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New classes of fractional integral inequalities and some recent results on random variables

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Abstract
In this paper, some applications of Continuous Uniform and Beta probability distributions are developed. Then, by means of Chebyshev and Hölder inequalities, some new results on integral inequalities are established. Finally, some concepts on ω-weighted continuous random variables are further considered to derive some more results.

Keywords
Fractional integral inequalities, Riemann-Liouville integral, random variable, normalized fractional moment, normalized fractional ω-weighted expectation.

AMS Subject Classification
26A33, 26D10.

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Contents

1. Introduction

The integral inequalities are very important in the probability theory, physics and applied sciences. For some applications of this theory, we refer the reader [1, 3–5, 7, 10, 11, 14]. In particular, we can find applications for Chebyshev's and Hölder's inequalities, see [2, 8, 16, 17]. In this sense, let us recall the following results: we begin by [7, 8] where, Z. Dahmani has introduced some fractional notions with some applications on expectations, variances and moments of continuous random variables. Other applications have been discussed in the papers [2, 12, 13]. Then, in [11] the authors established new identities and lower bounds for expectations and also some classical results have been generalised for any $\alpha > 0$ reformulated by the following theorem:

Theorem 1.1. *[11] Let X be a continuous random variable with support an interval $[a,b]$, $-\infty < a < b < \infty$, and density function f. Let ω be a positive continuous function on $[a,b]$. Then, the following equality holds for any $\alpha \geq 1$:*

$$E_{zg',\alpha,\omega} = E_{gh,\alpha,\omega} - E_{h,\alpha,\omega}E_{g,\alpha,\omega},$$

where $g \in C^1([a,b])$, with $|E_{zg',\alpha,\omega}| < \infty$, $h(x)$ is a given function and

$$z(t) = \frac{1}{(b-t)^{\alpha-1}\omega(t)f(t)} \times \int_a^t (b-u)^{\alpha-1}\omega(u)f(u)(E_{h,\alpha,\omega} - h(u))du.$$

with the condition that: $J_a^\alpha \omega f(b) = 1$.

Based on the above theorem, they also established with the same conditions the following inequality:

$$\frac{E^2_{zg',\alpha,\omega}}{E_{zh',\alpha,\omega}} \leq E_{[g-E_{g,\alpha,\omega}]^2,\alpha,\omega}, \quad \alpha > 0.$$

The purpose of this work is to establish some new identities and inequalities using the normalized concepts on continuous random variables. This paper is divided into three sections. In Section 2, we recall some basic facts about integral fractional calculus; in section 3, we give some new applications of fractional calculus on probabilistic random variables, we apply the obtained results and some fractional inequalities to establish new lower bounds. Finally, some excellent results of [11], are developed for any $\alpha > 0$ without the following condition:

$$J_a^\alpha \omega f(b) = 1.$$

2. Preliminaries

In this section, we will give some definitions and preliminary facts that will be used through out this paper.

Definition 2.1. *[15] The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[a,b]$is defined as*

$$J_a^\alpha[f(t)] = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau,\ \alpha > 0\ ,\ a < t \leq b,$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u}u^{\alpha-1}du.$

For $\alpha > 0$, $\beta > 0$, we have:

$$J_a^\alpha J_a^\beta[f(t)] = J_a^{\alpha+\beta}[f(t)].$$

$$J_a^\alpha J_a^\beta[f(t)] = J_a^\beta J_a^\alpha[f(t)].$$

The Euler Beta function is connected with the Euler Gamma function by:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ x > 0,\ y > 0,$$

where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$.

Let us now recall the following definitions [18] :

Definition 2.2. *The normalized fractional expectation of order $\alpha > 0$, for a random variable X with a p.d.f. f defined on $[a,b]$ is given by:*

$$E_\alpha(X) = \frac{1}{N_1\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1} t f(t)dt,\ \alpha > 0\ ,\ a < t \leq b,$$

where $N_1 = J_a^\alpha[f(b)]$.

