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Similarity measures on intuitionistic fuzzy matrices and its applications

Sankar Das*

Abstract

In this paper, the notions of intuitionistic fuzzy matrix with some operations and relations are studied. Also, we develop several similarity measures of intuitionistic fuzzy matrices and some related properties are investigated. Finally, an application of similarity measures on intuitionistic fuzzy matrices is presented here.

Keywords

Intuitionistic fuzzy set, Intuitionistic fuzzy matrices, Degree of similarity, Similarity measures on intuitionistic fuzzy matrices.

AMS Subject Classification

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Department of Mathematics, Kharagpur College, Kharagpur-721305, India.

***Corresponding author**: sankarkgp22@gmail.com

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Contents

1. Introduction

Till now fuzzy set (FS) theory hurriedly moving into the medullary of mathematics due to its grandiose application in various fields. Zadeh [18] first introduced the notion of FS to manage uncertainty in practical situation. In 1983, Hashimoto [4] gives the concept of fuzzy matrix (FM). After a period of time, some limitations of FS gets revealed when the measure of non-membership arrives in different real life situations. In these cases, individual measure of membership and non-membership are required. Keeping this type of situation in consideration, to removing limitations, intuitionistic fuzzy set (IFS) was proposed by Atanassov [2]. Later on, a lot of works on intuitionistic fuzzy matrices (IFMs) were done by different researchers [6, 7, 10, 12, 14]. The role of similarity is widely analyse by Cross et al. [3]. They insist the fundamental preface of ability and similarity in hypothesis and in applications in inferential argument using concept of FS theory. Analysis of the similarity is an elementary task when applying IFMs.

This paper is structured into the following sections: In section 2, several basic definitions related to IFMs are provided. In section 3, the notion of several similarity measures between IFMs are presented. In section 4, study some ground properties of IFMs. In section 5, an application of IFMs is given. We conclude the work in section 6.

List of Abbreviations:
FS: Fuzzy set
IFS: Intuitionistic fuzzy set
IFV: Intuitionistic fuzzy value
FM: Fuzzy matrix
IFMs: Intuitionistic fuzzy matrices
SIFM: Square intuitionistic fuzzy matrix
SM: Similarity measure
DMS: Degree of membership
DNS: Degree of non-membership .

2. Preliminaries

To eliminate the limitation of FS, including the measure of non-membership, Atanassov introduced IFS in 1986.

Definition 2.1. [18] *A FS $\mathbb{A}_1$ is defined on an universe U as $\mathbb{A}_1 = \{(r, \mu_{\mathbb{A}_1}(r)) : r \in U\}$, where $\mu_{\mathbb{A}_1}(r) \in [0,1]$ is the DMS of $r \in \mathbb{A}_1$ with $0 \le \mu_{\mathbb{A}_1}(r) \le 1\ \forall r \in U$.*

Definition 2.2. [2] *A IFS $\mathbb{A}_2$ is defined on an universe U as $\mathbb{A}_2 = \{r, (\mu_{\mathbb{A}_2}(r), \nu_{\mathbb{A}_2}(r)) : r \in U\}$, where $\mu_{\mathbb{A}_2}(r), \nu_{\mathbb{A}_2}(r) \in [0,1]$ are the DMS and DNS of $r \in \mathbb{A}_2$, respectively with $0 \le \mu_{\mathbb{A}_2}(r) + \nu_{\mathbb{A}_2}(r) \le 1$ $\forall r \in U$. Also $\forall r \in U$, $D_{\mathbb{A}_2}(r) = 1 - (\mu_{\mathbb{A}_2}(r) + \nu_{\mathbb{A}_2}(r))$ represent denial degree of $r \in \mathbb{A}_2$. Here, $\mu_{\mathbb{A}_2}(r)$ and $\nu_{\mathbb{A}_2}(r)$ are independent.*

Definition 2.3. *Let q_1 and q_2 be two real numbers with $q_1 \in [0,1]$ and $q_2 \in [0,1]$ with $0 \le q_1 + q_2 \le 1$. Then $q = \langle q_1, q_2 \rangle$ is called IFV.*

Definition 2.4. [4] *Let $U = \{u_{11}, u_{12}, ..., u_{rq}\}$ be the universe. A FM $\tilde{\mathbb{A}}_1'$ of order $r \times q$ is defined as $\tilde{\mathbb{A}}_1' = [u_{ij}, \langle t_a(u_{ij}) \rangle]$, where $t_a(u_{ij}) \in [0,1]$ is the DMS of u_{ij} in $\tilde{\mathbb{A}}'$ for $i = 1,2,..,r$ and $j = 1,2,..,q$ with $0 \le t_a(u_{ij}) \le 1$ for all i, j.*

Let us define IFM generalizing the concept of FM.

Definition 2.5. [6] *Let $U = \{u_{11}, u_{12}, ..., u_{rq}\}$ be the universe. An IFM $\tilde{\mathbb{A}}_1$ of order $r \times q$ is defined as $\tilde{\mathbb{A}}_1 = [u_{ij}, \langle t_a(u_{ij}), 1 - f_a(u_{ij}) \rangle]$, where $t_a(u_{ij}) \in [0,1]$ and $1 - f_a(u_{ij}) \in [0,1]$ are respectively the DMS and DNS of u_{ij} in $\tilde{\mathbb{A}}_1$ for $i = 1,2,..,r$ and $j = 1,2,..,q$ with $0 \le t_a(u_{ij}) + 1 - f_a(u_{ij}) \le 1$ for all i, j. An IFM $\tilde{\mathbb{A}}_1$ is said to SIFM if the number of rows and columns are equal.*

Definition 2.6. [16] *An identity IFM I of order m is the SIFM of order m with all diagonal entries $\langle 1,0 \rangle$ and non-diagonal entries $\langle 0,1 \rangle$.*

Definition 2.7. [16] *A null IFM O of order m is the SIFM of order m with all entries $\langle 0,1 \rangle$. It is to be noted that $\langle 1,0 \rangle$ is the greatest IFV and $\langle 0,1 \rangle$ is the least IFV. So, to define identity IFM and null IFM, least IFV and greatest IFV are needed.*

