k+1)(m+1)+t}\theta_i+c)\prod_{j=0}^{m}\frac{(\frac{b}{a})^{(k+1)j+t}\theta_i+c)}{(\frac{b}{a})^{(k+1)j+t+1}\theta_i+c}}{b((\frac{b}{a})^{(k+1)(m+1)+t}\theta_i+c)+c(\frac{a-b}{(\frac{b}{a})^t\theta_i+c})((\frac{b}{a})^t\theta_i+c)}\\
&=\frac{ax_{-2(k+1)+2t+i}((\frac{b}{a})^{(k+1)(m+1)+t}\theta_i+c)\prod_{j=0}^{m}\frac{(\frac{b}{a})^{(k+1)j+t}\theta_i+c)}{(\frac{b}{a})^{(k+1)j+t+1}\theta_i+c}}{b((\frac{b}{a})^{(k+1)(m+1)+t}\theta_i+c)+c(a-b)}\\
&=x_{-2(k+1)+2t+i}\frac{(\frac{b}{a})^{(k+1)(m+1)+t}\theta_i+c}{(\frac{b}{a})^{(k+1)(m+1)+t+1}\theta_i+c}\prod_{j=0}^{m}\frac{(\frac{b}{a})^{(k+1)j+t}\theta_i+c}{(\frac{b}{a})^{(k+1)j+t+1}\theta_i+c}\\
&=x_{-2(k+1)+2t+i}\prod_{j=0}^{m+1}\frac{(\frac{b}{a})^{(k+1)j+t}\theta_i+c}{(\frac{b}{a})^{(k+1)j+t+1}\theta_i+c}.\end{aligned}$$

This completes the proof. □

3. Global behavior of Equation (1.1)

In this section, we investigate the global behavior of Equation (1.1) with $a\neq b$, using the explicit formula of its solution.

We can write the Solution form (2.3) of Equation (1.1) as

$$x_{2(k+1)m+2t+i}=x_{-2(k+1)+2t+i}\prod_{j=0}^{m}\beta(j,t,i),\ m=-1,0,1,...,$$

where
$\beta(j,t,i)=\frac{(\frac{b}{a})^{(k+1)j+t}\theta_i+c}{(\frac{b}{a})^{(k+1)j+t+1}\theta_i+c}$, $0\le t\le k$ and $1\le i\le 2$.
We define the set $S=\{(t,i):0\le t\le k\quad\text{and}\quad 1\le i\le 2\}$.

Theorem 3.1. *Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a solution of Equation (1.1) such that for any $i\in\{1,2\}$, $\alpha_i\neq-\frac{b}{c\sum_{r=0}^{n}(\frac{a}{b})^r}$ for all $n\in\mathbb{N}$. If $\alpha_i=\frac{a-b}{c}$ for all $i\in\{1,2\}$, then $\{x_n\}_{n=-2k-1}^{\infty}$ is periodic with prime period $2k+2$.*

Proof. Assume that $\alpha_i=\frac{a-b}{c}$ for all $i\in\{1,2\}$. Then $\theta_i=0$ for all $i\in\{1,2\}$.
Therefore,

$$\begin{aligned}x_{2(k+1)m+2t+i}&=x_{-2(k+1)+2t+i}\prod_{j=0}^{m}\beta(j,t,i)\\&=x_{-2(k+1)+2t+i},\ m=-1,0,1,....\end{aligned}$$

This completes the proof. □

In the following theorem, suppose that $\alpha_i\neq\frac{a-b}{c}$ for all $i\in\{1,2\}$.