Definition 2.3. *The normalized fractional variance of order $\alpha > 0$ for a random variable X having a p.d.f. f on [a,b] is defined as*

$$V_\alpha(X) := \frac{1}{N_1\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}(t-E_\alpha(X))^2 f(t)dt,\ \alpha > 0.$$

Definition 2.4. *The normalized fractional moment of orders $r > 0$, $\alpha > 0$ for a continuous random variable X having a p.d.f. f defined on [a,b] is defined by*

$$E_\alpha(X^r) := \frac{1}{N_1\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1} t^r f(t)dt,\ \alpha > 0.$$

Based on the above definitions, we give the following properties:
Let $\alpha > 0$:

1* For any real number c, we have:

$$E_\alpha(c) = c.$$

2* The properties

$$E_\alpha(E_\alpha(X)) = E_\alpha(X),\quad Var_\alpha(X) = E_\alpha(X^2) - E_\alpha^2(X),$$

is also valid.

Let us now consider a positive continuous function ω defined on $[a,b]$. We recall the ω-concept:

Definition 2.5. *The normalized fractional ω-weighted expectation of order $\alpha > 0$, for a random variable X with a positive p.d.f. f defined on $[a,b]$ is defined as*

$$E_{X,\alpha,\omega}(b) := \frac{1}{N\Gamma(\alpha)}\int_a^b (b-\tau)^{\alpha-1}\tau\omega(\tau)f(\tau)d\tau,\ ,\ a < t \leq b,$$

where $N = J_a^\alpha[\omega f(b)]$.

3. Main Results

We begin this section by some applications:

3.1 Continuous Uniform Distribution

Let us take the continuous uniform distribution (CUD). So for any $x \in [a,b]$, we have $f(x) = (b-a)^{-1}$ which implies that:

$$J^\alpha f(b) = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}.$$

1 CUD Normalized Fractional Expectation:

Some easy calculations give:

$$E_\alpha(X) = \frac{b-a}{\alpha+1} + a.$$

By taking $\alpha = 1$ in the preceding result, then we obtain the classical expectation of X:

$$E_1(X) = \frac{b+a}{2} = E(X).$$

2 CUD Normalized Fractional Moment:

We have:

$$E_\alpha(X^2) = \frac{2(b-a)^2}{(\alpha+2)(\alpha+1)} + 2a\frac{b-a}{\alpha+1} + a^2.$$

If we take $\alpha = 1$ in the above formula, we get:

$$E_1(X^2) = \frac{a^2+b^2+ab}{3} = E(X^2).$$

3 CUD Normalized Fractional Variance:
Thanks to the properties (2*), we have:

$$Var_\alpha(X) := \sigma_\alpha^2(X) = \frac{\alpha(b-a)^2}{(\alpha+2)(\alpha+1)^2}.$$

If $\alpha = 1$, we obtain

$$\sigma_1^2(X) = \sigma^2(X) = \frac{(b-a)^2}{12}.$$

4 CUD Normalized Fractional Moment of order (r,α):
Particularly, where the $p.d.f$, of the uniform random X is defined on some positive real interval $[0,b]$, the fractional moment of X is given by:

$$E_\alpha(X^r) = \frac{\Gamma(\alpha+1)\Gamma(r+1)}{\Gamma(\alpha+r+1)} b^r,$$

for

$$N_1 = \frac{b^{\alpha-1}}{\Gamma(\alpha+1)}.$$

Notice that, if $\alpha = 1$, we obtain the classical moment of order r for the uniform distribution of X:

$$E_1(X^r) = \frac{\Gamma(r+1)}{\Gamma(r+2)} b^r = E(X^r).$$

3.2 Beta Distribution

Let consider now the Beta distribution (BD for short) which is defined, for any $x \in [0,1]$, by $f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$.
Using the preceding fractional definitions, we get:

1 BD Normalized Fractional Moment:

$$E_\alpha(X^r) = \frac{B(\alpha+b-1, a+r)}{B(\alpha+b-1, a)}.$$

2 BD Normalized Fractional Expectation:
Taking $r = 1$, in the above fractional moment formula, we obtain:

$$E_{X,\alpha} = \frac{a}{\alpha+b+a-1}.$$

Then if we take $\alpha = 1$ in the above formula, we get the classical expectation $E(X) = \frac{a}{a+b}$.