Definition 2.8. *Let $\tilde{\mathbb{A}}_1 = [u_{ij}, \langle t_a(u_{ij}), 1 - f_a(u_{ij}) \rangle]$ be a SIFM. Then multiplication by a IFV $q = \langle q_1, q_2 \rangle$ is defined as $q\tilde{\mathbb{A}}_1 = (\langle q_1 \wedge t_a(u_{ij}), q_2 \vee (1 - f_a(u_{ij})) \rangle))$.*

Definition 2.9. *Let $\tilde{\mathbb{A}}_1 = [u_{ij}, \langle t_a(u_{ij}), 1 - f_a(u_{ij}) \rangle]$ be a SIFM. Also, let $\langle a, b \rangle$ be a IFV. For $\langle t_a(u_{ij}), 1 - f_a(u_{ij}) \rangle \ge \langle a, b \rangle$, it means that $t_a(u_{ij}) \ge a$ and $(1 - f_a(u_{ij})) \le b$. For $\langle t_a(u_{ij}), 1 - f_a(u_{ij}) \rangle \ngeq \langle a, b \rangle$, it means that two inequalities $t_a(u_{ij}) \ge a$ and $(1 - f_a(u_{ij})) \le b$ do not hold at a time.*

Definition 2.10. *Let $\tilde{\mathbb{A}}_1 = [u_{ij}, \langle t_a(u_{ij}), 1 - f_a(u_{ij}) \rangle]$ and $\tilde{\mathbb{B}}_1 = [u_{ij}, \langle t_b(u_{ij}), 1 - f_b(u_{ij}) \rangle]$ be two SIFMs of order m. Then $\tilde{\mathbb{A}}_1 = \tilde{\mathbb{B}}_1$ iff $t_a(u_{ij}) = t_b(u_{ij})$ and $1 - f_a(u_{ij}) = 1 - f_b(u_{ij})$ for $i, j = 1,2,..,m$.*

Definition 2.11. *Let $\tilde{\mathbb{A}}_1 = [u_{ij}, \langle t_a(u_{ij}), 1 - f_a(u_{ij}) \rangle]$ and $\tilde{\mathbb{B}}_1 = [u_{ij}, \langle t_b(u_{ij}), 1 - f_b(u_{ij}) \rangle]$ be two SIFMs of order m. Then $\tilde{\mathbb{A}}_1 \le \tilde{\mathbb{B}}_1$ iff $t_a(u_{ij}) \le t_b(u_{ij})$ and $1 - f_a(u_{ij}) \ge 1 - f_b(u_{ij})$ for $i, j = 1,2,..,m$.*

Definition 2.12. *Let $\tilde{\mathbb{A}}_1 = [u_{ij}, \langle t_a(u_{ij}), 1 - f_a(u_{ij}) \rangle]$ and $\tilde{\mathbb{B}}_1 = [u_{ij}, \langle t_b(u_{ij}), 1 - f_b(u_{ij}) \rangle]$ be two SIFMs of order m.Then $\tilde{\mathbb{A}}_1 \ge \tilde{\mathbb{B}}_1$ iff $\tilde{\mathbb{B}}_1 \le \tilde{\mathbb{A}}_1$.*

Now, we present some basic operations on SIFMs.

Definition 2.13. *Let $\mathbb{A}_1$ and $\mathbb{B}_1$ be two SIFMs of order m. Then basic operations are as follows.*
(i) $\tilde{\mathbb{A}}_1 \cap \tilde{\mathbb{B}}_1 = [\langle min(t_a, t_b), max(1 - f_a, 1 - f_b) \rangle]$
(ii) $\tilde{\mathbb{A}}_1 \cup \tilde{\mathbb{B}}_1 = [\langle max(t_a, t_b), min(1 - f_a, 1 - f_b) \rangle]$
(iii) $\tilde{\mathbb{A}}_1 \oplus \tilde{\mathbb{B}}_1 = [\langle t_a + t_b - t_a t_b, (1 - f_a)(1 - f_b) \rangle]$
(iv) $\tilde{\mathbb{A}}_1 \odot \tilde{\mathbb{B}}_1 = [\langle t_a t_b, (1 - f_a) + (1 - f_b) - (1 - f_a)(1 - f_b) \rangle]$
(v) $\tilde{\mathbb{A}}_1 @ \tilde{\mathbb{B}}_1 = [\langle \frac{t_a + t_b}{2}, \frac{1 - f_a + 1 - f_b}{2} \rangle]$
(vi) $\tilde{\mathbb{A}}_1 \$ \tilde{\mathbb{B}}_1 = [\langle \sqrt{t_a t_b}, \sqrt{(1 - f_a)(1 - f_b)} \rangle]$
(vii) $\Box \tilde{\mathbb{A}}_1 = [\langle t_a, (1 - t_a) \rangle]$
(viii) $\diamond \tilde{\mathbb{A}}_1 = [\langle f_a, (1 - f_a) \rangle]$
(ix) $\tilde{\mathbb{A}}_1 \# \tilde{\mathbb{B}}_1 = [\langle 2\frac{t_a t_b}{t_a + t_b}, 2\frac{(1 - f_a)(1 - f_b)}{(1 - f_a) + (1 - f_b)} \rangle]$
(x) $\bar{\tilde{\mathbb{A}}}_1 = [\langle 1 - f_a, t_a \rangle]$
(xi) The residue of the matrix $\tilde{\mathbb{A}}_1$ is $\pi_{\tilde{\mathbb{A}}_1} = f_a - t_a$.

3. Similarity Measures (SMs) between IFMs

In the following, we will give the concept of several SMs between IFMs.

