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3-Successive C-edge coloring of graphs

U. Aswathy[1] and Charles Dominic[2*]

Abstract

The 3-*successive c-edge coloring number* $\overline{\psi}'_{3s}(G)$ of a graph G is the highest number of colors that can occur in a coloring of the edges of G such that every path on three edges has at most two colors. In this paper, we obtain some exact values of 3-successive c-edge coloring number. Also, we attempt to find bounds of $\overline{\psi}'_{3s}(G)$ for different product of graphs which includes Cartesian, direct, strong, rooted and corona. The 3-successive c-edge achromatic sum is the maximum sum of colors among all the 3-successive c-edge coloring of G with highest number of colors. We also determine the 3-successive c-edge achromatic sum for some classes of graphs.

Keywords

3-successive c-edge coloring, 3-successive c-edge coloring number, 3-successive c-edge achromatic sum, 3-consecutive edge coloring number, anti ramsey number.

AMS Subject Classification

05C15.

[1,2] *CHRIST (Department of Mathematics, CHRIST(Deemed to be University)), Bangalore-560029, Karnataka, India.*

*__Corresponding author__: [2] charlesdominicpu@gmail.com; [1] aswathy.u@res.christuniversity.in

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1. Introduction

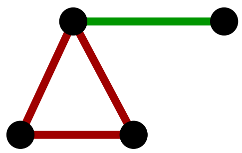
Three edges e_1, e_2 and e_3 in a graph $G = (V, E)$ are said to be successive if they form a path or a cycle of length 3. A coloring of the edges of G is called *3-consecutive edge coloring* if for any 3-successive edges e_1, e_2 and e_3, the edge e_2 receives the color of e_1 or e_3. The *3-consecutive edge coloring number* $\chi'_{3c}(G)$ of G is the highest number of colors conceded in such a coloring (see [3]).

A coloring of the edges of a graph G is called *3-successive c-edge coloring* if there exists no 3-colored path on three edges; that is, among every three successive edges there exist two having the same color. The 3-*successive c-edge coloring number* $\overline{\psi}'_{3s}(G)$ of G is the highest number of colors used in a 3- successive c-edge coloring.

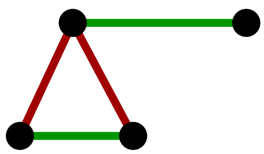
In an edge coloring a vertex v is called monochromatic if all edges incident to the vertex v have the same color (see [3]). The main difference between 3- successive c-edge coloring and 3-consecutive edge coloring is, in 3-consecutive edge coloring, for any arbitrary edge $e = uv$ either u or v will be monochromatic (see [3]), but this is not always true in 3-successive c- edge coloring, for example, see Figure 1. Clearly, any 3-consecutive edge coloring is a 3-successive c- edge coloring, and hence

$$\chi'_{3c}(G) \leq \overline{\psi}'_{3s}(G).$$

The anti-Ramsey number denoted by $ar(G_1, G_2)$ is defined as the highest number k, such that there exists an allocation of k colors to the edges of G_1, so that every copy of G_2 in G_1 has at least two edges with same color. The anti-Ramsey number $ar(G_1, P_3)$, where P_3 is the path of length 3 is similar to the definition of 3- successive c-edge coloring number (see [1]). S. Akhoondian et al. in [1] studied about the complexity of computing the anti-Ramsey number $ar(G_1, P_3)$.



3-consecutive edge coloring



3-successive c- edge coloring

Figure 1. 3-consecutive edge coloring and 3-successive c-edge coloring of a graph G

In this paper, we initiate a combinatorial study of 3-successive c-edge coloring number, and obtain some bounds besides finding the exact values of this parameter for some known graphs. For more definitions on graph theory, we refer the reader to the book [5].

2. Exact Values and Bounds

First, we obtain some preliminary results.

Proposition 2.1. *If $d(G) \geq 3$, then $\overline{\psi}'_{3s}(G) < m$, where $d(G)$ is the diameter and m is the number of edges in G.*

Proof. Assume $\overline{\psi}'_{3s}(G) = m$. Since $d(G) \geq 3$, there must be at least one set of 3-successive edges say e_1, e_2 and e_3 such that all of them have different colors, a contradiction. □

$\overline{\psi}'_{3s}$ can be straightforwardly determined for the following standard graphs :

- For P_n, a path graph of order $n \geq 2$, $\overline{\psi}'_{3s}(P_n) = \lceil n/2 \rceil$.
- For C_n, the cycle graph on $n \geq 4$ vertices, $\overline{\psi}'_{3s}(C_n) = \lfloor n/2 \rfloor$.
- If G is the complete graph on $n \geq 3$ vertices, then $\overline{\psi}'_{3s}(G) = 2$.
- For the complete bipartite graph $K_{m,n}$, $\overline{\psi}'_{3s}(K_{m,n}) = \max\{m, n\}$.
- If G is the Petersen graph, then $\overline{\psi}'_{3s}(G) = 3$.