3 BD Fractional Variance:
Taking into account that $J^\alpha f(b) = \frac{B(\alpha+b-1,a)}{\Gamma(\alpha)B(a,b)}$, we obtain

$$Var_\alpha(X) = \frac{B(\alpha+b-1, a+2)}{B(\alpha+b-1, a)} - \frac{a^2}{(\alpha+b+a-1)^2}.$$

We remark also that, if we take $\alpha = 1$, we get:

$$Var_1(X) = \frac{ab}{(a+b+1)(a+b)^2} = Var(X).$$

3.3 New Estimates of BD Normalized Fractional Moments

By Chebyshev fractional integral inequality, we can prove the following result:

Proposition 3.1. *Let m,n,p and q be positive real numbers, such that $(p-m)(q-n) \le 0$. Then, for any $\alpha \ge 1$ we have:*

$$B(p,q+\alpha-1)B(m,n+\alpha-1) \ge B(p,n+\alpha-1)B(m,q+\alpha-1).$$

Proof. The proof of this result is based on *Definition* 2 as well as on the Beta distribution p.d.f, and the 1-weighted version of Chebyshev fractional inequality given by:

$$J^\alpha p(1) J^\alpha pfg(1) - J^\alpha pf(1) J^\alpha pg(1) \ge 0,$$

for any $x \in [0,1]$ we take:

$$f(x) = x^{p-m}, \quad g(x) = (1-x)^{q-n}, \quad p(x) = x^{m-1}(1-x)^{n-1}.$$

Then, we obtain the desired result. □

Remark 3.2. *If $\alpha = 1$, then the above proposition generalizes Theorem 3.1 of [17].*

Based on the paper [2], we prove the following theorems:

Theorem 3.3. *Let X,Y,U and V be four random variables, such that$X \sim B(p,q)$, $Y \sim B(m,n)$, $U \sim B(p,n)$ and $V \sim B(m,q)$. If $(p-m)(q-n) \le 0$, then for any $\alpha \ge 1$, the inequality :*

$$\frac{E_\alpha(X^r)E_\alpha(Y^r)}{E_\alpha(U^r)E_\alpha(V^r)} \ge \frac{B(p,\alpha+n-1)B(m,\alpha+q-1)}{B(p,\alpha+q-1)B(m,\alpha+n-1)},$$

is valid.

Proof. In the following 1-weighted version of Chebyshev fractional inequality (see [9]),

$$J^\alpha p(1) J^\alpha pfg(1) - J^\alpha pf(1) J^\alpha pg(1) \ge 0,$$

for any $x \in [0,1]$ we take:

$$f(x) = x^{p-m}, \quad g(x) = (1-x)^{q-n}, p(x) = x^{r+m-1}(1-x)^{n-1}.$$

Then, it yields that

$$\begin{aligned} & B(p,\alpha+q-1)B(m,\alpha+n-1)E_\alpha(X^r)E_\alpha(Y^r) \\ - \quad & B(p,\alpha+n-1)B(m,\alpha+q-1)E_\alpha(U^r)E_\alpha(V^r) \ge 0, \end{aligned}$$

provided that: $(p-m)(q-n) \le 0$.