Definition 3.1. *A mapping $\tilde{S} : \tilde{\mathbb{A}}_1 \times \tilde{\mathbb{B}}_1 \to [0,1]$ is said to be SM between two IFMs $\tilde{\mathbb{A}}_1$ and $\tilde{\mathbb{B}}_1$ denoted by $\tilde{S}(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1)$, if $\tilde{S}(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1)$ satisfies:*
(i) $0 \le \tilde{S}(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) \le 1$;
(ii) $\tilde{S}(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = 1$, *iff* $\tilde{\mathbb{A}}_1 = \tilde{\mathbb{B}}_1$;
(iii) $\tilde{S}(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = \tilde{S}(\tilde{\mathbb{B}}_1, \tilde{\mathbb{A}}_1)$;
(iv) if $\tilde{\mathbb{A}}_1 \le \tilde{\mathbb{B}}_1 \le \tilde{\mathbb{C}}_1$, *then* $\tilde{S}(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) \ge \tilde{S}(\tilde{\mathbb{A}}_1, \tilde{\mathbb{C}}_1)$ *and* $\tilde{S}(\tilde{\mathbb{B}}_1, \tilde{\mathbb{C}}_1) \ge \tilde{S}(\tilde{\mathbb{A}}_1, \tilde{\mathbb{C}}_1)$.

Definition 3.2. *Let $\tilde{\mathbb{A}}_1 = [u_{ij}, \langle t_a(u_{ij}), 1 - f_a(u_{ij}) \rangle]_{r \times q}$ and $\tilde{\mathbb{B}}_1 = [u_{ij}, \langle t_b(u_{ij}), 1 - f_b(u_{ij}) \rangle]_{r \times q}$ be two IFMs of order $r \times q$ with the condition $0 \le t_a(u_{ij}) + 1 - f_a(u_{ij}) \le 1$. Then the SM of $\tilde{\mathbb{A}}_1$ and $\tilde{\mathbb{B}}_1$ is*

$$\tilde{S}_d^p(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = 1 - \frac{1}{\sqrt[p]{r \times q}} \sqrt[p]{\sum_{i=1}^{r} \sum_{j=1}^{q} \left(|\phi_a(i,j) - \phi_b(i,j)| \right)^p}; \ 1 \le p < \infty,$$

where $\phi_a(i,j) = (t_a(u_{ij}) + 1 - f_a(u_{ij}))/2$ and $\phi_b(i,j) = (t_b(u_{ij}) + 1 - f_b(u_{ij}))/2$, for all $u_{ij} \in U$.

For $\tilde{S}_d^p$, note that $\phi_a(i,j)$ is the middle value of $[t_a(u_{ij}), 1 - f_a(u_{ij})]$. Then we consider the fact, if middle values of every subintervals between IFMs are same, the SM between two IFMs is 1. Again, if $\tilde{S}_d^p(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = 1$, then $\tilde{\mathbb{A}}_1$ and $\tilde{\mathbb{B}}_1$ are identical. Here, $\tilde{S}_d^p(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1)$ satisfy the properties:
(i) $0 \le \tilde{S}_d^p(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) \le 1$,
(ii) $\tilde{S}_d^p(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = \tilde{S}_d^p(\tilde{\mathbb{B}}_1, \tilde{\mathbb{A}}_1)$
(iii) $\tilde{S}_d^p(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = 1$ *iff* $\tilde{\mathbb{A}}_1 = \tilde{\mathbb{B}}_1$.

The SM between two IFMs $\tilde{\mathbb{A}}_1$ and $\tilde{\mathbb{B}}_1$ can also be given in the following form.

Definition 3.3. *The SM between IFMs $\tilde{\mathbb{A}}_1$ and $\tilde{\mathbb{B}}_1$ is*

$$\tilde{S}_e^p(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = 1 - \frac{1}{\sqrt[p]{r \times q}} \sqrt[p]{\sum_{i=1}^{r} \sum_{j=1}^{q} \left(\phi_{t_{ab}}(i,j) + \phi_{f_{ab}}(i,j) \right)^p}; \ 1 \le$$

$p < \infty$
where $\phi_{t_{ab}}(i,j) = \frac{|t_a(u_{ij})-t_b(u_{ij})|}{2}$ *and* $\phi_{f_{ab}}(i,j) = \frac{|f_b(u_{ij})-f_a(u_{ij})|}{2}$ *for all* $u_{ij} \in U$.

We also proposed another types of SMs between IFMs as follows:
Let $\tilde{\mathbb{A}}_1 = [u_{ij}, \langle t_a(u_{ij}), 1-f_a(u_{ij}) \rangle]_{r\times q}$ and $\tilde{\mathbb{B}}_1 = [u_{ij}, \langle t_b(u_{ij}), 1-f_b(u_{ij}) \rangle]_{r\times q}$ be two IFMs. For $\tilde{\mathbb{A}}_1 = [t_a(u_{ij}), 1-f_a(u_{ij})]$, the median value of the membership and non-membership values is $m_a(u_{ij}) = (t_a(u_{ij}) + 1 - f_a(u_{ij}))/2$. In that case, the interval is divided into the subintervals denoted as $[t_a(u_{ij}), m_a(u_{ij})]$ and $[m_a(u_{ij}), 1-f_a(u_{ij})]$. Again we also find the median values of these subintervals are denoted by m_{a1} and m_{a2} which are defined as $m_{a1}(u_{ij}) = (t_a(u_{ij}) + m_a(u_{ij}))/2$, $m_{a2}(u_{ij}) = (m_a(u_{ij}) - (1-f_a(u_{ij}))/2$. Similarly, for an IFM $\tilde{\mathbb{B}}_1$, $m_b(u_{ij}) = (t_b(u_{ij}) + 1 - f_b(u_{ij}))/2$, $m_{b1}(u_{ij}) = (t_b(u_{ij}) + m_b(u_{ij}))/2$, $m_{b2}\ (u_{ij}) = (m_b(u_{ij}) - (1-f_b(u_{ij}))/2$.