The subsequent proposition characterizes simple graphs G for which $\overline{\psi}'_{3s}(G) = m$, where m is the number of edges in the graph G.

Proposition 2.2. *Let G be a simple graph with $m \geq 1$ edges. Then $\overline{\psi}'_{3s}(G) = m$ if and only if each component of G is a star graph $K_{1,n}$.*

Proof. We prove only the necessary part, as sufficiency is obvious. Assume $\overline{\psi}'_{3s}(G) = m$. If there exists a component of G which is not a star, then G contains 3-successive edges say e_1, e_2 and e_3 such that at least two of them have the same color, a contradiction. □

The following proposition characterizes connected graphs for which $\overline{\psi}'_{3s}(G) = 1$.

Proposition 2.3. *For any connected graph G, $\overline{\psi}'_{3s}(G) = 1$ if and only if G is K_2, the complete graph on two vertices.*

Proof. Let G be a connected graph which is not a K_2 and $\overline{\psi}'_{3s}(G) = 1$. Then there will be at least two edges e_1 and e_2. Color e_1 and all other edges except the edge e_2 with the color 1 and e_2 with the color 2. Clearly, this coloring yields a 3-successive c- edge coloring with two colors. This implies $\overline{\psi}'_{3s}(G) \geq 2$, a contradiction. □

Proposition 2.4. *If G is a connected graph with a cut vertex v, then $\overline{\psi}'_{3s}(G) \geq 2$.*

3. $\overline{\psi}'_{3s}(G)$ and the Diameter

It can be easily observed that $\overline{\psi}'_{3s}(G)$ is independent of the diameter of the graph G. In this section, we provide some graphs in which $diam(G) = 2$ and $\overline{\psi}'_{3s}(G) > 2$. Graph formed by connecting a universal vertex (a vertex which is adjacent to all other vertices of the graph G) to all vertices of the cycle graph C_n is called the Wheel graph and is denoted by W_{n+1}.

Proposition 3.1. *For the wheel graph W_{n+1},*

$$\overline{\psi}'_{3s}(W_{n+1}) = \begin{cases} \frac{n}{3}+1 & if\ n \equiv\ 0\ mod\ 3 \\ \frac{n-1}{3}+1 & if\ n \equiv\ 1\ mod\ 3 \\ \frac{n-2}{3}+1 & if\ n \equiv\ 2\ mod\ 3 \end{cases}$$

Proof. Let W_{n+1} denote the wheel graph. The 3-successive c-edge coloring number varies only at every third edge of the outer cycle C_n in the wheel graph W_{n+1}. If we color the edges incident to the universal vertex, say u, using 2 different colors, then the highest number of colors which can be used in the wheel graph W_{n+1} get restricted to 2. But the aim is to maximize the number of colors used. So, the approach begins by coloring all the edges incident to the universal vertex u by the same color, say 1.

Now, consider the outer cycle C_n. Variation of 3-successive c-edge coloring number depends on the number of vertices n. Thus we have 3 cases.

Case 1: When $n \equiv 0 \ mod \ 3$, a maximum of $\frac{n}{3}$ more distinct colors can be used in the outer cycle. The coloring sequence in this particular case is as follows, $2,1,1,3,1,1,4,\ldots,1,1$. Then, all the edges incident to the central vertex can be colored using the color 1. Hence, $\overline{\psi}'_{3s}(W_{n+1}) = \frac{n}{3} + 1 \ \ if \ n \equiv 0 \mod 3$.

Case 2: When $n \equiv 1 \mod 3$, a maximum of $\frac{n-1}{3}$ more distinct colors can be used to color the outer cycle C_n. The coloring sequence is $2,1,1,3,1,1,\ldots,1,1,1$. Then, including the extra color which is used to color the edges incident to the central vertex u, the $\overline{\psi}'_{3s}(W_{n+1}) = \frac{n-1}{3} + 1 \ \ if \ n \equiv 1 \ mod \ 3$.

Case 3: When $n \equiv 2 \mod 3$, $\frac{n-2}{3}$ more distinct colors can be used to color the outer cycle C_n. The coloring sequence would be $2,1,1,3,1,1,\ldots,1,1,1,1$. Hence,
$\overline{\psi}'_{3s}(W_{n+1}) = \frac{n-2}{3} + 1 \ \ if \ n \equiv 2 \ mod \ 3$. □

The Friendship graph F_n can be defined as the graph consisting of n-triangles attached with exactly one common vertex called the center(See[2]).