□

We present to the reader the following theorem:

Theorem 3.4. *Let X_i, $i = 1,2,...,8$ be continuos random variables, such that $X1 \sim B(\sigma,\delta)$, $X2 \sim B(\lambda-\sigma, \rho-\delta)$, $X3 \sim B(\varphi,\psi)$, $X4 \sim B(\lambda-\varphi, \rho-\psi)$, $X5 \sim B(\lambda-\varphi,\delta)$,*

$X6 \sim B(\varphi,\rho-\delta)$, $X7 \sim B(\lambda-\sigma,\psi)$, $X8 \sim B(\sigma,\rho-\psi)$.
If $(\lambda-\sigma-\varphi)(\rho-\delta-\psi) \leq 0$, *then,*

$$\begin{aligned} & B(\sigma,\alpha+\delta-1)B(\lambda-\sigma,\alpha+\rho-\delta-1)E_\alpha(X_1^r)E_\alpha(X_2^r) \\ +& B(\varphi,\alpha+\psi-1)B(\lambda-\varphi,\alpha+\rho-\psi-1)E_\alpha(X_3^r).E_\alpha(X_4^r) \\ \geq & B(\lambda-\varphi,\alpha+\delta-1)B(\varphi,\alpha+\rho-\delta-1)E_\alpha(X_5^r)\, E_\alpha(X_6^r) \\ +& B(\lambda-\sigma,\alpha+\psi-1)B(\sigma,\alpha+\rho-\psi-1)E_\alpha(X_7^r)E_\alpha(X_8^r), \end{aligned}$$

where $\lambda,\rho,\sigma,\delta,\varphi,\psi > 0$, $\alpha \geq 1$ *and* $r \in \mathbb{N}\backslash\{0\}$.

Proof. Replacing the functions:

$$\begin{array}{ll} p(x)=x^{r+\sigma-1}(1-x)^{\delta-1} & ,q(x)=x^{r+\varphi-1}(1-x)^{\psi-1} \\ f(x)=x^{\lambda-\sigma-\varphi} & ,g(x)=(1-x)^{\rho-\delta-\psi} \end{array}$$

where $x \in [0,1]$. In the 2-weighted version of Chebyshev fractional inequality (see [9]), given by:

$$\begin{aligned} & J^\alpha p(1)J^\alpha qfg(1)+J^\alpha q(1)J^\alpha pfg(1) \\ \geq \; & J^\alpha pf(1)J^\alpha qg(1)+J^\alpha qf(1)J^\alpha pg(1), \end{aligned}$$

we obtain the desired result. □

Thanks to Hölder fractional integral inequality (see [6]), we present to the reader the following result.

Theorem 3.5. *Let* $(p,q),(m,n) \in [0,\infty)^2$ *and* $a,b \geq 0$, *with* $a+b=1$. *Let* $X \sim B(ap+bm,aq+bn)$ *and* $Y \sim B(p,q)$. *Then*

$$\frac{E_\alpha(X^{ar})}{[E_\alpha(Y^r)]^a} \leq \frac{[B(p,q+\alpha-1)]^a[B(m,n+\alpha-1)]^b}{B(ap+bm,aq+bn+\alpha-1)}, r \in \mathbb{N}\backslash\{0\}.$$

Proof. We choose the positive mappings f,g defined over $[0,1]$ as follows:

$$f(t)=t^{r+p-1}(1-t)^{q-1}, \quad g(t)=t^{m-1}(1-t)^{n-1}, t \in [0,1]$$

for $p=\frac{1}{a}$, $q=\frac{1}{b}$, $(\frac{1}{p}+\frac{1}{q}=1$ and $p \geq 1)$. Substituting these mappings in the Hölder's fractional inequality,

$$J^\alpha(fg)(1) \leq \left[J^\alpha f^{\frac{1}{a}}(1)\right]^a \left[J^\alpha g^{\frac{1}{b}}(1)\right]^b.$$

Then notice that:

$$\frac{1}{\Gamma(\alpha)} = \frac{1}{[\Gamma(\alpha)]^a} \times \frac{1}{[\Gamma(\alpha)]^b},$$

we obtain the result. □

Remark 3.6. *If we take* $\alpha=1$, *then the above theorem reduces to* Theorem 2.12 *of [13].*