Theorem 3.2. *Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a solution of Equation (1.1) such that for any $1\le i\le 2$, $\alpha_i\neq-\frac{b}{c\sum_{r=0}^{n}(\frac{a}{b})^r}$ for all $n\in\mathbb{N}$. Then the following statements are true.*

1. *If $a<b$, then $\{x_n\}_{n=-2k-1}^{\infty}$ converges to 0.*
2. *If $a>b$, then $\{x_n\}_{n=-2k-1}^{\infty}$ converges to a period-$(2k+2)$ solution.*

Proof. 1. If $a < b$, then $\beta(j,t,i)$ converges to $\frac{a}{b} < 1$ as $j \to \infty$, for all $0 \le t \le k$ and $1 \le i \le 2$. So, for every pair $(t,i) \in S$ we have for a given $0 < \frac{a}{b} < \varepsilon < 1$ that, there exists $j_0(t,i) \in \mathbb{N}$ such that, $\beta(j,t,i) < \varepsilon$ for all $j \ge j_0(t,i)$. If we set $j_0 = \max_{(t,i)\in S} j_0(t,i)$, then for all $(t,i) \in S$ we get

$$\begin{aligned} \mid x_{2(k+1)m+2t+i} \mid &= \mid x_{-2(k+1)+2t+i} \mid \mid \prod_{j=0}^{m} \beta(j,t,i) \mid \\ &= \mid x_{-2(k+1)+2t+i} \mid \mid \prod_{j=0}^{j_0-1} \beta(j,t,i) \mid \mid \prod_{j=j_0}^{m} \beta(j,t,i) \mid \\ &< \mid x_{-2(k+1)+2t+i} \mid \mid \prod_{j=0}^{j_0-1} \beta(j,t,i) \mid \varepsilon^{m-j_0+1}. \end{aligned}$$

As m tends to infinity, the solution $\{x_n\}_{n=-2k-1}^{\infty}$ converges to 0.

2. If $a > b$, then $\beta(j,t,i) \to 1$ as $j \to \infty$, $0 \le t \le k$ and $1 \le i \le 2$. This implies that, for every pair $(t,i) \in S$, there exists $j_1(t,i) \in \mathbb{N}$ such that, $\beta(j,t,i) > 0$ for all $j \ge j_1(t,i)$. If we set $j_1 = \max_{(t,i)\in S} j_1(t,i)$, then for all $(t,i) \in S$ we get

$$\begin{aligned} x_{2(k+1)m+2t+i} &= x_{-2(k+1)+2t+i} \prod_{j=0}^{m} \beta(j,t,i) \\ &= x_{-2(k+1)+2t+i} \prod_{j=0}^{j_1-1} \beta(j,t,i) \exp\Big(\sum_{j=j_1}^{m} \ln(\beta(j,t,i))\Big). \end{aligned}$$

We shall test the convergence of the series $\sum_{j=j_1}^{\infty} \mid \ln(\beta(j,t,i)) \mid$. Since for all $0 \le t \le k$ and $1 \le i \le 2$ we have $\lim_{j\to\infty} \mid \frac{\ln(\beta(j+1,t,i))}{\ln(\beta(j,t,i))} \mid = \frac{0}{0}$, using L'Hospital's rule we obtain

$$\lim_{j\to\infty} \mid \frac{\ln\beta(j+1,t,i)}{\ln\beta(j,t,i)} \mid = (\frac{b}{a})^{k+1} < 1.$$

It follows from the ratio test that the series $\sum_{j=j_1}^{\infty} \mid \ln\beta(j,t,i) \mid$ is convergent. This ensures that there are $2k+2$ real numbers μ_{ti}, $0 \le t \le k$ and $1 \le i \le 2$ such that

$$\lim_{m\to\infty} x_{2(k+1)m+2t+i} = \mu_{ti}, \quad 0 \le t \le k \quad \text{and} \quad 1 \le i \le 2,$$

where

$$\mu_{ti} = x_{-2(k+1)+2t+i} \prod_{j=0}^{\infty} \frac{(\frac{b}{a})^{(k+1)j+t}\theta_i + c}{(\frac{b}{a})^{(k+1)j+t+1}\theta_i + c},$$
$$0 \le t \le k \quad \text{and} \quad 1 \le i \le 2.$$

□

Example (1) Figure (1) shows that if $a = 1.2$, $b = 2.5$, $c = 1$ $(a < b)$, then the solution $\{x_n\}_{n=-5}^{\infty}$ of Equation (1.1) with initial conditions $x_{-5} = -3$, $x_{-4} = 1.8$, $x_{-3} = 0.5$, $x_{-2} = -2.1$, $x_{-1} = 1.1$ and $x_0 = 1.4$ converges to zero.