Proposition 3.2. *Let F_n denote the friendship graph on $2n+1$ vertices. Then, $\overline{\psi}'_{3s}(F_n) = n+1$.*

Proof. Friendship graph can be considered as a graph in which n triangles are attached to the central vertex, say u. If we color the edges incident to u using two colors, then the maximum number of colors that we can use to color the edges will be 2. So, the technique is to color the edges incident to u using the same color. We know that a triangle can be colored with the maximum of two colors. So, each edge which forms the base of the triangle can be colored using a different color. Hence, $\overline{\psi}'_{3s}(F_n) = n+1$. □

4. 3-Successive c-edge coloring number of product graphs

In this section, we find the 3-Successive c-edge coloring number of some product graphs. First we find the 3-successive c-edge coloring number of strong product of the path P_n with the complete graph K_2.

Proposition 4.1. *For the strong product of a path P_n with K_2, $P_n \boxtimes K_2$, the 3-successive c-edge coloring number $\overline{\psi}'_{3s}(P_n \boxtimes K_2)= \lfloor \frac{n}{2} \rfloor +1$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the first copy of path P_n and let $v_1{}', v_2{}', \ldots, v_n{}'$ denote the vertices of the second copy of P_n in the graph G=$P_n \boxtimes K_2$. Assume that the edge $v_1 v_1{}'$ is colored with the color 1. And let all the other edges adjacent to the edge $v_1 v_1{}'$ be colored with the color 2. Then, a new color 3 can appear only on the edge $v_3 v_3{}'$. If we use a new color in between these edges, it contradicts the 3-successive c-edge coloring of the graph G. Consequently, a further new color 4 appears on the edge $v_5 v_5{}'$. If any new color appear in between then, it forms a rainbow P_3. □

The m^{th} super triangle is an equilateral triangular grid on m vertices on each side (See[6]).

Proposition 4.2. *If m denote the number of layers in the m^{th} super triangle, then the 3-successive edge coloring $\overline{\psi}'_{3s}(G)$=$\lfloor \frac{m}{2} \rfloor$+1.*

Proof. Let G denote the m^{th} super triangle with m layers. Then, when m=1 and m=2, the result is trivial.
Assume that the result is true for m=k. Now, we have to prove that the result is true for m=k+1. The $k+1^{th}$ layer contain k+1 vertices more than the k^{th} layer. Since, the result is true for the k^{th} super triangle, the $k+1^{th}$ super triangle can be colored with at least $\lfloor \frac{k}{2} \rfloor$. We have to prove that the $k+1^{th}$ super triangle can be colored with at most k+1 colors. Let v be the vertex of degree 2 in the m^{th} super triangle.
In the k^{th} super triangle, every new color appears at the distance of 2. When k is even, the value of $\lfloor \frac{k}{2} \rfloor$ and $\lfloor \frac{k+1}{2} \rfloor$ are same. If its not then, it contradicts the 3-successive c-edge coloring .
When k is odd, the value of $\lfloor \frac{k+1}{2} \rfloor$ is one more than $\lfloor \frac{k}{2} \rfloor$. And hence another color appear at a distance two. That is, a new color appear at the $k+1^{th}$ layer of the super triangle. □

The complete graph $K_4 - \{e\}$ is known as the diamond graph. The Necklace graph is the graph with s diamonds, denoted as N_s, is a 3-regular graph that can be obtained from a $3s$-cycle graph by appending s extra vertices, with each of these extra vertices is adjacent to 3- sequential cycle vertices (See[4]).

Proposition 4.3. *Let N_s denote the n^{th} necklace with 4s vertices; where s denotes the number of diamonds. Then, the 3- successive c-edge coloring number $\overline{\psi}'_{3s}(N_s) \geq s+1$.*

Proof. Consider a set of sequential vertices from the 3s-cycle graph as u, v and w and let x be the exta vertex added in the 3s-cycle graph to form a diamond in the necklace graph Ns. Since there are s-diamonds in the necklace graph and we can color each xv type edges distinctly with s-colors and the remaining edges with one more extra color. Clearly this coloring produces a 3-successive c-edge coloring with $s+1$ colors therefore, the result follows. □

Let $P_n \square P_m$ be the Cartesian product of the path P_n with the path P_m.

Proposition 4.4. *Let G denote the graph $P_n \square P_m$ and $\alpha_0(G)$ denote the vertex covering number of G. Then, $\overline{\psi}'_{3s}(G) = \alpha_0(G) = \lfloor \frac{mn}{2} \rfloor$.*

Proof. Let $G_1, G_2, \ldots, G_m$ denote the m-copies of the path P_n in G. We attempt to prove the proposition using mathematical induction on m. For $n, m = 1$, the result is trivial since G is the path P_2.