3.4 Normalized fractional inequalities for continuous random variable

Theorem 3.7. *Let X be a continuous random variable with support an interval* $[a,b]$, $-\infty < a < b < \infty$ *and density function f. Let* ω *be a positive continuous function on* $[a,b]$. *Then, for any* $\alpha \geq 1$, *the following equality holds*

$$E_{zg',\alpha,\omega} = E_{gh,\alpha,\omega} - E_{h,\alpha,\omega}E_{g,\alpha,\omega}, \tag{3.1}$$

where $g \in C^1([a,b])$, *with* $|E_{zg',\alpha,\omega}| < \infty$, $h(x)$ *is a given function and*

$$\begin{aligned} z(t) &= \frac{1}{(b-t)^{\alpha-1}\omega(t)f(t)} \\ &\times \int_a^t (b-u)^{\alpha-1}\omega(u)f(u)(E_{h,\alpha,\omega}-h(u))du. \end{aligned}$$

Proof. By *Definition 5*, we write:

$$\begin{aligned} & E_{zg',\alpha,\omega} \\ =& \frac{1}{N\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}g'(t)\frac{1}{(b-t)^{\alpha-1}\omega(t)f(t)} \\ \times & \int_a^t \begin{pmatrix} (b-u)^{\alpha-1}\omega(u)f(u) \\ \times(E_{h,\alpha,\omega}-h(u)) \end{pmatrix} du \times \omega(t)f(t)dt. \\ =& \frac{1}{N\Gamma(\alpha)}\int_a^b g'(t)\int_a^t \begin{pmatrix} (b-u)^{\alpha-1}\omega(u)f(u) \\ \times(E_{h,\alpha,\omega}-h(u)) \end{pmatrix} dudt. \end{aligned}$$

Integration by part gives:

$$\begin{aligned} & E_{zg',\alpha,\omega} \\ =& \frac{1}{N\Gamma(\alpha)}\left\{g(t)\int_a^t \begin{pmatrix} (b-u)^{\alpha-1}\omega(u)f(u) \\ \times(E_{h,\alpha,\omega}-h(u)) \end{pmatrix} du|_{t=a}^{t=b}\right\} \\ -& \frac{1}{N\Gamma(\alpha)}\left\{\int_a^b \begin{pmatrix} g(t)(b-t)^{\alpha-1}\omega(t)f(t) \\ \times(E_{h,\alpha,\omega}-h(t)) \end{pmatrix} dt\right\} \\ =& \frac{1}{N\Gamma(\alpha)}\left\{g(b)\int_a^b \begin{pmatrix} (b-u)^{\alpha-1}\omega(u)f(u) \\ \times(E_{h,\alpha,\omega}-h(u)) \end{pmatrix} du\right\} \\ -& \frac{1}{N\Gamma(\alpha)}\left\{\int_a^b \begin{pmatrix} g(t)(b-t)^{\alpha-1}\omega(t)f(t) \\ \times(E_{h,\alpha,\omega}-h(t)) \end{pmatrix} dt\right\}. \end{aligned}$$

Therefore, since $N=J_a^\alpha[\omega f(b)]$, we get

$$\begin{aligned} & E_{zg',\alpha,\omega} \\ =& g(b)\left\{\begin{array}{c} \frac{E_{h,\alpha,\omega}}{N\Gamma(\alpha)}\int_a^b (b-u)^{\alpha-1}\omega(u)f(u)du \\ \\ -\frac{1}{N\Gamma(\alpha)}\int_a^b (b-u)^{\alpha-1}\omega(u)f(u)h(u)du \end{array}\right\} \\ -& \left\{\begin{array}{c} \frac{E_{h,\alpha,\omega}}{N\Gamma(\alpha)}\int_a^b g(t)(b-t)^{\alpha-1}\omega(t)f(t)dt \\ \\ -\frac{1}{N\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}\omega(t)f(t)h(t)du \end{array}\right\} \\ =& g(b)\left\{\frac{E_{h,\alpha,\omega}}{N}J_a^\alpha\omega f(b)-E_{h,\alpha,\omega}\right\} \\ & -E_{h,\alpha,\omega}E_{g,\alpha,\omega}+E_{gh,\alpha,\omega}. \end{aligned}$$