Example (2) Figure (2) shows that if $a = 3$, $b = 1$, $c = 2.5$ $(a > b)$, then the solution $\{x_n\}_{n=-7}^{\infty}$ of Equation (1.1) with initial conditions $x_{-7} = -2.1$, $x_{-6} = 1.5$, $x_{-5} = 0.2$, $x_{-4} = -2.1$ $x_{-3} = -2.1$, $x_{-2} = -1.3$, $x_{-1} = 1.2$ and $x_0 = 1.8$ converges to a period-8 solution.

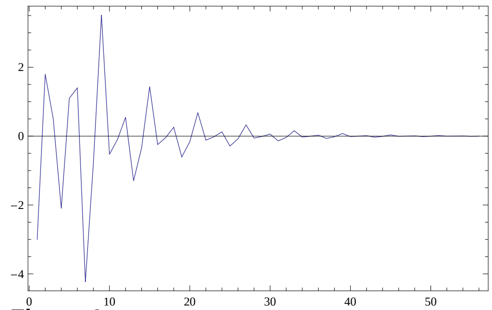

Figure 1.
$x_{n+1} = \frac{1.2x_{n-5}}{2.5+\prod_{l=0}^{2} x_{n-2l-1}}$

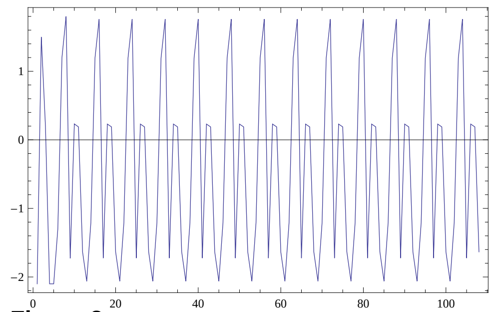

Figure 2.
$x_{n+1} = \frac{3x_{n-7}}{1+2.5\prod_{l=0}^{3} x_{n-2l-1}}$

Example (3) Figure (3) shows that if $a = 1$, $b = 1.5$, $c = 2$ $(\frac{a-b}{c} = -0.25 = \alpha_i,\ i = 1,2)$, then the solution $\{x_n\}_{n=-3}^{\infty}$ of Equation (1.1) with initial conditions $x_{-3} = -0.2$, $x_{-2} = -0.1$, $x_{-1} = 1.25$ and $x_0 = 2.5$ is a period-4 solution.

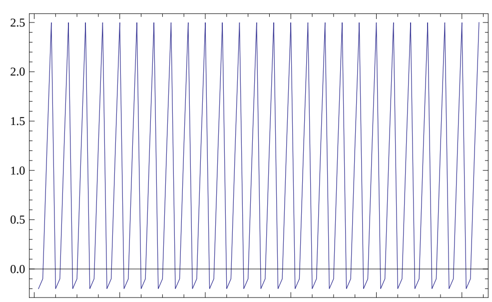

Figure 3. $x_{n+1} = \frac{x_{n-3}}{1.5+2x_{n-1}x_{n-3}}$

4. Special Cases

In this section, we introduce the solutions, show the existence of periodic solutions and discuss the global behavior of some special cases of Equation (1.1).

4.1 Case $a=b$

In this subsection, we study the equation

$$x_{n+1}=\frac{ax_{n-2k-1}}{a+c\prod_{l=0}^{k}x_{n-2l-1}},\quad n=0,1,\ldots,\tag{4.1}$$

where a,c are positive real numbers, the initial conditions $x_{-2k-1},x_{-2k},\ldots,x_{-1},x_0$ are real numbers and k is a nonnegative integer.