Consider the case when $m = 2$ and consider the path P_n. Let S denote the minimum vertex cover of the graph $P_n \square P_2$. Let G' and G'' denote the 2 copies of the path P_n in $P_n \square P_2$. Mark the vertices of G' as $v_1, v_2, \ldots, v_n$ and the vertices of G'' as $v_1', v_2', \ldots, v_n'$. Then $S = \{v_1, v_2', v_3, \ldots, v_n\}$ when n is odd and $S = \{v_1, v_2', v_3, \ldots, v_n'\}$ when n is even. Color the edges of G in such a way that all the edges incident to v_1 receives the same color. Now, clearly, $N(v_1) \cap N(v_2') \neq \phi$. The edges incident to $v_1 \cup v_2'$ can be colored with at most two colors. That is, the edges incident to v_2' is colored with another color. Similarly, $N(v_1) \cap N(v_2') \neq \phi$ and $N(v_2') \cap N(v_3) \neq \phi$. Hence $v_1 \cup v_2' \cup v_3$ can be colored with at most three colors. Continuing like this, the maximum number of colors that can be used in 3- successive c-edge coloring of G is $\alpha_0(G)$. In this particular case, $\alpha_0(G) = n$, since S contains n vertices.

Assume that the result is true for $m = k$. We have to prove the result for $m = k+1$. Let $G_1, G_2, \ldots G_m$ denote the m copies of P_n. Let S denote the vertex cover of $G = P_n \square P_m$ and let $v_1, v_2, \ldots$ denote the vertices in S. Then, clearly for any $v_i \in S$, $N(v_i) \cap N(v_{i+1}) \neq \phi$. Thus, the edges incident to the vertices in $N(v_i) \cup N(v_{i+1})$ can be colored with at most two colors. The colors can be assigned to the edges of G in such a way that all the edges incident to a vertex $v_i \in S$ are given the same color. This assignment maximizes the number of colors used in the coloring of G and consequently $\overline{\psi}'_{3s}(G) = \alpha_0(G)$. This yields the 3-successive c-edge coloring of G. □

Definition 4.5. *Consider an even cycle C_n of order $n \geq 4$. Let $v_1, v_2, \ldots, v_n$ be the vertices of C_n. The graph C_n^k, (where $2 \leq k \leq n$), is obtained by taking k-copies of the cycle with vertices denoted by $v_1^1, v_2^1, \ldots, v_n^1$, $v_1^2, v_2^2, \ldots, v_n^2$, ..., $v_1^k, v_2^k, \ldots, v_n^k$, and concantenating the vertex $v^1_{\frac{n}{2}+1}$ with the vertex v_1^2, and again with this graph concantenating the vertex $v^2_{\frac{n}{2}+1}$ with the vertex v_1^3 and continuing similarly.*

For example see the graph C_6^3 in Figure-2.

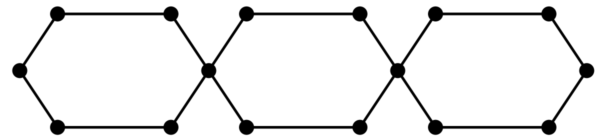


Figure 2. The graph C_6^3

Theorem 4.6. *Let C_n be an even cycle and let C_n^k be the graph obtained by taking k copies of C_n and concatenating one vertex in the first copy of C_n with the $n/2+1$ vertex of the second copy and continuing similarly. Then, $\overline{\psi}'_{3s}(C_n^k) \leq k \times \lfloor \frac{n}{2} \rfloor$.*

Proof. We have from proposition-2.1 that $\overline{\psi}'_{3s}(C_n) = \lfloor n/2 \rfloor$. Therefore, each cycle C_n in C_n^k can be colored with at most $n/2$ colors. Since we have k copies of C_n in C_n^k. Therefore, the upper bound follows. □

A vertex subset $I \subseteq V$ in which no two vertices are adjacent is called an independent set. The highest number of vertices in such a set is known as the vertex independence number of G and is denoted by $\beta_0(G)$.

Proposition 4.7. *For the prism graph $C_n \square P_2$, $\overline{\psi}'_{3s}(C_n \square P_2) = \begin{cases} n \; if \; n \; is \; even \\ 2\lfloor \frac{n}{2} \rfloor \; if \; n \; is \; odd \end{cases}$.*

Proof. Let G be the prism graph $C_n \square P_2$ with $2n$ vertices.