Hence the result,

$$E_{zg',\alpha,\omega} = E_{gh,\alpha,\omega} - E_{h,\alpha,\omega}E_{g,\alpha,\omega}.$$

□

Now, taking three continuous functions g,h,f defined on $[a,b]$, we prove the following result:

Theorem 3.8. *Let g,h and f be three continuous functions on $[a,b]$, and let $\omega:[a,b]\to\mathbb{R}^+$ be a continuous function, then*

$$J_a^\alpha\left[(g-\frac{1}{N}J_a^\alpha\omega gf(b))\times(h-\frac{1}{N}J_a^\alpha\omega hf(b))\omega f\right](b) = J_a^\alpha\omega ghf(b)-\frac{1}{N}\left[J_a^\alpha\omega gf(b)J_a^\alpha\omega hf(b)\right] \quad (3.2)$$

is valid, for any $\alpha\geq 1$, where $N=J_a^\alpha[\omega f(b)]$.

Proof. We have

$$\begin{aligned}
&J_a^\alpha\left[(g-\frac{1}{N}J_a^\alpha\omega gf(b))\times(h-\frac{1}{N}J_a^\alpha\omega hf(b))\omega f\right](b)\\
=\;& \frac{1}{\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}\begin{pmatrix}\left(g(t)-\frac{1}{N}J_a^\alpha\omega gf(b)\right)\\ \times\left(h(t)-\frac{1}{N}J_a^\alpha\omega hf(b)\right)\\ \times\omega(t)f(t)\end{pmatrix}dt\\
=\;& \frac{1}{\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}\omega(t)g(t)h(t)f(t)dt\\
&-\frac{J_a^\alpha\omega hf(b)}{N\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}\omega(t)g(t)f(t)dt\\
-\;& \frac{J_a^\alpha\omega gf(b)}{N\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}\omega(t)h(t)f(t)dt\\
+\;& \frac{J_a^\alpha\omega gf(b)J_a^\alpha\omega hf(b)}{N^2\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}\omega(t)f(t)dt\\
=\;& J_a^\alpha\omega ghf(b)-\frac{2}{N}J_a^\alpha\omega gf(b)J_a^\alpha\omega hf(b)\\
&+\frac{1}{N^2}J_a^\alpha\omega gf(b)J_a^\alpha\omega hf(b)J_a^\alpha\omega f(b).
\end{aligned}$$

where $N=J_a^\alpha[\omega f(b)]$. Hence the result,

$$\begin{aligned}
&J_a^\alpha\left[(g-\frac{1}{N}J_a^\alpha\omega gf(b))\times(h-\frac{1}{N}J_a^\alpha\omega hf(b))f\right](b)\\
=\;& J_a^\alpha\omega ghf(b)-\frac{1}{N}\left[J_a^\alpha\omega gf(b)J_a^\alpha\omega hf(b)\right].
\end{aligned}$$

□

Theorem 3.9. *Let X be a continuous random variable with support an interval $[a,b]$, $-\infty<a<b<\infty$, having a pdf f. Then, for any $\alpha\geq 1$, we have*

$$\frac{E^2_{|zg',\alpha,\omega|}}{E_{zh',\alpha,\omega}}\leq E_{[g-E_{g,\alpha,\omega}]^2,\alpha,\omega},\quad \alpha>0. \quad (3.3)$$

where $g\in C^1([a,b])$, with $|E_{zg',\alpha,\omega}|<\infty$, $h(x)$ is a given function and z is given by

$$\begin{aligned}
z(t) &= \frac{1}{(b-t)^{\alpha-1}\omega(t)f(t)}\\
&\times\int_a^t(b-u)^{\alpha-1}\omega(u)f(u)(E_{h,\alpha,\omega}-h(u))du.
\end{aligned}$$