Theorem 4.1. *Let $x_{-2k-1},x_{-2k},\ldots,x_{-1}$ and x_0 be real numbers such that for any $i\in\{1,2\}$, $\alpha_i\neq-\frac{a}{c(n+1)}$ for all $n\in\mathbb{N}$. Then the solution $\{x_n\}_{n=-2k-1}^{\infty}$ of Equation (4.1) is*

$$x_n=\begin{cases} x_{-2k-1}\prod_{j=0}^{\frac{n-1}{2(k+1)}}\frac{a+c\alpha_1(k+1)j}{a+c\alpha_1((k+1)j+1)}, & n=1,(2k+3),\ldots,\\ x_{-2k}\prod_{j=0}^{\frac{n-2}{2(k+1)}}\frac{a+c\alpha_2(k+1)j}{a+c\alpha_2((k+1)j+1)}, & n=2,(2k+4),\ldots,\\ x_{-2k+1}\prod_{j=0}^{\frac{n-3}{2(k+1)}}\frac{a+c\alpha_1(k+1)j+1)}{a+c\alpha_1((k+1)j+2)}, & n=3,(2k+5),\ldots,\\ x_{-2k+2}\prod_{j=0}^{\frac{n-4}{2(k+1)}}\frac{a+c\alpha_2(k+1)j+1)}{a+c\alpha_2((k+1)j+2)}, & n=4,(2k+6),\ldots,\\ \vdots & \\ x_{-1}\prod_{j=0}^{\frac{n-(2k+1)}{2(k+1)}}\frac{a+c\alpha_1(k+1)j+k)}{a+c\alpha_1((k+1)j+k+1)}, & n=2k+1,(4k+3),\ldots,\\ x_0\prod_{j=0}^{\frac{n-2(k+1)}{2(k+1)}}\frac{a+c\alpha_2(k+1)j+k)}{a+c\alpha_2((k+1)j+k+1)}, & n=2(k+1),4(k+1),\ldots,\end{cases}\tag{4.2}$$

where $\alpha_i=\prod_{l=0}^{k}x_{-2(l+1)+i}$, $i=1,2$.

For simplicity, let $\gamma(j,t,i)=\frac{a+c\alpha_i((k+1)j+t)}{a+c\alpha_i((k+1)j+t+1)}$. Then we can write the Solution (4.2) as

$$x_{2(k+1)m+2t+i}=x_{-2(k+1)+2t+i}\prod_{j=0}^{m}\gamma(j,t,i),$$

where $\gamma(j,t,i)=\frac{a+c\alpha_i((k+1)j+t)}{a+c\alpha_i((k+1)j+t+1)}$, $0\le t\le k$ and $1\le i\le 2$.

Theorem 4.2. *Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a nontrivial solution of Equation (4.1) such that for any $i\in\{1,2\}$, $\alpha_i\neq-\frac{a}{c(n+1)}$ for all $n\in\mathbb{N}$. If $\alpha_i=0$ for all $i\in\{1,2\}$, then $\{x_n\}_{n=-2k-1}^{\infty}$ is periodic with prime period $2k+2$.*

Proof. Assume that $\alpha_i=0$ for all $i\in\{1,2\}$. Then $\gamma(j,t,i)=1$ for all $0\le t\le k$ and $1\le i\le 2$. Therefore,

$$\begin{aligned}x_{2(k+1)m+2t+i}&=x_{-2(k+1)+2t+i}\prod_{j=0}^{m}\gamma(j,t,i)\\&=x_{-2(k+1)+2t+i},\ m=-1,0,1,\ldots.\end{aligned}$$

This completes the proof. □

In the following theorem, suppose that $\alpha_i\neq 0$ for all $i\in\{1,2\}$.