Case 1: Assume that n is even. Let C_1 and C_2 be the inner and outer cycles in G. We know that a cycle C_n on n vertices can be colored using $\lfloor \frac{n}{2} \rfloor$ colors. So the cycle C_1 has n vertices, hence can be colored using $\lfloor \frac{n}{2} \rfloor$ colors. Similarly, the outer cycle C_2 has n vertices and the edges can be colored using another $\lfloor \frac{n}{2} \rfloor$ colors. If any new colors are used in the edges joining the inner and outer cycles, it would form a three colored path, which is a contradiction to the definiton of 3-successive c-edge coloring. Hence

$$\overline{\psi}'_{3s}(C_n \square P_2) \leq n \tag{4.1}$$

Consider an independent set in G with maximum cardinality. That is consider a β_0-set. Let it be $v_1, v_2, \ldots, v_n$. Now color the edges incident at each $v_i, 1 \leq i \leq n$ by the color i. One can observe that this coloring produces a 3-successive c-edge coloring of G and hence

$$\overline{\psi}'_{3s}(C_n \square P_2) \geq n \tag{4.2}$$

From 4.1 and 4.2 the result follows.

Case 2: Assume that n is odd. Let C_1 and C_2 be the inner and outer cycles in G. We know that each cycle on n vertices can be colored using $\lfloor\frac{n}{2}\rfloor$ colors. So the cycle C_1 has n vertices, where n is odd. Hence can be colored using $\lfloor\frac{n}{2}\rfloor$ colors. Similarly, the outer cycle C_2 has n vertices and the edges can be colored using another $\lfloor\frac{n}{2}\rfloor$ colors. If any new colors are used in the edges joining the inner and outer cycles, it would form a three colored path, which is a contradiction. Hence, $C_n\Box P_2$ can be colored using at most $2\lfloor\frac{n}{2}\rfloor$ colors. Therefore,

$$\overline{\psi}'_{3s}(C_n\Box P_2) \leq 2\lfloor\frac{n}{2}\rfloor \tag{4.3}$$

Consider a β_0-set in G. Let it be $u_1, u_2, \ldots, u_{n-1}$. Now color the edges incident at each $u_i, 1 \leq i \leq n$ by the color i. Again consider the colorless edges in G and color these edges by the color 1. Now this coloring gives a 3-successive c-edge coloring of G and hence

$$\overline{\psi}'_{3s}(C_n\Box P_2) \geq 2\lfloor\frac{n}{2}\rfloor \tag{4.4}$$

From 4.3 and 4.4 the result follows. □

Let G_1 and G_2 be two graphs. Then the *corona product*, of G_1 and G_2 is defined in [7], as the graph G obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and by joining each vertex of the i-th copy of G_2 to the i-th vertex of G_1, where $1 \leq i \leq |V(G)|$ and is denoted as $G = G_1 \circ G_2$.

Proposition 4.8. *If G is the graph obtained by taking the corona product of the complete graph with the path P_m, $m \leq 4$, then $\overline{\psi}'_{3s}(G) = n+1$.*

Proof. Let G be the graph obtained by taking the corona product of the graph K_n with the path P_m; $m \leq 4$. We know that the any complete graph K_n can be colored with at most 2 colors in a 3-successive c-edge coloring. Also any path P_m; $m \leq 4$ can be colored with at most 2 colors in a 3-successive c-edge coloring.
For $m = 2$, color the complete graph K_n using exactly one color, say c_1. Every copy of P_2 corresponding to each vertex in K_n can be colored using one extra color, given that the edges joining each copy P_2 with corresponding vertex is given the color c_1. Corresponding to the n vertices of K_n, n different colors can be used for each copy of P_2 in G. This gives $\overline{\psi}'_{3s}(G) = n+1$.
Similar argument can be done for P_3 and P_4. □

The edge coloring of the graph G is the coloring of the edges of G in such way that no two adjacent edges receives the same color. The chromatic index, $\chi'(G)$ is the minimum number of colors required in such a coloring. The following proposition gives the relationship between $\chi'(G)$ and $\overline{\psi}'_{3s}(G^*)$, where G^* is the graph obtained by sub-dividing each edges of the given graph G exactly once.

Proposition 4.9. *Let G^* be the graph obtained by sub-dividing each edge of the given graph G exactly once. Then, $\Delta(G) \leq \chi'(G) \leq \overline{\psi}'_{3s}(G^*)$.*

Proof. Consider and edge coloring $C : E(G) \to N$ of G. Let $\chi'(G) = k$ and $1, 2, \ldots$ be the number of colors used in such a coloring. Now, consider G^* where each edge of G is sub-divided exactly once. Now, color the sub-divided edges by the same color we have used to color the corresponding edge in G. Clearly, this coloring yields a 3-successive c-edge coloring of G and hence the proof follows. □

Theorem 4.10. *Let G be any connected graph of order n, where, $n \geq 3$. Let $v_1, v_2, \ldots, v_n$ be the vertices of G and let H be any connected graph of order m, where $m \geq 2$. Let G^* be the graph obtained by taking n copies of H corresponding to each vertex of G, say $H_1, H_2 \ldots H_n$ and by adding a single edge between each vertex v_i of G and a vertex of H_i, $1 \leq i \leq n$. Then, $n+1 \leq \overline{\psi}'_{3s}(G^*)$.*