Proof. Thanks to (3.1), we observe that

$$\begin{aligned}
E_{zg',\alpha,\omega} &= E_{gh,\alpha,\omega}-E_{h,\alpha,\omega}E_{g,\alpha,\omega}.\\
&= \frac{1}{N}J_a^\alpha\omega ghf(b)-\frac{1}{N^2}J_a^\alpha\omega hf(b)J_a^\alpha\omega gf(b).\\
&= \frac{1}{N}\left[J_a^\alpha\omega ghf(b)-\frac{1}{N}J_a^\alpha\omega hf(b)J_a^\alpha\omega gf(b)\right].
\end{aligned}$$

By using (3.2), we observe that

$$\begin{aligned}
&E_{zg',\alpha,\omega}\\
=\;& \frac{1}{N}\left(J_a^\alpha\begin{bmatrix}(g-\frac{1}{N}J_a^\alpha\omega gf(b))\\ \times(h-\frac{1}{N}J_a^\alpha\omega hf(b))\omega f\end{bmatrix}(b)\right)\\
=\;& \frac{1}{N\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}\times\omega(t)f(t)\times\\
&\left(g(t)-\frac{1}{N}J_a^\alpha\omega gf(b)\right)\left(h(t)-\frac{1}{N}J_a^\alpha\omega hf(b)\right)dt.
\end{aligned}$$

Thanks to Cauchy-Shwarz inequality, we obtain

$$\begin{aligned}
&E^2_{|zg',\alpha,\omega|} && (3.4)\\
\leq\;& \left(\frac{1}{N\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}\begin{bmatrix}(g(t)-\frac{1}{N}J_a^\alpha\omega gf(b))^2\\ \times\omega(t)f(t)\end{bmatrix}dt\right)\\
\times\;& \left(\frac{1}{N\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}\begin{bmatrix}(h(t)-\frac{1}{N}J_a^\alpha\omega hf(b))^2\\ \times\omega(t)f(t)\end{bmatrix}dt\right)\\
=\;& E_{\left(g-\frac{1}{N}J_a^\alpha\omega gf(b)\right)^2,\alpha,\omega}E_{\left(g-\frac{1}{N}J_a^\alpha\omega hf(b)\right)^2,\alpha,\omega}\\
=\;& E_{\left(g-E_{g,\alpha,\omega}\right)^2,\alpha,\omega}E_{\left(h-E_{h,\alpha,\omega}\right)^2,\alpha,\omega}. && (3.5)
\end{aligned}$$

Again, taking $g=h$ in (3.1), we get

$$\begin{aligned}
E_{zh',\alpha,\omega} &= E_{h^2,\alpha,\omega}-(E_{h,\alpha,\omega})^2\\
&= E_{h^2,\alpha,\omega}-2(E_{h,\alpha,\omega})^2+(E_{h,\alpha,\omega})^2\\
&= E_{h^2,\alpha,\omega}-2(E_{h,\alpha,\omega})^2+(E_{h,\alpha,\omega})^2\frac{J_a^\alpha\omega f(b)}{J_a^\alpha\omega f(b)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E_{zh',\alpha,\omega} &= \frac{1}{N}J_a^\alpha(h^2\omega f)(b)-\frac{2}{N}E_{h,\alpha,\omega}J_a^\alpha(\omega hf)(b)\\
&\quad+\frac{1}{N}(E_{h,\alpha,\omega})^2J_a^\alpha\omega f(b)\\
&= \frac{1}{N}\left[J_a^\alpha\left(h^2-2E_{h,\alpha,\omega}h+(E_{h,\alpha,\omega})^2\right)\omega f(b)\right]\\
&= \frac{1}{N}\left[J_a^\alpha\left(h-E_{h,\alpha,\omega}\right)^2\omega f\right](b)\\
&= E_{(h-E_{h,\alpha,\omega})^2,\alpha,\omega}. \quad (3.6)
\end{aligned}$$