Theorem 4.3. *Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a solution of Equation (4.1) such that for any $i\in\{1,2\}$, $\alpha_i\neq\frac{a}{c(n+1)}$ for all $n\in\mathbb{N}$. Then $\{x_n\}_{n=-2k-1}^{\infty}$ converges to 0.*

Proof. It is clear that $\gamma(j,t,i)\to 1$ as $j\to\infty$, $0\le t\le k$ and $1\le i\le 2$. This implies that, for every pair $(t,i)\in S$ there exists $j_2(t,i)\in\mathbb{N}$ such that, $\gamma(j,t,i)>0$ for all $j\ge j_2(t,i)$. If we set $j_2=\max_{(t,i)\in S}j_2(t,i)$, then for all $(t,i)\in S$ we get

$$\begin{aligned}x_{2(k+1)m+2t+i}&=x_{-2(k+1)+2t+i}\prod_{j=0}^{m}\gamma(j,t,i)\\&=x_{-2(k+1)+2t+i}\prod_{j=0}^{j_2-1}\gamma(j,t,i)\exp\Big(-\sum_{j=j_2}^{m}\ln\frac{1}{\gamma(j,t,i)}\Big).\end{aligned}$$

We shall show that

$$\sum_{j=j_2}^{\infty}\ln\frac{1}{\gamma(j,t,i)}=\sum_{j=j_2}^{\infty}\ln\frac{a+c\alpha_i((k+1)j+t+1)}{a+c\alpha_i((k+1)j+t)}=\infty$$

by considering the series $\sum_{j=j_2}^{\infty}\frac{c\alpha_i}{a+c\alpha_i((k+1)j+t)}$. As

$$\lim_{j\to\infty}\frac{\ln(1/\gamma(j,t,i))}{c\alpha_i/(a+c\alpha_i((k+1)j+t))}=1,$$

using the limit comparison test, we get $\sum_{j=j_2}^{\infty}\ln\frac{1}{\gamma(j,t,i)}=\infty$. Then

$$x_{2(k+1)m+2t+i}=x_{-2(k+1)+2t+i}\prod_{j=0}^{j_2-1}\gamma(j,t,i)\exp\Big(-\sum_{j=j_2}^{m}\ln\frac{1}{\gamma(j,t,i)}\Big)$$

converges to 0 as $m\to\infty$. Therefore, $\{x_n\}_{n=-2k-1}^{\infty}$ converges to 0. □

4.2 Case $a=b=c$

When $a=b=c$, Equation (1.1) reduced to the equation

$$x_{n+1}=\frac{x_{n-2k-1}}{1+\prod_{l=0}^{k}x_{n-2l-1}},\quad n=0,1,\ldots,\tag{4.3}$$

where the initial conditions $x_{-2k-1},x_{-2k},\ldots,x_{-1},x_0$ are real numbers and k is a nonnegative integer.

Theorem 4.4. *Let $x_{-2k-1},x_{-2k},\ldots,x_{-1}$ and x_0 be real numbers such that for any $i\in\{1,2\}$, $\alpha_i\neq-\frac{1}{n+1}$ for all $n\in\mathbb{N}$. Then the solution $\{x_n\}_{n=-2k-1}^{\infty}$ of Equation (4.3) is*

$$x_n=\begin{cases} x_{-2k-1}\prod_{j=0}^{\frac{n-1}{2(k+1)}}\frac{1+\alpha_1(k+1)j}{1+\alpha_1((k+1)j+1)}, & n=1,(2k+3),\ldots,\\ x_{-2k}\prod_{j=0}^{\frac{n-2}{2(k+1)}}\frac{1+\alpha_2(k+1)j}{1+\alpha_2((k+1)j+1)}, & n=2,(2k+4),\ldots,\\ x_{-2k+1}\prod_{j=0}^{\frac{n-3}{2(k+1)}}\frac{1+\alpha_1(k+1)j+1)}{1+\alpha_1((k+1)j+2)}, & n=3,(2k+5),\ldots,\\ x_{-2k+2}\prod_{j=0}^{\frac{n-4}{2(k+1)}}\frac{1+\alpha_2(k+1)j+1)}{1+\alpha_2((k+1)j+2)}, & n=4,(2k+6),\ldots,\\ \vdots & \\ x_{-1}\prod_{j=0}^{\frac{n-(2k+1)}{2(k+1)}}\frac{1+\alpha_1(k+1)j+k)}{1+\alpha_1((k+1)j+k+1)}, & n=2k+1,(4k+3),\ldots,\\ x_0\prod_{j=0}^{\frac{n-2(k+1)}{2(k+1)}}\frac{1+\alpha_2(k+1)j+k)}{1+\alpha_2((k+1)j+k+1)}, & n=2(k+1),4(k+1),\ldots,\end{cases}\tag{4.4}$$