Proof. Color all the edges of G in G^* by the color, say 1. Let $h_1, h_2 \ldots h_n$ be the vertices of $H_1, H_2, \ldots, H_n$ which are adjacent to the vertices $v_1, v_2, \ldots, v_n$ of G in G^*. Now color the edges $v_1h_1, v_2h_2, \ldots, v_nh_n$ by the same color 1. Again consider the remaining edges of $H_1, H_2, \ldots H_n$ in G^*. Color all the edges in H_1 by the color 2, and of H_2 by the color 3 and so on. Continuing like this, we obtain a 3-successive c-edge coloring of G^* with $n+1$ colors. Therefore, $n+1 \leq \overline{\psi}'_{3s}(G^*)$. □

The above upper bound holds if G is the complete graph K_n of order $n \geq 3$ and H is the path P_2.

Fan graphs F_{mn} are graph obtained by taking the graph join $\bar{K}_m$,(the totally disconnected graph) on m vertices and the path P_n (the path graph) on n vertices.

Proposition 4.11. *Let $F_{1,n}$ denote the fan graph where n denotes the number of vertices in the path P_n. Then,*

$$\overline{\psi}'_{3s}(F_{1,n}) = \begin{cases} \frac{n}{3}+1 & if\ n \equiv\ 0\ mod\ 3 \\ \frac{n-1}{3}+1 & if\ n \equiv\ 1\ mod\ 3 \\ \frac{n-2}{3}+1 & if\ n \equiv\ 2\ mod\ 3 \end{cases}$$

Proof. Let u denote the universal vertex of $F_{1,n}$.

Observation: If we give two different colors to the edges incident at the universal vertex u, then it is not possible to use one new color to any edges in $F_{1,n}$. Hence, in this coloring the maximum number of colors to be used get restricted to 2. Therefore, to maximize the number of colors used, we color the edges incident to u with the same color, say 1.

Color all the edges incident to u by color, say 1. Now the vertices $v_1, \dots, v_n$ of the path P_n remains to be colored. Since all the vertices of the path P_n forms three successive edges with the edges incident to u, the coloring has to be done in such a way that the coloring sequence does not form three successive edges with the path P_n, that is, between any two different colors used, the intermediate two edges should receive the common color 1. Therefore, the path P_n should be colored in the coloring sequence $2,1,1,3,1,1,\dots$. This color sequence generates the 3-successive c- edge coloring of the fan graph $F_{1,n}$. □

The Pineapple graphs are graphs obtained by coalescing a vertex of the complete graph K_m with the star $K_{1,n}$. Therefore, the order of the Pineapple graph is $m+n$ and the size is $\frac{m^2-m+2n}{2}$

Proposition 4.12. *Let G be the pineapple graph K_m^n. Then* $\overline{\psi}'_{3s}(K_m^n) = n+1$.

Proof. The pineapple graph can be obtained by concatenating a vertex of the complete graph K_m with the star $K_{1,n}$. We know that a star $K_{1,n}$ can be colored with at most n color, since it does not contain any three successive edges. To color the pineapple graph K_m^n, we color the star attached to the vertex of K_m using n colors. We then color the complete graph K_m using the $n+1^{th}$ color. If we color in the alternate way of coloring the complete graph K_m using 2 colors, then we cannot use any new color to color the star, as it forms successive edges with all the edges of the complete graph K_m. This concludes that $\overline{\psi}'_{3s}(K_m^n) = n+1$. □

5. Graphs with 3-Successive C-edge coloring number 2

In this section, we provide some graphs for which 3-Successive c-edge coloring number 2.

Proposition 5.1. *The 3-successive c- edge coloring number of the join of C_n, $3 \le n \le 5$ with K_2,* $\overline{\psi}'_{3s}(C_n \vee K_2) = 2$.

Proof. Let u_1 and u_2 be the vertices of the graph K_2 in $C_n \vee K_2$ and let $v_1, v_2, \dots, v_n$ be the vertices of the cycle C_n in $C_n \vee K_2$. If we assign two distinct colors to the edges incident at u_1 and u_2, then it is not possible to use one new color to any edges in $C_n \vee K_2$. Therefore, to maximize the number of colors the edges incident at u_1 and u_2 must be colored with the same color, say 1. Again if we color the colorless edges in $C_n \vee K_2$ using two more colors we get three successive edges with three distinct colors, a contradiction to the definition. Hence $\overline{\psi}'_{3s}(C_n \vee K_2) = 2$. □

It is an open problem to charaterize graphs for which $\overline{\psi}'_{3s}(G) = 2$. We provide one more class of graphs for which $\overline{\psi}'_{3s}(G) = 2$

Proposition 5.2. *The 3-successive c-edge coloring number of join of the path graph P_n, $2 \le n \le 4$ with K_2,* $\overline{\psi}'_{3s}(P_n \vee K_2) = 2$.