Definition 3.4. *Let* $\phi_{s1}(u_{ij}) = |m_{a1}(u_{ij}) - m_{b1}(u_{ij})|/2$ *and* $\phi_{s2}(u_{ij}) = |m_{a2}(u_{ij}) - m_{b2}(u_{ij})|/2$. *Then the SM is defined as* $\tilde{S}^p_s(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = 1 - \frac{1}{\sqrt[p]{r\times q}} \sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q} \big(\phi_{s1}(u_{ij}) + \phi_{s2}(u_{ij})\big)^p}$; $1 \le p < \infty$.
For all $\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1 \in IFMs$,
(i) $0 \le \tilde{S}^p_s(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) \le 1$
(ii) $\tilde{S}^p_s(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = 1$ *iff* $\tilde{\mathbb{A}}_1 = \tilde{\mathbb{B}}_1$, *where* $\omega_m \neq 0$
(iii) $\tilde{S}^p_s(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = \tilde{S}^p_s(\tilde{\mathbb{B}}_1, \tilde{\mathbb{A}}_1)$

Definition 3.5. *Let* $\phi_1(i,j) = \phi_{t_{ab}}(i,j) + \phi_{f_{ab}}(i,j)$, $\phi_2(i,j) = |\phi_a(i,j) + \phi_b(i,j)|$, $l_a(i,j) = ((1-f_a(u_{ij})) - t_a(u_{ij}))/2$ *and* $l_b(i,j) = ((1-f_b(u_{ij})) - t_b(u_{ij}))/2$. *The dissimilarity degree of the length is* $\phi_3 = max(l_a(i,j), l_b(i,j)) - min(l_a(i,j), l_b(i,j))$. *Then* $\tilde{S}^p_h(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = 1 - \frac{1}{\sqrt[p]{r\times q}} \sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q} \big(\sum_{m=1}^{3} \omega_m \phi_m(i,j)\big)^p}$ *where* $0 \le \omega_m \le 1$, $\sum_{m=1}^{3} \omega_m = 1$ *and* $1 \le p < \infty$.
For all $\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1 \in$*IFMs, we have*
(i) $0 \le S^p_h(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) \le 1$
(ii) $\tilde{S}^p_h(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = 1$ *iff* $\tilde{\mathbb{A}}_1 = \tilde{\mathbb{B}}_1$, *assuming* $\omega_m \neq 0$
(iii) $\tilde{S}^p_h(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = \tilde{S}^p_h(\tilde{\mathbb{B}}_1, \tilde{\mathbb{A}}_1)$.
The SM $\tilde{S}^p_h(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1)$ *contain more information on IFMs then the definition of (1), so* $\tilde{S}^p_h(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1)$ *gives better results about SM of IFMs. As a result, we can find more logical results by* $\tilde{S}^p_h(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1)$ *in same cases.*

Definition 3.6. *Let weight of* $u_{ij} \in U$ *is* ω_{ij}, *where* $\omega_{ij} \in [0,1]$, $i = 1,2,\dots,r$, $j = 1,2,\dots,q$ *and* $\sum_{i=1}^{r}\sum_{j=1}^{q} \omega_{ij} = 1$. *The SM between IFMs* $\tilde{\mathbb{A}}_1$ *and* $\tilde{\mathbb{B}}_1$ *can be obtained in the form*
$\tilde{S}^p_\omega(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = 1 - \frac{1}{\sqrt[p]{r\times q}} \sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q} \omega_{ij} \big(\sum_{m=1}^{3} \omega_m \phi_m(i,j)\big)^p}$
where $\sum_{i=1}^{r}\sum_{j=1}^{q} \omega_{ij} = 1$, $\sum_{m=1}^{3} \omega_m = 1$.
Likewise, for $\tilde{S}^p_\omega(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1)$, *the following propositions holds. For all* $\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1 \in IFMs$,
(i) $0 \le \tilde{S}^p_\omega(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) \le 1$
(ii) $\tilde{S}^p_\omega(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = 1$ *iff* $\tilde{\mathbb{A}}_1 = \tilde{\mathbb{B}}_1$, *where* $\omega_m \neq 0$
(iii) $\tilde{S}^p_\omega(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = \tilde{S}^p_\omega(\tilde{\mathbb{B}}_1, \tilde{\mathbb{A}}_1)$.

Definition 3.7. *Another definition of SM has been proposed here based on the normalizer absolute difference between the DMS and DNS values is*

$$\tilde{S}_1(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = \frac{\sum_{i=1}^{r}\sum_{j=1}^{q} |t_a(u_{ij}) - t_b(u_{ij})| + |f_b(u_{ij}) - f_a(u_{ij})|}{\sum_{i=1}^{r}\sum_{j=1}^{q} (t_a(u_{ij}) - t_b(u_{ij})) + (f_b(u_{ij}) - f_a(u_{ij}))}.$$

This SM satisfy the following properties
(i) $0 \le \tilde{S}_1(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) \le 1$
(ii) $\tilde{S}_1(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = 1$ *iff* $\tilde{\mathbb{A}}_1 = \tilde{\mathbb{B}}_1$,
(iii) $\tilde{S}_1(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) = \tilde{S}_1(\tilde{\mathbb{B}}_1, \tilde{\mathbb{A}}_1)$.

Proposition 3.8. *If* $\tilde{\mathbb{A}}_1$ *and* $\tilde{\mathbb{B}}_1$ *be two IFMs. Then* $\tilde{S}^p_d(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) \le \tilde{S}^p_e(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1)$; $1 \le p < \infty$.

Proof. Since, $|\phi_a(i,j) - \phi_b(i,j)| \le |t_a(u_{ij}) - t_b(u_{ij})|/2$
$+ |f_b(u_{ij}) - f_a(u_{ij})|/2$
$\le \phi_{t_{ab}}(i,j) + \phi_{f_{ab}}(i,j)$. Then
$\frac{1}{\sqrt[p]{r\times q}} \sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q} (|\phi_a(i,j) - \phi_b(i,j)|)^p}$
$\le \frac{1}{\sqrt[p]{r\times q}} \sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q} (\phi_{t_{ab}}(i,j) + \phi_{f_{ab}}(i,j))^p}$
or, $1 - \frac{1}{\sqrt[p]{r\times q}} \sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q} \big(|\phi_a(i,j) - \phi_b(i,j)|\big)^p}$
$\ge 1 - \frac{1}{\sqrt[p]{r\times q}} \sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q} (\phi_{t_{ab}}(i,j) + \phi_{f_{ab}}(i,j))^p}$
or, $\tilde{S}^p_d(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) \le \tilde{S}^p_e(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1)$; $1 \le p < \infty$. □

Proposition 3.9. *If* $\tilde{\mathbb{A}}_1$ *and* $\tilde{\mathbb{B}}_1$ *be two IFMs. Then* $\tilde{S}^p_s(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) \ge \tilde{S}^p_e(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1)$; $1 \le p < \infty$.