where $\alpha_i=\prod_{l=0}^{k}x_{-2(l+1)+i}$, $i=1,2$.

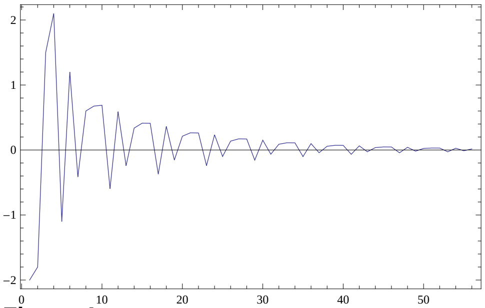

Figure 4.
$x_{n+1} = \frac{1.3x_{n-5}}{2+1.3\prod_{l=0}^{2} x_{n-2l-1}}$

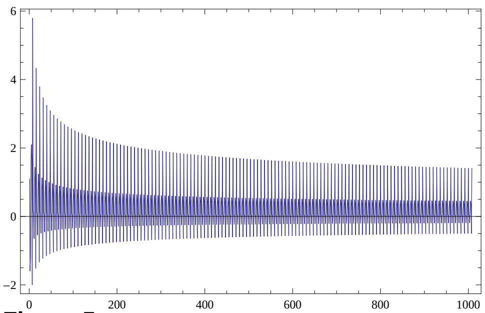

Figure 5.
$x_{n+1} = \frac{x_{n-7}}{1+\prod_{l=0}^{3} x_{n-2l-1}}$

Note: To study the global behavior of Equation (4.3), we note that Equation (4.3) is the same as Equation (4.1) with $a = c$. So the proof of the following two theorems will be omitted.

Theorem 4.5. *Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a nontrivial solution of Equation (4.3) such that for any $i \in \{1,2\}$, $\alpha_i \neq -\frac{1}{n+1}$ for all $n \in \mathbb{N}$. If $\alpha_i = 0$ for all $i \in \{1,2\}$, then $\{x_n\}_{n=-2k-1}^{\infty}$ is periodic with prime period $2k+2$.*

In the following theorem, suppose that $\alpha_i \neq 0$ for all $i \in \{1,2\}$.

Theorem 4.6. *Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a solution of Equation (4.3) such that for any $i \in \{1,2\}$, $\alpha_i \neq -\frac{1}{n+1}$ for all $n \in \mathbb{N}$. Then $\{x_n\}_{n=-2k-1}^{\infty}$ converges to 0.*

Example (4) Figure (4) shows that if $a = c = 1.3$, $b = 2$, then the solution $\{x_n\}_{n=-5}^{\infty}$ of Equation (4.3) with initial conditions $x_{-5} = -2$, $x_{-4} = -1.8$, $x_{-3} = 1.5$, $x_{-2} = 2.1$, $x_{-1} = -1.1$ and $x_0 = 1.2$ converges to zero.
Example (5) Figure (5) shows that if $a = b = c$, then the solution $\{x_n\}_{n=-7}^{\infty}$ of Equation (4.3) with initial conditions $x_{-7} = 1.1$, $x_{-6} = -1.6$, $x_{-5} = -1.2$, $x_{-4} = 1.5$ $x_{-3} = 2.1$, $x_{-2} = 1.7$, $x_{-1} = -2$ and $x_0 = 5.8$ converges to zero.

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