Proof. Let u_1 and u_2 be the vertices of the graph K_2 in $P_n \vee K_2$ and let $v_1, v_2, \dots, v_n$ be the vertices of the cycle P_n in $P_n \vee K_2$. If we assign two distinct colors to the edges incident at u_1 and u_2 then it is not possible to use one new color to colr any colorless edges in $P_n \vee K_2$. Therefore, inorder to maximize the number of colors, the edges incident at u_1 and u_2 must be colored with the same color, say 1. Again if we color the colorless edges in $C_n \vee K_2$ using two more colors we get three successive edges with three distinct colors, a contradiction to the definition. Hence $\overline{\psi}'_{3s}(P_n \vee K_2) = 2$. □

6. 3-Successsice C-edge Achromatic Sum

Definition 6.1. *The 3-successive c-edge achromatic sum $\sum_{3sa}$ is the maximum sum among all the 3-successive c-edge coloring of G with maximum colors.*

Definition 6.2. *The 3-successive c-edge achromatic polynomial is the number of different ways of 3-successive c-edge coloring of a graph G with λ colors.*

For example, the path of length 3 the 3-successive c-edge achromatic sum is 5. The 3-successive c-edge achromatic polynomial is 3.

Proposition 6.3. *For the complete graph K_n of order $n \ge 2$, the 3-successive c- edge achromatic sum is n^2-n-2.*

Proof. Let G be the complete graph on n vertices. From the observations, we have $\overline{\psi}'_{3s}(G) = 2$. Color the edges of G in such a way that just an edge receives the color 1 and all the other edges receives the color 2. Then the 3-successive c-edge achromatic sum is consequently $n^2 - n - 2$. □

Proposition 6.4. *Let K_m^n denote the pineapple graph. Then the 3-successive c-edge achromatic sum*

$$\sum_{3sa}(K_m^n) = \frac{n^2+n+(m^2-m)(n+1)}{2}$$

.

Proof. We have discussed that the pineapple graph can be obtained by concatenating a star at any vertex of the complete graph K_m. We have mentioned the 3-successive c-edge coloring of pineapple graph earlier. But, to obtain the maximum 3-successive c-edge achromatic sum, we color the complete graph using the $n+1^{th}$ color. We know the K_m has $\frac{m(m-1)}{2}$ edges. So, coloring all the edges of the complete graph using $n+1^{th}$ color gives the achromatic sum of the complete graph K_m in K_m^n to be $\frac{m(m-1)(n+1)}{2}$. And color the edges in the star $K_{1,n}$ with the remaining n colors. The sum of the n colors in the star would add up to $\frac{n(n+1)}{2}$. So, the achromatic sum of $K_m^n = \frac{n^2+n+(m^2-m)(n+1)}{2}$. □

Proposition 6.5. *Let F_n denote the friendship graph on $2n+1$ vertices. Then, the 3-successive c-edge achromatic sum $\sum_{3sa} F_n = \frac{5n^2+5n}{2}$.*

Proof. The coloring of the friendship graph is mentioned above. Our objective is to maximize the achromatic sum. So, we color all the edges incident to the central vertex, say u, using the $n+1^{th}$ color. There would be $2n$ edges incident to the central vertex u. So that would sum up to $2n(n+1)$. Each edge which forms the base of the triangle is given a different color and hence it would add up to $\frac{n(n+1)}{2}$. So adding both and condensing we get $\frac{5n^2+5n}{2}$. □

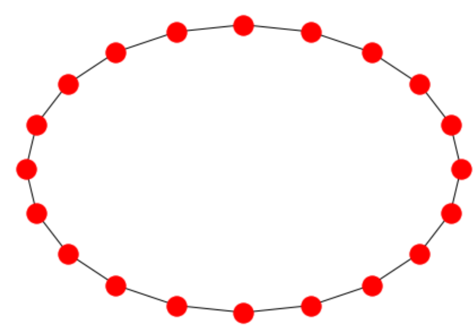
7. Python Programes to Compute 3-Successive c-edge Achromatic Sum

This section deals with some python programs to calculate the 3-successive c-edge achromatic sum of some standard graphs such as Cycles, Complete graphs and Pineapple graphs. Most of the modules used here are takes from the text books Doing maths with Python Amit Saha (see [8]) and Graph Theory Using python .