Proof. since, $\phi_{s1}(i,j) = |m_{a1}(u_{ij}) - m_{b1}(u_{ij})|$
$\le 3|(t_a(u_{ij}) - t_b(u_{ij}))|/8 + |f_b(u_{ij}) - f_a(u_{ij})|/8$
and $\phi_{s2}(i,j) = |m_{a2}(u_{ij}) - m_{b2}(u_{ij})|$
$\le |(t_a(u_{ij}) - t_b(u_{ij}))|/8 + 3|f_b(u_{ij}) - f_a(u_{ij})|/8$.
Then $\phi_{s1}(i,j) + \phi_{s2}(i,j) \le \phi_{t_{ab}}(i,j) + \phi_{f_{ab}}(i,j)$
$1 - \frac{1}{\sqrt[p]{r\times q}} \sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q} \big(|\phi_{s1}(i,j) + \phi_{s2}(i,j)|\big)^p}$
$\ge 1 - \frac{1}{\sqrt[p]{r\times q}} \sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q} \big(\phi_{t_{ab}}(i,j) + \phi_{f_{ab}}(i,j)\big)^p}$
or, $\tilde{S}^p_s(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) \ge \tilde{S}^p_e(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1)$; $1 \le p < \infty$. □

Proposition 3.10. *If* $\tilde{\mathbb{A}}_1$ *and* $\tilde{\mathbb{B}}_1$ *be two IFMs. Then* $\tilde{S}^p_h(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1) \ge \tilde{S}^p_e(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1)$; $1 \le p < \infty$.

Proof. Since,

$$\sum_{m=1}^{3}\omega_m\phi(i,j) = \omega_1\phi_1(i,j)+\omega_2\phi_2(i,j)+\omega_3\phi_3(i,j)$$
$$\leq \omega_1(\phi_{t_{ab}}(i,j)+\phi_{f_{ab}}(i,j))+\omega_2(\phi_{t_{ab}}(i,j)+\phi_{f_{ab}}(i,j))+(1-\omega_1-\omega_2)(\phi_{t_{ab}}(i,j)+\phi_{f_{ab}}(i,j))$$
$$= \phi_{t_{ab}}(i,j)+\phi_{f_{ab}}(i,j)$$

or, $\sum_{i=1}^{r}\sum_{j=1}^{q}\big(\sum_{m=1}^{3}\omega_m\phi(i,j)\big)^p \leq \sum_{i=1}^{r}\sum_{j=1}^{q}\big(\phi_{t_{ab}}(i,j)+\phi_{f_{ab}}(i,j)\big)^p$
or,

$$1-\sum_{i=1}^{r}\sum_{j=1}^{q}\big(\sum_{m=1}^{3}\omega_m\phi(i,j)\big)^p \geq 1-\sum_{i=1}^{r}\sum_{j=1}^{q}(\phi_{t_{ab}}(i,j)+\phi_{f_{ab}}(i,j))^p$$

or, $\tilde{S}_h^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1) \geq \tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1); 1\leq p<\infty$. □

Proposition 3.11. *If $\tilde{\mathbb{A}}_1$ and $\tilde{\mathbb{B}}_1$ be two IFMs. Then $\tilde{S}_s^p(\tilde{\mathbb{A}}_1, \tilde{\mathbb{B}}_1)=1$ iff $\tilde{\mathbb{A}}_1=\tilde{\mathbb{B}}_1; 1\leq p<\infty$.*

Proof. Since, $\tilde{S}_s^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1)=1$, we have from the definition (3.4).
$m_{a1}(u_{ij})=m_{b1}(u_{ij})$ and $m_{a2}(u_{ij})=m_{b2}(u_{ij})$
$3t_a(u_{ij})+(1-f_a(u_{ij}))=3t_b(u_{ij})+(1-f_b(u_{ij}))$
or, $3(t_a(u_{ij})-t_b(u_{ij}))+3((1-f_a(u_{ij}))-(1-f_b(u_{ij})))=0$
or, $t_a(u_{ij})-t_b(u_{ij})=0$ and $(1-f_a(u_{ij}))-(1-f_b(u_{ij}))=0$
or, $t_a(u_{ij})=t_b(u_{ij})$ and $(1-f_a(u_{ij}))=(1-f_b(u_{ij}))$ Therefore, $\tilde{\mathbb{A}}_1=\tilde{\mathbb{B}}_1$. □

4. Some properties of IFMs based on SM

Here, we study some ground properties of IFMs through the following propositions.

Proposition 4.1. *For two IFMs $\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1$ of order $r\times q$.*
(i) $0\leq \tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1)\leq 1$
(ii) $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1)=1$ iff $\tilde{\mathbb{A}}_1=\tilde{\mathbb{B}}_1$
(iii) $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1)=\tilde{S}_e^p(\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1)$
(iv) $\tilde{S}_e^p(\bar{\tilde{\mathbb{A}}}_1,\bar{\tilde{\mathbb{B}}}_1)=\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1)$

Proof. (**i**) Since IFMs $\tilde{\mathbb{A}}_1$ and $\tilde{\mathbb{B}}_1$ must be satisfy the conditions $0\leq t_a(u_{ij})+(1-f_a(u_{ij}))\leq 1$ and $0\leq t_b(u_{ij})+(1-f_b(u_{ij}))\leq 1$. Then from the definition $0\leq \tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1)\leq 1$.
(**ii**) If $\tilde{\mathbb{A}}_1=\tilde{\mathbb{B}}_1$,

$$\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1)=1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}\big(\phi_{t_{ab}}(i,j)+\phi_{f_{ab}}(i,j)\big)^p}=1.$$