Using (3.5) and (3.6), we deduce the result

$$\frac{E^2_{|zg',\alpha,\omega|}}{E_{zh',\alpha,\omega}}\leq E_{[g-E_{g,\alpha,\omega}]^2,\alpha,\omega},\ \alpha>0.$$

The proof of this theorem is complete. □

References

[1] R. P. Agawal, N. Elezovic, J. Pecaric, On some inequalities for Beta and Gamma functions via some classical inequalities. *J. Inequal. and Appl.*, 5(2005), 593–613.

[2] M. Bezziou, Z. Dahmani, *Random Inequalities via Riemann-Liouville fractional integration*, Under View, 2019.

[3] T. Caccoulos, V. Papathanasiou, On upper bounds for the variance of functions of random variables, *Stat. Proba. Lett.*, 3(1985), 175–184.

[4] T. Caccoulos, V. Papathanasiou, Caracterizations of distribution by variance bounds, *Stat. Proba. Lett.*, 7(1989), 351–356.

[5] T. Caccoulos, V. Papathanasiou, Characterizations of distributions by generalizations of variance bounds and simple proofs of the CLT, *Journal of Statistical Planningand and Inference*, 63(1997), 157–171.

[6] Z. Dahmani, tAbout some integral inequalities using Riemann-Liouville integrals, *General Mathematics*, 20(4)(2012), 63-69.

[7] Z. Dahmani, New applications of fractional calculus on probabilistic random variables, *Acta Math. Univ. ComenianaeVol. LXXXVI*, 2 (2017), 299–307.

[8] Z. Dahmani, New identities and lower bounds for CUD and Beta distributions, *Romai J.*, 15(1)(2019), 25–35.

[9] Z. Dahmani, New inequalities in fractional integrals, *International Journal of Nonlinear Sciences*, 9(4)(2010), 493–497.

[10] Z. Dahmani, A.E. Bouziane, M. Houas, M.Z. Sarikaya, New *w-weighted concepts for continuous random variables with applications*, *Note Di Math.,* 37(1)(2017), 23–40.

[11] M. Doubbi Bounoua, Z. Dahmani, Z. Bekkouche, Further results and applications on continuous random variables, *Malaya Journal of Matematik*, 7(3)(2019), 429–435.

[12] S.S. Dragomir, R.P. Agarwal, N.S. Barnett, Inequalities for Beta and Gamma functions via some classical and new integral inequalities, *J. of lnequal. and Appl.,* 5(2000), 103–165.

[13] S.S. Dragomir, P. Kumar, S.P. Singh, Mathematical inequalities with applications to the Beta and Gamma mappings, *Indian Jour. Math.,* 42(3)(2000), 1–19.

[14] F. Goodarzi, M. Amini, G.R. Mohtashami Borzadaran, On upper bounds for the variance of functions of randomvariables with weighted distributions, *Labachevskii Journal of Mathematics*, 37(4)(2016), 422–435.

[15] R. Gorenflo, F. Mainardi, *Fractional Calculus: Integral and Diferential Equations of Fractional Order*, Springer Verlag, Wien, (1997), 223–276.

[16] P. Kumar, Inequality involving moments of a continuous random variable defined over a finite interval, *Computers and Mathematics with Applications*, 48(2004), 257–273.

[17] P. Kumar, S.P. Singh, S.S. Dragomir, Some inequalities involving Beta and Gamma functions, *Nonlinear Analysis Forum*, 6(1)(2001), 143–150.

[18] I. Sliman, Z. Dahmani: *Normalized fractional inequalities for continous random variables*. Under View, 2019.

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