Program 7.1. Python Program to calculate the 3-successive c-edge achromatic sum of cycles with vertices more than six.

```
import networkx as gp
import matplotlib.pyplot as mplot
def even_v():
    evensum = (m**2 + 2*m)/4
    print('3 S C- achromatic sum =
    {0}'.format(evensum))
    G = gp.cycle_graph(m)
    gp.draw_circular(G)
    mplot.show()
def odd_v():
    oddsum =( m**2 + 2*m -3)/4
    print('3 S C- achromatic sum =
    {0}'.format(oddsum))
    G1 = gp.cycle_graph(m)
    gp.draw_circular(G1)
    mplot.show()
if __name__ == '__main__':
    while True:
        m = int(input('Number of vertices
        in the cycle C_n: '))
        if (m % 2) == 0:
                even_v()
        else:
                odd_v()
        answer = input('Do you want to exit?
        (y) for yes ')
        if answer == 'y':
              break
```

Output: Number of vertices in the cycle C_n: 20
3 S C- chromatic sum = 110.0



Do you want to exit? (y) for yes y

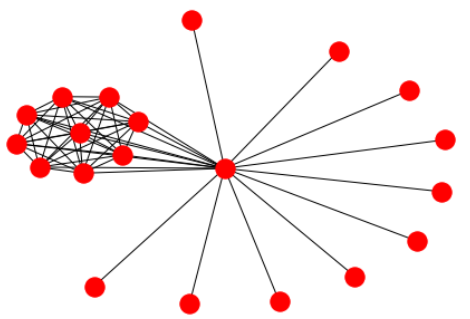
Program 7.2. Python Program to find the 3-successive c-edge achromatic sum of Pineapple graphs with atleast 10 vertices.

```
import networkx as gp
import matplotlib.pyplot as mplot
def pineapple():
    sum = (n*n+n+(m*m-m)*(n+1))/2
    print('3 S C- achromatic sum of this
    pineapple graph = {0}'.format(sum))
    a = gp.star_graph(n)
    b = gp.complete_graph(m)
    a= gp.relabel_nodes(a, { n: str(n)
    if n==0 else 'a-'+str(n) for n in
    a.nodes })
    b= gp.relabel_nodes(b, { n: str(n)
    if n==0 else 'b-'+str(n) for n in
    b.nodes })
    c = gp.compose(a,b)
    gp.draw(c)
    mplot.show()
if __name__ == '__main__':

    while True:
        m = int(input('Number of
        vertices in the complete graph
        K_m: '))
        n = int(input('Number of
```

```
        vertices in the star graph K1n:
        '))
        pineapple()
        answer = input('Do you want to
        exit? (y) for yes ')
        if answer == 'y':
                break
```

Output: Number of vertices in the complete graph K_m: 10
Number of vertices in the star graph K_1n: 10
3 S C- achromatic sum of this pineapple graph = 550.0



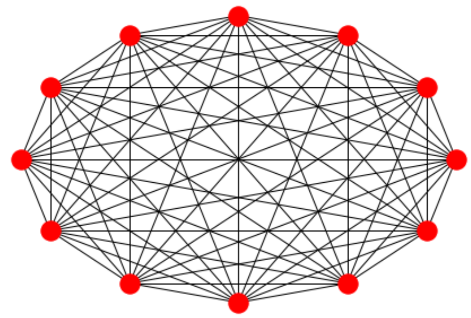
Do you want to exit? (y) for yes y

Program 7.3. Python program to calculate the 3-successive c-edge achromatic sum of complete graphs.

```
import networkx as gp
import matplotlib.pyplot as mplot
def vertices_v():
    evensum = m**2-m-1
    print('3 S C- Achromatic sum =
    {0}'.format(evensum))
    G = gp.complete_graph(m)
    gp.draw_circular(G)
    mplot.show()
if __name__ == '__main__':

    while True:
        m = int(input('Number of vertices in
        the complete graph K_n: '))
        vertices_v()
        answer = input('Do you want to exit?
        (y) for yes ')
        if answer == 'y':
                break
```

Output: Number of vertices in the complete graph K_n: 12
3 S C- Achromatic sum = 131



Do you want to exit? (y) for yes y

8. Conclusion and Open Problems

In this paper, we initiated the study of 3-successive c-edge Colorings of graphs. We found exact values of $\overline{\psi}'_{3s}$ for several classes of graphs. In section 3, we determined the 3-successive c-edge colorings of product graphs. It is an open problem to characterize the connected graphs for which $\overline{\psi}'_{3s}(G) = 2$. In section 4, we introduced the concept of 3-successive c-edge achromatic sum and found the 3-successive c-edge achromatic sum of certain classes of graphs. Section 5, deals with certain python programes to compute 3-successive c-edge achromatic sum of cycles, complete graphs and Pineapple graphs. It is again open to find out general formulas for 3-successive c-edge achromatic sum of product graphs such as Cartesian, Lexicographic etc.

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