Conversely, if $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1)=1$
or,

$$1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}\Big(\frac{|t_a(u_{ij})-t_b(u_{ij})|}{2}+\frac{|f_b(u_{ij})-f_a(u_{ij})|}{2}\Big)^p};$$
$1\leq p<\infty$

or, $\sum_{i=1}^{r}\sum_{j=1}^{q}\Big(\frac{|t_a(u_{ij})-t_b(u_{ij})|}{2}+\frac{|f_b(u_{ij})-f_a(u_{ij})|}{2}\Big)^p$
or, $\frac{|t_a(u_{ij})-t_b(u_{ij})|}{2}=0$ and $\frac{|f_b(u_{ij})-f_a(u_{ij})|}{2}=0$
or, $t_a(u_{ij})=t_b(u_{ij})$ and $f_a(u_{ij})=f_b(u_{ij})$
or, $\tilde{\mathbb{A}}_1=\tilde{\mathbb{B}}_1$.
Therefore, $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1)=1$ iff $\tilde{\mathbb{A}}_1=\tilde{\mathbb{B}}_1$.
(**iii**) $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1)=$

$$1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}\Big(\frac{|t_a(u_{ij})-t_b(u_{ij})|}{2}+\frac{|f_b(u_{ij})-f_a(u_{ij})|}{2}\Big)^p}$$
$$=1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}\Big(\frac{|t_b(u_{ij})-t_a(u_{ij})|}{2}+\frac{|f_b(u_{ij})-f_a(u_{ij})|}{2}\Big)^p}$$
$$=\tilde{S}_e^p(\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1).$$

(**iv**) Let $\tilde{\mathbb{A}}_1=[u_{ij},\langle t_a(u_{ij}),\ 1-f_a(u_{ij})\rangle]_{r\times q}$ and $\tilde{\mathbb{B}}_1=[u_{ij},\langle t_b(u_{ij}),\ 1-f_b(u_{ij})\rangle]_{r\times q}$ and

Let $\bar{\tilde{\mathbb{A}}}_1=[u_{ij},\langle 1-f_a(u_{ij}),\ t_a(u_{ij})\rangle]_{r\times q}$ and $\bar{\tilde{\mathbb{B}}}_1=[u_{ij},\langle 1-f_b(u_{ij}),t_b(u_{ij})\rangle]_{r\times q}$.

Then $\tilde{S}_e^p(\bar{\tilde{\mathbb{A}}}_1,\bar{\tilde{\mathbb{B}}}_1)$

$$=1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}\Big(\frac{|f_b-f_a|}{2}+\frac{|t_a-t_b|}{2}\Big)^p}=\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\ \tilde{\mathbb{B}}_1).$$

□

Proposition 4.2. *Let $\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1$ and $\tilde{\mathbb{C}}_1$ be three IFMs of same order. Then*
(i) If $\tilde{\mathbb{A}}_1\leq\tilde{\mathbb{B}}_1\leq\tilde{\mathbb{C}}_1$, then $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{C}}_1)\leq\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1)$ and $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{C}}_1)\leq\tilde{S}_e^p(\tilde{\mathbb{B}}_1,\tilde{\mathbb{C}}_1)$
(ii) $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1\cap\tilde{\mathbb{C}}_1)\leq\tilde{S}_e^p(\tilde{\mathbb{A}}_1\cap\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1\cap\tilde{\mathbb{C}}_1)$
(iii) $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1\cup\tilde{\mathbb{C}}_1)\leq\tilde{S}_e^p(\tilde{\mathbb{A}}_1\cup\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1\cup\tilde{\mathbb{C}}_1)$
(iv) $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1+\tilde{\mathbb{C}}_1)\leq\tilde{S}_e^p(\tilde{\mathbb{A}}_1+\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1+\tilde{\mathbb{C}}_1)$
(v) $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1.\tilde{\mathbb{C}}_1)\leq\tilde{S}_e^p(\tilde{\mathbb{A}}_1.\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1.\tilde{\mathbb{C}}_1)$
(vi) $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1@\tilde{\mathbb{C}}_1)\leq\tilde{S}_e^p(\tilde{\mathbb{A}}_1@\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1@\tilde{\mathbb{C}}_1)$
(vii) $\tilde{S}_e^p(\tilde{\mathbb{A}}_1\cap\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1\cup\tilde{\mathbb{C}}_1)=\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1)$
(viii) $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1\$\tilde{\mathbb{C}}_1)\leq\tilde{S}_e^p(\tilde{\mathbb{A}}_1\$\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1\$\tilde{\mathbb{C}}_1)$

Proof. (**i**) Since, $\tilde{\mathbb{A}}_1\leq\tilde{\mathbb{B}}_1\leq\tilde{\mathbb{C}}_1$, there exits
$t_a(u_{ij})\leq t_b(u_{ij})\leq t_c(u_{ij})$ and $1-f_a(u_{ij})\leq 1-f_b(u_{ij})\leq 1-f_c(u_{ij})$.
Let $\phi_{ac}(i,j)=\phi_{t_{ac}}(i,j)+\phi_{f_{ac}}(i,j)$ and $\phi_{ab}(i,j)=\phi_{t_{ab}}(i,j)+\phi_{f_{ab}}(i,j)$
then $\phi_{ab}(i,j)\leq\phi_{ac}(i,j)$

$$\text{or, } 1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}\big(\phi_{t_{ab}}(i,j)+\phi_{f_{ab}}(i,j)\big)^p}$$
$$\geq 1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}\big(\phi_{t_{ac}}(i,j)+\phi_{f_{ac}}(i,j)\big)^p}$$

or, $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{C}}_1)\leq\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1)$.
Similarly, we also prove that $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{C}}_1)\leq\tilde{S}_e^p(\tilde{\mathbb{B}}_1,\tilde{\mathbb{C}}_1)$.
(**ii**) From definition,
$\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\ \tilde{\mathbb{B}}_1\cap\tilde{\mathbb{C}}_1)$

$$=1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}(\phi_{t_{a,b\cap c}}(i,j)+\phi_{f_{a,b\cap c}}(i,j))^p} \text{ and}$$

$\tilde{S}_e^p(\tilde{\mathbb{A}}_1\cap\tilde{\mathbb{B}}_1,\ \tilde{\mathbb{A}}_1\cap\tilde{\mathbb{C}}_1)$

$$=1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}(\phi_{t_{a\cap b,a\cap c}}(i,j)+\phi_{f_{a\cap b,a\cap c}}(i,j))^p}.$$

Let $\phi_{a,b\cap c}(i,j)=\phi_{t_{a,b\cap c}}(i,j)+\phi_{f_{a,b\cap c}}(i,j)$ and $\phi_{a\cap b,a\cap c}(i,j)=$

$\phi_{t_{a\cap b,a\cap c}}(i,j)+\phi_{f_{a\cap b,a\cap c}}(i,j)$.
$\phi_{a,b\cap c}(i,j)-\phi_{a\cap b,a\cap c}(i,j)=$
$\frac{1}{2}[|t_a-(t_b\wedge t_c)|-|(t_a\wedge t_b)-(t_a\wedge t_c)|$
$+|(1-f_a)-((1-f_a)\vee(1-f_c))|$
$-|((1-f_a)\vee(1-f_b))-((1-f_a)\vee(1-f_c))|]$.

Case I: Let $t_a\le t_b\le t_c$ and $1-f_a\le 1-f_b\le 1-f_c$. Then $\phi_{a,b\cap c}(i,j)\ge\phi_{a\cap b,a\cap c}(i,j)$

or, $1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}\left(\phi_{t_{a,b\cap c}}(i,j)+\phi_{f_{a,b\cap c}}(i,j)\right)^p}$

$\le 1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}\left(\phi_{t_{a\cap b,a\cap c}}(i,j)+\phi_{f_{a\cap b,a\cap c}}(i,j)\right)^p}$

or, $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1\cap\tilde{\mathbb{C}}_1)\le\tilde{S}_e^p(\tilde{\mathbb{A}}_1\cap\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1\cap\tilde{\mathbb{C}}_1)$.

Case II: Let $t_a\le t_b\le t_c$ and $1-f_a\ge 1-f_b\ge 1-f_c$. Then $\phi_{a,b\cap c}(i,j)\ge\phi_{a\cap b,a\cap c}(i,j)$
or, $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1\cap\tilde{\mathbb{C}}_1)\le\tilde{S}_e^p(\tilde{\mathbb{A}}_1\cap\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1\cap\tilde{\mathbb{C}}_1)$.

Case III: Let $t_a\ge t_b\ge t_c$ and $1-f_a\le 1-f_b\le 1-f_c$. Then $\phi_{a,b\cap c}(i,j)\ge\phi_{a\cap b,a\cap c}(i,j)$ or, $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1\cap\tilde{\mathbb{C}}_1)\le\tilde{S}_e^p(\tilde{\mathbb{A}}_1\cap\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1\cap\tilde{\mathbb{C}}_1)$.

Case IV: Let $t_a\ge t_b\ge t_c$ and $1-f_a\ge 1-f_b\ge 1-f_c$. Then also $\phi_{a,b\cap c}(i,j)\ge\phi_{a\cap b,a\cap c}(i,j)$ or, $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1\cap\tilde{\mathbb{C}}_1)\le\tilde{S}_e^p(\tilde{\mathbb{A}}_1\cap\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1\cap\tilde{\mathbb{C}}_1)$.

(**iii**) Proof is similar to (ii).

(**iv**) $\phi_{a,b+c}(i,j)=\phi_{t_{a,b+c}}(i,j)+\phi_{f_{a,b+c}}(i,j)$ and $\phi_{a+b,a+c}(i,j)=\phi_{t_{a+b,a+c}}(i,j)+\phi_{f_{a+b,a+c}}(i,j)$.
Now, $\phi_{a,b+c}(i,j)-\phi_{a+b,a+c}(i,j)=$
$\frac{1}{2}[|(t_a-t_b)-t_c(1-t_b)|-|(1-t_a)(t_b-t_c)|$
$+|(1-f_a)-(1-f_b)(1-f_c)|-|(1-f_a)((1-f_b)-(1-f_c))|]$.
Let $t_a\le t_b\le t_c$ and $1-f_a\le 1-f_b\le 1-f_c$.
Then $\phi_{a,b+c}(i,j)\ge\phi_{a+b,a+c}(i,j)$

or, $1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}\left(\phi_{t_{a,b+c}}(i,j)+\phi_{f_{a,b+c}}(i,j)\right)^p}$

$\le 1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}\left(\phi_{t_{a+b,a+c}}(i,j)+\phi_{f_{a+b,a+c}}(i,j)\right)^p}$

or, $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1+\tilde{\mathbb{C}}_1)\le\tilde{S}_e^p(\tilde{\mathbb{A}}_1+\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1+\tilde{\mathbb{C}}_1)$.
In this way we can prove it for other cases.

(**v**) Proof is similar to (iv).

(**vi**) $\phi_{a,b@c}(i,j)=\phi_{t_{a,b@c}}(i,j)+\phi_{f_{a,b@c}}(i,j)$
and $\phi_{a@b,a@c}(i,j)=\phi_{t_{a@b,a@c}}(i,j)+\phi_{f_{a@b,a@c}}(i,j)$.
Now, $\phi_{a,b@c}(i,j)-\phi_{a@b,a@c}(i,j)=$
$\frac{1}{4}[|(2t_a-t_b.t_c)|-|t_b.t_c|+|2(1-f_a)-$
$(1-f_b)-(1-f_c)|-|(1-f_b)-(1-f_c)|]$.
Let $t_a\le t_b\le t_c$ and $1-f_a\le 1-f_b\le 1-f_c$.
Then $\phi_{a,b@c}(i,j)\ge\phi_{a@b,a@c}(i,j)$

or, $1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}\left(\phi_{t_{a,b@c}}(i,j)+\phi_{f_{a,b@c}}(i,j)\right)^p}$

$\le 1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}\left(\phi_{t_{a@b,a@c}}(i,j)+\phi_{f_{a@b,a@c}}(i,j)\right)^p}$

or, $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1@\tilde{\mathbb{C}}_1)\le\tilde{S}_e^p(\tilde{\mathbb{A}}_1@\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1@\tilde{\mathbb{C}}_1)$. In this way we can prove it for other cases.

(**vii**) $\phi_{a\cap b,a\cup b}(i,j)=\phi_{t_{a\cap b,a\cup b}}(i,j)+\phi_{f_{a\cap b,a\cup b}}(i,j)$
$=\frac{1}{2}[|t_a\wedge t_b-t_a\vee t_b|+|(1-f_a)\vee(1-f_b)-(1-f_a)\wedge(1-f_b)|]$.
If $t_a\le t_b$ and $1-f_a\le 1-f_b$, then $\phi_{a\cap b,a\cup b}(i,j)=\frac{1}{2}[|t_a-t_b|+|f_b-f_a|]=\phi_{a,b}(i,j)$.
Thus $\tilde{S}_e^p(\tilde{\mathbb{A}}_1\cap\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1\cup\tilde{\mathbb{C}}_1)=\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1)$.
Similarly we prove for other cases.

(**viii**) $\phi_{a,b\$c}(i,j)=\phi_{t_{a,b\$c}}(i,j)+\phi_{f_{a,b\$c}}(i,j)$ and $\phi_{a\$b,a\$c}(i,j)=\phi_{t_{a\$b,a\$c}}(i,j)+\phi_{f_{a\$b,a\$c}}(i,j)$.
Now, $\phi_{a,b\$c}(i,j)-\phi_{a\$b,a\$c}(i,j)=$
$\frac{1}{2}[|t_a-\sqrt{t_b.t_c}|-|\sqrt{t_a.t_b}-\sqrt{t_a.t_c}|$
$-|(1-f_a)+\sqrt{f_c-f_b}|-|\sqrt{f_b-f_a}-\sqrt{f_c-f_a}|]$.
Let $t_a\le t_b\le t_c$ and $1-f_a\le 1-f_b\le 1-f_c$. Then $\phi_{a,b\$c}(i,j)\ge\phi_{a\$b,a\$c}(i,j)$

or, $1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}\left(\phi_{t_{a,b\$c}}(i,j)+\phi_{f_{a,b\$c}}(i,j)\right)^p}$

$\le 1-\frac{1}{\sqrt[p]{r\times q}}\sqrt[p]{\sum_{i=1}^{r}\sum_{j=1}^{q}\left(\phi_{t_{a\$b,a\$c}}(i,j)+\phi_{f_{a\$b,a\$c}}(i,j)\right)^p}$

or, $\tilde{S}_e^p(\tilde{\mathbb{A}}_1,\tilde{\mathbb{B}}_1\$\tilde{\mathbb{C}}_1)\le\tilde{S}_e^p(\tilde{\mathbb{A}}_1\$\tilde{\mathbb{B}}_1,\tilde{\mathbb{A}}_1\$\tilde{\mathbb{C}}_1)$.
Similarly we prove it for other cases. □

5. Application

In multi-criteria decision making problem, identify the best institution from a set of institutions with the help of ideal alternative, it will be in our own choice. Suppose three institutions $\tilde{\mathbb{A}}_1,\tilde{\mathbb{A}}_2$ and $\tilde{\mathbb{A}}_3$ selected by the students on the basis of three parameters: good faculty (C_1), strong library (C_2) and distance to institution (C_3). Suppose when a institution fulfill all the parameters, the ideal alternative is of the form of IFM $\tilde{\mathbb{A}}^*=[u_{ij},\langle t_a(u_{ij}),1-f_a(u_{ij})\rangle]$, where $t_a(u_{ij})=1$ and $1-f_a(u_{ij})=0$ for $i,j=1,2,3$. The students identify institutions $\tilde{\mathbb{A}}_i$, $(i=1,2,3)$ based on the parameters C_j, $(j=1,2,3)$, it will be obtained through questionnaire in the public domain. Suppose when a student ask about the institution $\tilde{\mathbb{A}}_1$ based on the parameter C_1, he/she may answer for good faculty is 60 percent and not good is 20 percent. We expressed it $u_{11}=\langle .6,.2\rangle$. Similarly obtained answers based on other parameters. Apply same process for the other institutions. Let, on the basis of parameters we can obtain three IFMs in the form:

$$\tilde{\mathbb{A}}_1=\begin{pmatrix}\langle .6,.2\rangle & \langle .5,.3\rangle & \langle .7,.3\rangle\\ \langle .5,.3\rangle & \langle .3,.6\rangle & \langle .8,.1\rangle\\ \langle .2,.5\rangle & \langle .4,.2\rangle & \langle .5,.5\rangle\end{pmatrix}$$

$$\tilde{\mathbb{A}}_2=\begin{pmatrix}\langle .4,.2\rangle & \langle .2,.5\rangle & \langle .3,.4\rangle\\ \langle .5,.4\rangle & \langle .6,.3\rangle & \langle .5,.3\rangle\\ \langle .2,.6\rangle & \langle .3,.3\rangle & \langle .3,.2\rangle\end{pmatrix}$$

$$\tilde{\mathbb{A}}_3=\begin{pmatrix}\langle .2,.4\rangle & \langle .5,.3\rangle & \langle .3,.2\rangle\\ \langle .8,.2\rangle & \langle .4,.3\rangle & \langle .2,.6\rangle\\ \langle .5,.1\rangle & \langle .7,.1\rangle & \langle .3,.2\rangle\end{pmatrix}$$

By using definition 3.2, the SMs between $\tilde{\mathbb{A}}_i (i = 1,2,3)$ and $\tilde{\mathbb{A}}^*$ are: $\tilde{S}_d^p(\tilde{\mathbb{A}}^*, \tilde{\mathbb{A}}_1) = 0.172$, $\tilde{S}_d^p(\tilde{\mathbb{A}}^*, \tilde{\mathbb{A}}_2) = 0.169$, $\tilde{S}_d^p(\tilde{\mathbb{A}}^*, \tilde{\mathbb{A}}_3) = 0.174$. The ranking order of the three institutions according to the SM is $\tilde{\mathbb{A}}_3 \succ \tilde{\mathbb{A}}_1 \succ \tilde{\mathbb{A}}_2$. Hence, $\tilde{\mathbb{A}}_3$ is the best institution.

6. Conclusion

Here, we have presented some new process to measure the similarity between IFMs. At first we give notion of IFMs and then introduced several SMs to calculate the similarity between IFMs. We have highlighted some results related to these. Finally, an application of the developed approach through a numerical example is presented.

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